ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 2, Summer 1987

THE DISTANCE TO THE ANALYTIC TOEPLITZ OPERATORS

BY

KENNETH R. DAVIDSON[1]

The algebra $\mathcal{T}(H^\infty)$ of all analytic Toeplitz operators is a reflexive, maximal abelian subalgebra of $\mathcal{B}(H^2)$. Thus there are three natural measures of how far an operator A is from $\mathcal{T}(H^\infty)$, namely

$$d(A) = \inf_{h \in H^\infty} \|A - T_h\|,$$
$$\delta(A) = \sup_{h \in H^\infty, \|h\|_\infty = 1} \|AT_h - T_h A\|$$

and

$$\beta(A) = \sup_{\omega \text{ inner}} \|P_\omega^\perp A P_\omega\|$$

where $P_\omega = T_\omega T_\omega^*$ is the orthogonal projection onto any invariant subspace of $\mathcal{T}(H^\infty)$. It is immediate that all three of these measures vanishes precisely on $\mathcal{T}(H^\infty)$. It will be shown that they are comparable. More precisely:

THEOREM 1. *Let A belong to $\mathcal{B}(H^2)$. If A is lower triangular,*

$$\beta(A) \le d(A) \le \delta(A) \le 2d(A) \le 18\beta(A).$$

In general,

$$\tfrac{1}{2}d(A) \le \delta(A) \le 2d(A) \quad \text{and} \quad \beta(A) \le d(A) \le 19\beta(A).$$

THEOREM 2. *Let $\mathcal{T}$ be a unital, weak* closed subalgebra of Toeplitz operators. Then for any A in $\mathcal{B}(H^2)$,*

$$d(A, \mathcal{T}) \le 39 \sup\{\|P^\perp AP\| : P \in \text{Lat}\, \mathcal{T}\}.$$

Received May 29, 1985.
[1]This work was supported partially by a grant from National Science and Engineering Council of Canada.

Let L^2 denote the space of Lebesgue square integrable functions on the unit circle, and let H^2 be the closed span in L^2 of the polynomials. Given a bounded, measurable function f on the circle ($f \in L^\infty$), let M_f be the bounded operator given by $(M_f h)(x) = f(x)h(x)$. The Toeplitz operator of f is the operator on H^2 given by $T_f = P_{H^2}T_f|_{H^2}$. The subalgebra of L^∞ of all functions with all negative Fourier coefficient equal to zero is H^∞.

The Toeplitz operator T_z is unitarily equivalent to the unilateral shift. The weak* closed algebra it generates is $\mathcal{T}(H^\infty) = \{T_h \colon h \in H^\infty\}$, which is precisely $\{A\colon AT_z = T_z A\} = \{T_z\}'$. Beurling's Theorem describes the invariant subspaces of T_z as $\{\omega H^2\colon \omega \text{ are inner}\}$ where an inner function is an element ω in H^∞ such that $|\omega| = 1$ in L^∞. A theorem of Sarason [10] shows that

$$\mathcal{T}(H^\infty) = \{T\colon T\omega H^2 \subseteq \omega H^2 \text{ for all } \omega \text{ inner}\}.$$

See [5] for an exposition of these ideas.

A reflexive algebra $\mathcal{T}$ with the property that $d(A, \mathcal{T}) \leq C \sup \|P^\perp AP\|$ as P runs over the orthogonal projections onto the invariant subspaces of $\mathcal{T}$ is called hyper-reflexive [2]. For example, nest algebras [1], nice von Neumann algebras [3], and abelian, unital weak* closed algebras of normal operators [9] are hyper-reflexive. However, not all reflexive algebras have this property [6], and even reflexive algebras with commutative subspace lattice need not by hyper-reflexive [4].

Sarason [10] showed that every unital, weak* closed algebra of Toeplitz operators is reflexive. R. Olin and J. Thomson raised the question of whether $\mathcal{T}(H^\infty)$ is in fact hyper-reflexive (personal conversation), and indeed it is. In the case of von Neumann algebras, Christensen [3] noted the relation between the derivation estimate δ and the subspace estimate β. This plays a crucial role here.

