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THE PERMUTATION INDEX OF p-DEFECT ZERO CHARACTERS

BY

ELIOT JACOBSON

1. Introduction

Artin's induction theorem asserts that any rational valued character of a finite group can be written as a rational linear combination of transitive permutation characters with cyclic stabilizer. The question of which denominators can occur has led to such notions as the "Artin exponent", and the "permutation index". The principal aim of this work is to study a certain divisibility property of the permutation index for p-defect zero characters.

To be more specific, let G be a finite group and let $\chi \in \operatorname{irr}(G)$. Let $\operatorname{sp}(\chi) = \Sigma_\sigma \chi^\sigma$ where σ ranges through the Galois group $\operatorname{Gal}(\mathbf{Q}(\chi), \mathbf{Q})$. Then $\operatorname{sp}(\chi)$ is a rational valued character of G, hence by Artin's theorem, $|G|\operatorname{sp}(\chi)$ in a $\mathbf{Z}$-linear combination of permutation characters. The "permutation index" of χ, denoted $n(\chi)$, is the least positive integer so that $n(\chi)\operatorname{sp}(\chi)$ is such a combination.

The p-Defect Zero Conjecture. Let G be a finite group, and let p be a prime. Let $\chi \in \operatorname{irr}(G)$ satisfy $p \nmid |G|/\chi(1)$ (χ has p-defect zero). Then $p \nmid n(\chi)$.

This natural conjecture arose by consideration of Solomon's similar result for the Schur index [5]. Gluck's work [1] is applicable to some extent; his ideas led to Proposition 4. We discuss the significance of his methods in Theorem F. Parks' work [4] is also helpful in understanding the problem, however the configuration that makes his proof go through is nowhere in sight. Thus a proof for even solvable G seems quite remote at present. We present here several cases in which the conjecture holds. We use these cases as evidence towards the general result, hoping to inspire greater interest in this and similar problems.

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2. Main theorems

Throughout this paper, G will denote a fixed finite group and $P(G)$ will denote its ring of generalized permutation characters. If $H \leq G$ and $\chi \in \operatorname{irr}(G)$, the notation $\operatorname{irr}(H/\chi)$ denotes the set of irreducible constituents of χ_H.

We begin with the supersolvable case. Then $n(\chi)$ always divides the character inner product $[\chi, \psi]$ for any $\psi \in P(G)$. Taking ψ to be the regular character gives $n(\chi) | \chi(1)$ (see [4]).

THEOREM A. *If G is supersolvable, and $\chi \in \operatorname{irr}(G)$ has p-defect zero, then* $p \nmid n(\chi)$.

Proof. Let P be p-Sylow in G. Then $n(\chi)$ divides $[\chi, 1_P^G] = [\chi_P, 1_P] = \chi(1)/|P|$, as χ vanishes on p-singular elements. ■

We now state several well-known results, the first of which is problem 5.22 of [3], and the rest of which can be found in Section 2 of [4].

If $\chi \in \operatorname{irr}(G)$ and $N \trianglelefteq G$ with $N \subseteq \ker \chi$, then $n_{G/N}(\chi)$ is the permutation index of χ, viewed as a character of G/N.

PROPOSITION 1. *Let $\chi \in \operatorname{irr}(G)$ with $N = \ker \chi$. Then $n(\chi) = n_{G/N}(\chi)$.*

Let $\chi \in \operatorname{irr}(G)$ and suppose that $N \trianglelefteq G$. Choose $\nu \in \operatorname{irr}(N)$ such that $[\chi, \nu^G] > 0$. Put $M = \mathscr{I}_G(\nu)$, the inertia group of ν, and let $\mu \in \operatorname{irr}(M/\nu)$, so that by Clifford's theorem, $\chi = \mu^G$. Let $T = \{g \in G | \exists \sigma \in \operatorname{Gal}(\mathbf{C}, \mathbf{Q}(\chi))$ with $\nu^g = \nu^\sigma\}$. Let $\theta = \mu^T$. We have:

PROPOSITION 2. *Consider the setting of the preceding paragraph. Then*:
(a) $\operatorname{sp}(\theta)^G = \operatorname{sp}(\chi)$.
(b) $\operatorname{sp}(\mu)^G = |T : M| \operatorname{sp}(\chi)$, *and $n(\chi)$ divides $n(\mu)|G : M|$.*
(c) *If not all irreducible constituents of χ_N are Galois conjugate, then $n(\chi)$ divides $n(\theta)$, and $T \neq G$.*
(d) *$M \trianglelefteq T$, and* $T/M \simeq \operatorname{Gal}(\mathbf{Q}(\mu), \mathbf{Q}(\theta))$.