First we collect the well known estimates.

LEMMA 3. *Let $\mathcal{T}$ be an algebra of operators, and let $\mathcal{T}'$ be its commutant. Let A be any operator. Then*

$$\sup_{P \in \operatorname{Lat} \mathcal{T}} \|P^\perp AP\| \leq d(A, \mathcal{T})$$

and

$$\sup_{B \in \mathcal{T}', \|B\| \leq 1} \|AB - BA\| \leq 2d(A, \mathcal{T}).$$

Proof. Let P be a projection onto an invariant subspace of $\mathcal{T}$. For any T in $\mathcal{T}$,

$$\|P^\perp AP\| = \|P^\perp(A - T)P\| \leq \|A - T\|.$$

So the first estimate follows. Given B in $\mathcal{T}'$ and T in $\mathcal{T}$,

$$\|AB - BA\| = \|(A - T)B - B(A - T)\| \leq 2\|B\|\,\|A - T\|$$

so the second estimate follows. □

Thus $\beta(A) \leq d(A)$ and $\delta(A) \leq 2d(A)$. One more estimate is easy.

LEMMA 4. *For A in $\mathcal{B}(H^2)$, $\beta(A) \leq \delta(A)$.*

Proof. Every invariant projection of $\mathcal{T}(H^\infty)$ has the form $P_\omega = T_\omega T_\omega^*$ for some inner function ω. Since T_ω^* takes ωH^2 isometrically onto H^2,

$$\|P_\omega^\perp A P_\omega\| = \|P_\omega^\perp A T_\omega\| = \|P_\omega^\perp (T_\omega A - A T_\omega)\| \leq \delta(A).$$

Now take the supremum over all inner functions. □

It will be convenient to verify that the situation is ideal for Toeplitz operators. Let $P_n = P_{z^n}$ denote the projection onto $z^n H^2$.

LEMMA 5. *For f in L^∞, $d(T_f) = \beta(T_f) = \delta(T_f) = d(f, H^\infty)$.*

Proof. By Nehari's Theorem [5], $d(T_f) = d(f, H^\infty) = \|H_f\|$ where H_f is the Hankel operator $P_{H^2}^\perp M_f P_{H^2}$ as an operator from H^2 to $H^{2\perp}$. A moment's thought reveals that $\|P_n^\perp T_f P_n\| = \|H_f P_n^\perp\|$. Hence

$$\beta(T_f) \leq d(T_f) = \|H_f\| = \lim_{n\to\infty} \|H_f P_n^\perp\| = \lim_{n\to\infty} \|P_n^\perp T_f P_n\| \leq \beta(T_f).$$

So $\beta(T_f) = d(T_f)$. Then if h belongs to H^∞ and $\|h\|_\infty \leq 1$,

$$\|T_f T_h - T_h T_f\| = \|T_{fh} - T_h T_f\| = \|P_{H^2} M_h P_{H^2}^\perp M_f P_{H^2}\| \leq \|H_f\| = d(T_f).$$

So $d(T_f) = \beta(T_f) \leq \delta(T_f) \leq d(T_f)$. □

The next portion of our proof relies on the following result of Arveson [1, Prop. 5.2].

PROPOSITION A. *There is a linear projection π of $\mathcal{B}(H^2)$ onto the space of Toeplitz operators $\{T_f\colon f \in L^\infty\}$ such that:*
(1) *$\pi(I) = I$ and $\|\pi\| = 1$.*
(2) *$\pi(T_h A) = \pi(A T_h) = \pi(A) T_h$ for all A in $\mathcal{B}(H^2)$, $h \in H^\infty$.*
(3) *$\pi(A)$ belongs to the weak* closed convex hull of $\{T_{z^n}^* A T_{z^n},\ n \geq N\}$.*
(4) *If A is lower triangular, $\pi(A)$ belongs to $\mathcal{T}(H^\infty)$.*

Remark. Although (3) is not stated in Prop. 5.2. of [1], it is a consequence of the proof of Props. 5.1 and 5.2.

LEMMA 6. *For A in $\mathscr{B}(H^2)$, $d(\pi(A)) \leq \beta(A)$ and $\|A - \pi(A)\| \leq \delta(A)$.*

Proof. Let ω be an inner function. By property (3),

$$\begin{aligned}\|P_\omega^\perp \pi(A) P_\omega\| &\leq \sup_n \|P_\omega^\perp T_{z^n}^* A T_{z^n} P_\omega\| \\ &= \sup_n \|T_{z^n}^*\left(P_{\omega z^n}^\perp A P_{\omega z^n}\right) T_{z^n}\| \\ &\leq \beta(A).\end{aligned}$$

Hence by Lemma 5, $d(\pi(A)) = \beta(\pi(A)) \leq \beta(A)$. Likewise,

$$\begin{aligned}\|A - \pi(A)\| &\leq \sup_n \|A - T_{z^n}^* A T_{z^n}\| \\ &= \sup_n \|T_{z^n}^*(T_{z^n} A - A T_{z^n})\| \\ &\leq \delta(A).\end{aligned}$$

□

COROLLARY 7. *If A is lower triangular, $d(A) \leq \delta(A)$. In general,*

$$d(A) \leq \delta(A) + \beta(A).$$

Proof. If A is lower triangular, property (4) guarantees that $\pi(A)$ belongs to $\mathscr{T}(H^\infty)$. So $d(A) \leq \|A - \pi(A)\| \leq \delta(A)$ by Lemma 6. In general,

$$d(A) \leq \|A - \pi(A)\| + d(\pi(A), \mathscr{T}(H^\infty)) \leq \delta(A) + \beta(A)$$

by Lemmas 5 and 6. □

LEMMA 8. *Let A belong to $\mathscr{B}(H^2)$. If $\pi(A) = 0$ or A is lower triangular, then $d(A) \leq 9\beta(A)$. In general, $d(A) \leq 19\beta(A)$.*

Proof. If A is lower triangular, then $\pi(A)$ belongs to $\mathscr{T}(H^\infty)$. So A can be replaced by $A - \pi(A)$, and thus one can assume $\pi(A) = 0$. By part (3) of Proposition A, if $\pi(A) = 0$, then 0 belongs to the weak* closed convex hull of $\{T_{z^n}^* A T_{z^n}\}$. Assuming $A \neq 0$, normalize so that $\|A\| = 1$. By Lemma 6, $d(A) \leq \|A\| = 1 \leq \delta(A)$.

Fix $\varepsilon > 0$. Choose an integer N and a unit vector $x = P_N^\perp x$ so that $\|Ax\| > 1 - \varepsilon$. Replace N by a larger integer if necessary so that

$$\|P_N^\perp A x\| > 1 - \varepsilon, \quad \|P_N A x\| < \varepsilon \quad \text{and} \quad \|P_N A^* A x\| < \varepsilon.$$

Let $y = \|P_N^\perp Ax\|^{-1} P_N^\perp Ax$. By hypothesis, 0 belongs to the convex hull of

$$\{(T_{z^n}^* A T_{z^n} x, y),\ n > N\}.$$

So choose an integer $n > N$ so that $\operatorname{Re}(Az^n x, z^n y) < \varepsilon$.

Let $0 < a < 1$, and let ω be the inner function

$$\omega = \frac{a - z^n}{1 - az^n} = a - (1 - a^2)\sum_{k=1}^{\infty} a^{k-1} z^{kn}.$$