PROPOSITION 3. *Let $\chi \in \operatorname{irr}(G)$. Suppose there is an integer n and a prime p such that the p-part of $n(\chi)$ does not divide n. Then there is a p-quasi-elementary subgroup $L \leq G$ such that $n \operatorname{sp}(\chi)_L \notin P(L)$.*

Our first result handles the solvable case when $\chi(1)$ is a p-power.

THEOREM B. *Let G be a solvable group. If $\chi \in \operatorname{irr}(G)$ with $\chi(1) = |G|_p$, then $p \nmid n(\chi)$.*

Proof. Let G be a counterexample of least order. By Proposition 1, χ is faithful. Now, since each irreducible constituent of χ restricted to $\mathcal{O}_p(G)$ also has p-defect zero, we have $\mathcal{O}_p(G) = 1$. Let $N = \mathcal{O}_{p'}(G)$. By Proposition 2(c), all irreducible constituents of χ_N are Galois conjugate, hence faithful. As $(N, \chi(1)) = 1$, these constituents are linear, so that N is cyclic. By Hall-Higman, $N = \mathbf{C}_G(N)$, so $\chi = \lambda^G$ for any $\lambda \in \operatorname{irr}(N/\chi)$ and hence N is a Hall p'-subgroup. As G/N is a p-group, G is supersolvable. By Theorem A, $p \nmid n(\chi)$, a contradiction. ■

We now work towards a characterization of the structure of a minimal solvable counterexample. As we shall see, the structure obtained in the previous result is typical.

PROPOSITION 4. *Let L be a p-quasi-elementary group with an elementary abelian p-Sylow. Then every rational valued character of L is in $P(L)$.*

Proof. It suffices to show that for any $\chi \in \operatorname{irr}(L)$ we have $\operatorname{sp}(\chi) \in P(L)$, that is, that $n(\chi) = 1$. Assume the result is false, and let L be of least order for which there is $\chi \in \operatorname{irr}(L)$ with $n(\chi) > 1$. By minimality, and Proposition 1, χ is faithful. Denote the cyclic normal p-complement of L by $M = \langle x \rangle$.

If not all constituents of χ_M are Galois conjugate, then by 2(c), there is $T < L$ and $\theta \in \operatorname{irr}(T)$ such that $n(\chi) | n(\theta)$. By minimality of L, $n(\theta) = 1$, contradicting $n(\chi) > 1$. In particular, if $\lambda \in \operatorname{irr}(M)$ with $[\lambda, \chi_M] > 0$, then λ is faithful, hence $\mathscr{I} = \mathscr{I}_L(\lambda) = \mathbf{C}_L(x)$, an abelian, normal (since $T = L$) subgroup. By Clifford's theorem, there is $\nu \in \operatorname{irr}(\mathscr{I})$ with $\nu^L = \chi$ and $\nu_M = \lambda$. Let P be p-Sylow in L, and let $P_x \subseteq P$ be a p-Sylow subgroup of $\mathscr{I}$. Since P is elementary abelian, choose $H \subseteq P$ with $HP_x = P$ and $H \cap P_x = \{1\}$. Then $\mathscr{I}H = L$ and $\mathscr{I} \cap H = \{1\}$. We consider 1_H^L. Since L is supersolvable, we have that $n(\chi)$ divides $[\chi, 1_H^L]$. However, Frobenius reciprocity gives

$$[\chi, 1_H^L] = [\nu^L, 1_H^L] = [\nu_H^L, 1_H] = [\nu_{\mathscr{I} \cap H}^H, 1_H] = [1_{\{1\}}^H, 1_H] = 1,$$

contradicting $n(\chi) > 1$. ■

PROPOSITION 5. *Suppose that for some $\chi \in \operatorname{irr}(G)$ and prime p, one has $p | n(\chi)$. Then a p-Sylow subgroup of G is not elementary abelian.*

Proof. Compare this result with [3, 10.9]. Let n be the p'-part of $|G|$. Assume a p-Sylow subgroup of G is elementary abelian. By Proposition 3, there is a p-quasi-elementary subgroup $L \leq G$ such that $n \cdot \operatorname{sp}(\chi)_L \notin P(L)$. However, a p-Sylow subgroup of L is elementary abelian, and $n \cdot \operatorname{sp}(\chi)$ is rational valued. This contradicts the conclusion of Proposition 4. ■

In fact, one can show that if a p-Sylow subgroup of G is abelian of exponent p^a, then $p^a \nmid n(\chi)$ for all $\chi \in \operatorname{irr}(G)$.

The next three propositions are standard.

PROPOSITION 6. *Suppose $N \lhd_{\neq} G$ with G/N a p-group. Suppose $\chi \in \operatorname{irr}(G)$ and $\Theta \in \operatorname{irr}(N/\chi)$, with θ invariant in G. Write $\chi_N = e\theta$. Then $e < |G:N|$.*

Proof. Apply [3, 6.19] and induction. ■

PROPOSITION 7. *Let $N \trianglelefteq G$ with G/N a p-group. Let $\chi \in \operatorname{irr}(G)$ be of p-defect zero and let $\theta \in \operatorname{irr}(N/\chi)$. Then $N = \mathscr{I}_G(\theta)$, so that $\theta^G = \chi$.*

Proof. Let $\mathscr{I} = \mathscr{I}_G(\theta)$ and suppose $N \neq \mathscr{I}$. Choose $\mu \in \operatorname{irr}(\mathscr{I}/\theta)$ with $\mu^G = \chi$. Plainly, μ is of p-defect zero. Now, θ is invariant in $\mathscr{I}$ and $\mathscr{I}/N$ is a p-group, so by Proposition 6, $\mu_N = e\theta$ where $e < |\mathscr{I}:N|$. Then

$$\frac{\chi(1)_p}{\theta(1)_p} = \frac{|G:\mathscr{I}|\mu(1)_p}{\theta(1)_p} = |G:\mathscr{I}|e < |G:N|,$$

so that

$$\chi(1)_p < |G:N| \cdot \theta(1)_p \leq |G|_p,$$

a contradiction. ■

PROPOSITION 8. *Let $\chi \in \operatorname{irr}(G)$ have p-defect zero, and let $M, K \trianglelefteq G$ with $K \leq M$ and M/K a p-group. Let $\gamma \in \operatorname{irr}(K/\chi)$. Then $\mathscr{I}_G(\gamma) \cap M = K$.*

Proof. Apply Proposition 7 with $G = M$ and $N = K$. ■

THEOREM C. *Let G be a solvable group such that the p-defect zero conjecture is true for every proper subgroup of G, but there is $\chi \in \operatorname{irr}(G)$ of p-defect zero with $p \mid n(\chi)$. Then:*

(a) *$\mathcal{O}^p(G)$ is a normal p-complement.*

(b) *$\mathcal{O}_p(G) = 1$.*

(c) *A p-Sylow subgroup of G is abelian, but not elementary abelian.*

(d) *Every abelian normal subgroup of G is cyclic.*

Proof. (a) Set $N = \mathcal{O}^p(G)$ and assume that $p \mid |N|$. Let $K \subseteq M \subseteq N$ be such that M/K is p-chief in G with $p \nmid |N:M|$. Let $\gamma \in \operatorname{irr}(K/\chi)$. By Proposition 8, we have $\mathscr{I} \cap M = K$, where $\mathscr{I} = \mathscr{I}_G(\gamma)$.

Let H/K be a complement of M/K in N/K, which exist by Schur-Zassenhaus. Let $T = \mathscr{I} \cap N$. We argue that $T \subseteq H$. First we establish that $p \nmid |T:K|$.

If $p \mid |T : K|$ then let P/K be p-Sylow in T/K. As M/K is p-Sylow in N/K and $M/K \trianglelefteq N/K$, we have $P \subseteq M$. But then

$$P \subseteq M \cap T \subseteq M \cap N \cap \mathcal{I} = K$$

a contradiction.

Now, by Proposition 2(c), (d) and minimality of G, we see that $\mathcal{I} \trianglelefteq G$ and $G/\mathcal{I}$ is abelian, being isomorphic to a section of the Galois group of a cyclotomic field. Then $T \trianglelefteq N$ and T/K is a p'-subgroup of N/K. As H/K is p'-Hall and $T/K \trianglelefteq N/K$ we have $T/K \subseteq H/K$, that is, $T \subseteq H$.

The proof of (a) is completed by observing that $N/T = N/\mathcal{I} \cap N \simeq IN/\mathcal{I} \leq G/\mathcal{I}$, hence N/T is abelian. As $H/T \leq N/T$ we have $H \trianglelefteq N$. But $|N : H|$ is a p-power, contradicting $N = \mathcal{O}^p(G)$.

(b) As $\mathcal{O}_p(G) \trianglelefteq G$, the irreducible constituents of the restriction of χ to $\mathcal{O}_p(G)$ all have p-defect zero. As $\mathcal{O}_p(G)$ is a p-group, this is possible only if $\mathcal{O}_p(G) = 1$.