And let $\mathscr{k}$ be the unit vector (analogous to a kernel function) given by

$$\mathscr{k} = (1 - a^2)^{1/2} \sum_{k=0}^{\infty} a^k z^{kn}.$$

For $0 \le l < n$ and $j \ge 0$, one readily obtains that $(\mathscr{k} z^l, \omega z^j) = 0$. Thus $\mathscr{k} z^l$ is orthogonal to ωH^2 for $0 \le l < n$. For notational convenience, set $b = (1 - a^2)^{1/2}$. Consider the unit vectors

$$\begin{aligned} \omega x &= ax - b^2 \sum_{k=1}^{\infty} a^{k-1} z^{kn} x \\ &= ax - b\mathscr{k} z^n x \\ &= ax - b^2 z^n x - ab\mathscr{k} z^{2n} x \end{aligned}$$

and

$$\mathscr{k} y = b \sum_{k=0}^{\infty} a^k z^{kn} y = by + a\mathscr{k} z^n y = by + abz^n y + a^2 z^{2n} \mathscr{k} y.$$

The latter function is a unit vector because the sum is an orthogonal direct sum. Since $\mathscr{k} y$ belongs to the span of $\{\mathscr{k} z^l, 0 \le l \le N\}$, it follows that $\mathscr{k} y$ is orthogonal to ωH^2. Thus

$$\begin{aligned} \beta(A) &\ge \|P_\omega^\perp A P_\omega\| \\ &\ge |(A\omega x, \mathscr{k} y)| \\ &\ge |(Aax, by) - (Ab^2 z^n x, abz^n y)| - |(Aax, a\mathscr{k} z^n y)| \\ &\quad - |(Ab\mathscr{k} z^n x, by)| - |(Ab\mathscr{k} z^n x, a^2 \mathscr{k} y)| - |(Aab\mathscr{k} z^{2n} x, abz^n y)| \\ &\ge \big(ab(1 - \varepsilon) - ab^3\varepsilon\big) - \big(a^2\|P_n Ax\| + b^2\|P_n A^* y\| + a^2 b + a^2 b^2\big) \\ &= ab(1 - a - ab) + O(\varepsilon). \end{aligned}$$

Now let ε tend to zero, and take $a = 1/4$ to obtain $\beta(A) > 1/9$. Thus $d(A) \le 9\beta(A)$.

For a general A in $\mathscr{B}(H^2)$,

$$\begin{aligned} d(A) &\leq d(A - \pi(A)) + d(\pi(A)) \\ &\leq 9\beta(A - \pi(A)) + \beta(\pi(A)) \\ &\leq 9\beta(A) + 10\beta(\pi(A)) \\ &\leq 19\beta(A). \end{aligned}$$ □

Proof of Theorem 1. Let A be lower triangular. By Lemmas 3 and 8, $\beta(A) \leq d(A) \leq 9\beta(A)$. By Corollary 7 and Lemma 3, $d(A) \leq \delta(A) \leq 2d(A)$. For general A, the same lemmas yield $\beta(A) \leq d(A) \leq 19\beta(A)$, and $\frac{1}{2}\delta(A) \leq d(A) \leq \delta(A) + \beta(A)$. By Lemma 4, $d(A) \leq 2\delta(A)$. □

Now we turn to the second theorem. In Olin and Thomson's paper [8], they remark that their first theorem has a simple proof in the case of the unilateral shift based on Szego's Theorem:

PROPOSITION B. *Let ϕ be a weak* continuous functional on $\mathscr{T}(H^\infty)$. Given $\varepsilon > 0$, there are vectors x and y in H^2 such that $\|x\|\,\|y\| < (1 + \varepsilon)\|\phi\|$ so that $\phi(T_h) = (T_h x, y)$ for all h in H^∞.*

Theorem 2 will follow from this general result.

LEMMA 9. *Let $\mathscr{A}$ be a hyper-reflexive algebra with distance constant C_1. Suppose that every weak* continuous functional ϕ on $\mathscr{A}$, there are vectors x and y with $\|x\|\,\|y\| \leq C_2\|\phi\|$ such that $\phi(A) = (Ax, y)$ for A in $\mathscr{A}$. Then every unital, weak* closed subalgebra of $\mathscr{A}$ is hyper-reflexive with constant $C_1 + C_2 + C_1C_2$.*

Proof. Let $\mathscr{B}$ be a unital, weak* closed subalgebra of $\mathscr{A}$. For T in $\mathscr{B}(\mathscr{H})$, let $d(T) = d(T, \mathscr{B})$ and

$$\beta(T) = \sup\{\|P^\perp TP\| : P \in \operatorname{Lat} \mathscr{B}\}.$$

Then $d(T, \mathscr{A}) \leq C_1 \sup\{\|P^\perp TP\| : P \in \operatorname{Lat} \mathscr{A}\} \leq C_1\beta(T)$. Let A belong to $\mathscr{A}$ such that $\|T - A\| \leq C_1\beta(T)$.

By the Hahn-Banach Theorem, there is a weak* continuous linear functional ϕ on $\mathscr{A}$ of norm one which annihilates $\mathscr{B}$ such that $\phi(A) = d(A, \mathscr{B})$. Let x and y be vectors in $\mathscr{H}$, so that $\|x\|\,\|y\| \leq C_2$ and $\phi(A) = (Ax, y)$ for A in $\mathscr{A}$. Let P be the orthogonal projection onto the $\mathscr{B}$ invariant subspace $\overline{\mathscr{B}x}$. Since ϕ annihilates $\mathscr{B}$, $Px = x$ and $P^\perp y = y$. Thus

$$\beta(A) \geq \|P^\perp AP\| \geq \frac{|(Ax, y)|}{\|x\|\,\|y\|} \geq C_2^{-1} d(A, \mathscr{B}).$$

Now $\beta(A) \le \beta(T) + \beta(A - T) \le (C_1 + 1)\beta(T)$. Hence

$$d(T) \le \|T - A\| + d(A, \mathscr{B}) \le (C_1 + C_2 + C_1C_2)\beta(T). \qquad \square$$

Remark. This result can be reformulated in terms of subspaces. To do this, compare with Larson's paper [7].

Proof of Theorem 2. If $T_fT_g = T_gT_f = T_{fg}$, then f and g both belong to either H^∞ or $\overline{H^\infty}$. So any algebra $\mathscr{T}$ consisting solely of Toeplitz operators is either contained in $\mathscr{T}(H^\infty)$ or $\mathscr{T}(\overline{H^\infty})$. So the proof is now immediate from Theorem 1, Proposition B, and Lemma 9. $\square$

Now we give some examples to show that the inequalities of Theorem 1 are strict. Let P be the rank one projection onto the constants. Then

$$\beta(P) \le d(P) = d(P, \mathbf{C}I) = 1/2,$$
$$\delta(P) = \sup_{\|h\|\le 1} \|P^\perp T_h P\| = \|P^\perp T_z P\| = 1.$$

Taking

$$\omega = \frac{1/\sqrt{2} - z}{1 - z/\sqrt{2}} \quad \text{and} \quad k = \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{2}^{\,k+1}},$$

then $k = P_\omega^\perp k$, so

$$\beta(P) \ge |(P\omega, k)| = 1/2.$$

So

$$\beta(P) = d(P) = 1/2\delta(P).$$

If T_f is a Toeplitz operator with f not in H^∞, by Lemma 5, $\beta(T_f) = d(T_f) = \delta(T_f)$.

Let D be the diagonal matrix $D = \operatorname{diag}(1, -1, 0, 0, \ldots)$. As above, $d(D) = d(D, \mathbf{C}I) = 1$. Thus $\delta(D) \le 2$, and $\|(DT_z - T_zD)1\| = 2$; so $\delta(D) = 2$. It will be shown that $\beta(D) < 1$. If ω is inner, then

$$\omega = a + bz + z^2h \quad \text{and} \quad |a|^2 + |b|^2 \le 1.$$

Let x be a unit vector in ωH^2. Then

$$x = c\omega + d\omega z + f\omega z^2 = ac + (bc + ad)z + \cdots$$

where c, d are scalars and f belongs to H^2. So

$$Dx = ac - (bc + ad)z.$$

To maximize $\|P_\omega^\perp DP_\omega x\|$, one may assume $|c|^2 + |d|^2 = 1$. Now $(Dx, \omega z^n) = 0$ for $n \geq 2$, and

$$(Dx, \omega) = |a|^2 c - \bar{b}(bc + ad), \qquad (Dx, \omega z) = -\bar{a}(bc + ad).$$