(c) Let $\theta \in \operatorname{irr}(N/\chi)$. By Proposition 7, $N = \mathcal{I}_G(\theta)$. By Proposition 2(c), (d) and the minimality of G we have $G/N \simeq \operatorname{Gal}(\mathbf{Q}(\theta)/\mathbf{Q}(\chi))$, an abelian group. As G/N is isomorphic to a p-Sylow subgroup of G, the result follows from Proposition 5.

(d) The linear irreducible constituents of χ restricted to such a subgroup are all Galois conjugate, hence faithful, by Proposition 2. ■

THEOREM D. *Suppose that $|G|$ is odd and that every q-Sylow subgroup of G for $q \neq p$ is abelian. If $\chi \in \operatorname{irr}(G)$ has p-defect zero, then $p \nmid n(\chi)$.*

Proof. Let G be a counterexample of minimum order. By Theorem C, every abelian normal subgroup of G is cyclic. Let $N \trianglelefteq G$ with $p \nmid |N|$ and G/N abelian of order $|G|_p$. By [3, 6.22, 6.23], G is now an M-group. By [2, Theorem B], G is now supersolvable. This contradicts Theorem A. ■

THEOREM E. *Suppose there is $M \trianglelefteq G$ with M abelian and G/M supersolvable (G is "abelian by supersolvable"). If $\chi \in \operatorname{irr}(G)$ has p-defect zero, then $p \nmid n(\chi)$.*

Proof. Assume the result false and let G be a counterexample of least cardinality. Note that subgroups of G are again abelian by supersolvable. By Theorem C, we have that M is cyclic, hence G is supersolvable, contradicting Theorem A. ■

It is interesting to see what light Gluck's criterion sheds on this conjecture. Let $n = |G|_{p'}$ and let $\chi \in \operatorname{irr}(G)$ have p-defect zero. According to the main theorem of [1], we have $n\,\operatorname{sp}(\chi) \in P(G)$ if and only if for every p'-element $z \in G$, then class function $n\,\operatorname{sp}(\chi)_z$ defined on a p-Sylow subgroup P_z of

$\mathbf{C}_G(z)$ extends to a permutation character on a p-Sylow subgroup $\hat{P}_z$ of $N_G\langle z\rangle$ which is constant on $N_G\langle z\rangle$-conjugate elements in $\hat{P}_z$. Since χ vanishes on p-singular elements of G, we have

$$n\,\mathrm{sp}(\chi)_z = n\cdot |P_z|^{-1}\cdot \mathrm{sp}(\chi)(z)\cdot 1_{\{1\}}^{P_z}.$$

Observe that as $\chi(z)|\mathrm{cl}(z)|/\chi(1)$ is an algebraic integer, summing over $\mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$ shows that

$$\frac{|\mathrm{cl}(z)|}{\chi(1)}\cdot \mathrm{sp}(\chi)(z)$$

is an integer. Since χ has p-defect zero, it follows that $|P_z|$ divides $\mathrm{sp}(\chi)(z)$, hence $n\,\mathrm{sp}(\chi)_z$ is an integer multiple of the regular character on P_z. We wish to extend this character to $\hat{P}_z$. The most natural way occurs if $|\hat{P}_z|$ divides $\mathrm{sp}(\chi)(z)$, then $n\,\mathrm{sp}(\chi)_z$ extends to an integer multiple of the regular character on $\hat{P}_z$. This gives a practical test for specific examples, which we state as follows.

THEOREM F. *Let G be a finite group, and let $\chi \in \mathrm{irr}(G)$ have p-defect zero. If $|N_G\langle z\rangle|_p$ divides $\mathrm{sp}(\chi)(z)$ for every p-regular element $z \in G$, then $p \nmid n(\chi)$.*

Added in proof. Tom Wolf and Olaf Manz recently established the truth of the conjecture for groups of order $p^a q^b$.

REFERENCES

1. D. GLUCK, *A characterization of generalized permutation characters*, J. Algebra, vol. 63 (1980), pp. 541–547.
2. I.M. ISAACS, *Abelian normal subgroups of M-groups*, Math. Z., vol. 182 (1983), pp. 205–221.
3. ______, *Character theory of finite groups*, Academic Press, New York, 1976.
4. A. PARKS, *Generalized permutation characters of solvable groups*, J. Algebra, vol. 93 (1985), pp. 445–474.
5. L. SOLOMON, *On Schur's index and the solutions of $G^n = 1$ in a finite group*, Math. Z., vol. 78 (1962), pp. 122–125.

OHIO UNIVERSITY
ATHENS OHIO