Thus

$$\|P_\omega^\perp DP_\omega \omega\|^2 = |a|2 + |b|^2 - \left(|a|^2 - |b|^2\right)^2 - |ab|^2,$$
$$\|P_\omega^\perp DP_\omega \omega z\|^2 = |a|^2 - |ab|^2 - |a|^4 = |a|^2(1 - |a|^2 - |b|^2).$$

Fix b and maximize both terms over a, to obtain

$$\|P_\omega^\perp D\omega\|^2 \leq \begin{cases} \frac{1}{4} + \frac{3}{2}|b|^2 - \frac{3}{4}|b|^4, & 0 \leq |b|^2 \leq \frac{1}{3} \\ 2|b|^2(1 - |b|^2), & \frac{1}{3} \leq |b|^2 \leq 1 \end{cases}$$

and

$$\|P_\omega^\perp D\omega z\|^2 \leq \tfrac{1}{4}\left(1 - |b|^2\right)^2.$$

Thus by the Cauchy-Schwartz inequality,

$$\begin{aligned} &\left\|P_\omega^\perp D(c\omega + d\omega z)\right\|^2 \\ &\quad\leq \left(|c|\,\|P_\omega^\perp D\omega\| + |d|\,\|P_\omega^\perp D\omega z\|\right)^2 \\ &\quad\leq \|P_\omega^\perp D\omega\|^2 + \|P_\omega^\perp D\omega z\|^2 \\ &\quad\leq \begin{cases} \frac{1}{2} + |b|^2 - \frac{1}{2}|b|^4 \leq \frac{7}{9} < \frac{9}{11} & \text{if } 0 \leq |b|^2 \leq \frac{1}{3} \\ \dfrac{1 + 10|b|^2 - 11|b|^4}{4} \leq \dfrac{9}{11} & \text{if } \frac{1}{3} \leq |b|^2 \leq 1. \end{cases} \end{aligned}$$

Hence $\beta(D) \leq 3/\sqrt{11} < d(D)$.

I do not known any example for which $\delta(T) < d(T)$.

Added in proof. J. Kraus and D. Larson, *Reflexivity and distance estimates*, Proc. London Math. Soc. (to appear) also prove Lemma 9 (their Theorem 3.3). They explicitly raise the question resolved in this paper (Problem 3.8).

REFERENCES

1. W. AVERSON, *Interpolation problems in nest algebras*, J. Functional Analysis, vol. 20 (1975), 208–233.
2. ______, *Ten lectures on operator algebras*, C.B.M.S. Lecture Note Series #55, Amer. Math. Soc., Providence, 1984.
3. E. CHRISTENSEN, *Extensions of derivations*, J. Functional Analysis, vol. 27 (1978), 234–247.

4. K.R. DAVIDSON and S.C. POWER, *Failure of the distance formula*, J. London Math. Soc. (2), vol. 32 (1985), pp. 157–165.
5. R.G. DOUGLAS, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.
6. J. KRAUS and D. LARSON, *Some applications of a technique for constructing reflexive operator algebras*, J. Operator Theory, vol. 13 (1985), pp. 227–236.
7. D. LARSON, *Hyperreflexivity and a dual product construction*, Trans. Amer. Math. Soc., vol. 294 (1986), pp. 79–88.
8. R. OLIN and J. THOMSON, *Algebras of subnormal operators*, J. Functional Analysis, 37 (1980), pp. 271–301.
9. S. ROSENOER, *Distance estimates for von Neumann algebras*, Proc. Amer. Math. Soc., vol. 86 (1982), pp. 248–252.
10. D. SARASON, *Invariant subspaces and unstarred operator algebras*, Pacific J. Math., vol. 17 (1966), pp. 511–517.

UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO