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HIRONAKA GROUP SCHEMES

BY

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We will be working in the category of k-schemes where k is a field. A Hironaka group scheme is a closed subgroup scheme H of an affine space $\mathbf{A}^n$ which is invariant under scalar multiplication by $\mathbf{G}_m$; i.e., H is a cone. Hironaka [1] introduced such group schemes as the stabilizer in $\mathbf{A}^n$ of a closed cone in $\mathbf{A}^n$. As a Hironaka group scheme is its own stabilizer any Hironaka group scheme arises this way.

We intend to classify roughly all Hironaka group schemes and explain their structure. Our presentation avoids Dieudonné modules and Hopf algebras. If $\operatorname{char}(k) = 0$, a Hironaka group scheme is just a vector subspace. Hence in this paper we will assume that $\operatorname{char}(k)$ is a prime p.

Let $H \subset \mathbf{A}^n$ be a Hironaka group scheme. Then the quotient $\mathbf{A}^n/H$ is an algebraic group whose formation commutes with base extension. Furthermore we have an induced action of $\mathbf{G}_m$ on $\mathbf{A}^n/H$. Here $\mathbf{G}_m$ acts via automorphisms of this quotient group. The central result is:

THEOREM. *$\mathbf{A}^n/H$ is $\mathbf{G}_m$-equivariantly isomorphic to a finite sum $\oplus\, \mathbf{A}^{m_q}(q)$ where the q's and m_*'s are positive integers and $\mathbf{A}^{m_q}(q)$ is the affine space $\mathbf{A}^{m_q}$ with the $\mathbf{G}_m$-action given by $t^*x = t^q x$.*

Proof. Let B be the ring of regular functions on $\mathbf{A}^n/H$ and let m^* be the comultiplication. Let P be the k-subspace of primitive element in B; i.e.,

$$P = \{ f \in B \mid m^*f = f \otimes 1 + 1 \otimes f \}.$$

By linear algebra the formation of P commutes with base extension and P is contained in the maximal ideal m of functions vanishing at the identity of $\mathbf{A}^n/H$.

LEMMA. *The induced k-linear mapping d: $P \to m/m^2$ is surjective.*

Proof. By base extension it is enough to check this when k is algebraically closed. Now $\mathbf{A}^n/H$ is an algebraic group variety which is commutative and its

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pth-power mapping is trivial as these properties are inherited from $\mathbf{A}^n$. Thus by Proposition 11 of Section 11 of [3], $\mathbf{A}^n/H$ is isomorphic to affine space $\mathbf{A}^m$ as an algebraic group. Now the linear functions on $\mathbf{A}^m$ are contained in P and map surjective onto its tangent space at zero. This proves the lemma.

Next as $\mathbf{G}_m$ acts by group automorphisms on $\mathbf{A}^n/H$, P is a sub-$\mathbf{G}_m$-module of B and m/m^2 is naturally a $\mathbf{G}_m$-module such that the differential d is $\mathbf{G}_m$-equivariant. As the representations of $\mathbf{G}_m$ are completely reducible into lines, we may find a set $S = \coprod_q \{f_{q,i}\}_{1 \leq i \leq m_q}$ of $\mathbf{G}_m$-eigenvectors in P such that $d(S)$ is a basis of m/m^2 where $\alpha^* f_{q,i} = T^q \otimes f_{q,i}$ where α^* is the coaction of $\mathbf{G}_m$ on B. As $f_{q,i}$ is contained in $k[\mathbf{A}^n]$ and vanishes at 0, the integer q must be positive.

The assignment $x \mapsto \{f_{q,i}(x)\}$ gives a morphism $\psi\colon \mathbf{A}^n/H \to \oplus \mathbf{A}^{m_q}(q)$. As the f's are primitive, ψ is a homomorphism and, as they are eigenvectors, ψ is $\mathbf{G}_m$-equivariant. By construction ψ induces an isomorphism on tangent spaces and hence ψ is étale. We need to see that ψ is an isomorphism.

To see this it is enough to show that the kernel $\operatorname{Ker}(\psi)$ of ψ is the identity e of $\mathbf{A}^n/H$. Now $\operatorname{Ker}(\psi) - \{e\}$ and $\{e\}$ are two disjoint $\mathbf{G}_m$-invariant closed subschemes of $\mathbf{G}_m$. By invariant theory [2] these sets are separated by $\mathbf{G}_m$-invariants in B. On the other hand the only $\mathbf{G}_m$-invariants in $B \subset k[\mathbf{A}^n]$ are the constants. Hence $\operatorname{Ker}(\psi) - \{e\}$ is empty. This proves that $\operatorname{Ker}(\psi) = \{e\}$, which was all we needed. Q.E.D.

By the theorem, H is the kernel of a $\mathbf{G}_m$-equivariant surjective homomorphism

$$f\colon \mathbf{A}^n \to \oplus \mathbf{A}^{m_q}(q).$$

The point is that the only such homomorphism are the obvious ones.

LEMMA. *A $\mathbf{G}_n$-equivariant non-zero homomorphism $g\colon \mathbf{A}^n \to \mathbf{A}^1(q)$ has the form*

$$g(x_1, \ldots, x_n) = \Sigma a_i x_i^{p^i}$$

where the a's are constant and $p^i = q$.

Proof. $g(x)$ is homogeneous of degree q by equivariance. We also need $g(x+y) = g(x) + g(y)$. An exercise with binomial coefficients gives the result. Q.E.D.

By the lemma, f is given by a formula

$$f(x_1, \ldots, x_n) = \left(f_{q,i}(x_1, \ldots, x_n)_{1 \leq i \leq m_q}\right)$$

where $q = p^i$ and $f_{q,i} = \Sigma a_{q,i,r} x_r^q$. As f is a surjective homomorphism, it is flat. Hence the $\{f_{q,i}\}$ are a $k[\mathbf{A}^n]$-sequence.

We can easily see the grosser structure of a Hironaka group scheme by base extending to a perfect field using

COROLLARY. *If k is perfect, we can decompose $\mathbf{A}^n$ into a direct sum (not bold!)*

$$\mathbf{A}_0 \oplus \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_d \oplus B$$

of vector subspaces so that

$$H = 0 \oplus (\mathrm{Frob})^{-1}(0) \oplus \cdots \oplus (\mathrm{Frob})^{d-1}(0) \oplus B$$

where $(\mathrm{Frob}^i)^{-1}(0)$ is the kernel of the ith Frobenius Frob^i on $\mathbf{A}_i$.

Furthermore the numbers $(\dim \mathbf{A}_1, \ldots, \dim \mathbf{A}_d, \dim B)$ depend only on H as a group scheme.

Proof. As k is perfect we may write each $f_{q,i}$ as $(l_{q,i})^q$ where $l_{q,i}$ is a linear function. As the f's are an $k[\mathbf{A}^n]$-sequence, the l's are. Hence they are linearly independent. Thus we may assume that $l_{1,1} \ldots l_{1,m_1} \ldots l_{pd,m_{p^d}}$ are the first bunch of coordinates in $\mathbf{A}^n$. Thus $p_{i,*}(x) = x_j^{p^i}$ for some index j. The first result is now clear. The second remark follows as we can find the numbers once we know $\mathrm{rank}(\mathrm{Frob}^j)^{-1}(0))$ where now Frob^j is the jth Frobenius on H itself for enough values of j. Recall the rank of a finite group scheme S is $\dim_k \Gamma(S, \mathcal{O}_S)$. Q.E.D.

Returning to the general case we can compute the above numerical invariants without base extension because the rank of the kernel of the Forbenius's does change under base extension. Next we will see how we can simplify the equations $\{f_{q,i} = 0\}$ of H in the general case.

We can easily compute the invariants of $H \subset \mathbf{A}^n$ from its equation $\{f_{q,i}\}_{1 \le i \le m_q}$ with $q = p^j$ for $1 \le j \le d$. Then the invariants are

$$\left(m_{p^1}, \ldots, m_{p^d}, n - \sum_{j=0}^{d} m_{p^i} \right)$$

where the last number is just the dimension of H. As $f_{1,1}, \ldots, f_{1,m_1}$ are linearly independent functions on $\mathbf{A}^n$, we can assume that they are the first m_1 coordinates if $f_{p,1} = \Sigma_i a_i x_i^p$. Then we may replace $f_{p,1}$ by $\Sigma_{i \ge m_1} a_i x_i^p$ where we have to subtract pth powers of the previous f's. Then one coordinate a_i of $f_{p,1}$ is non-zero; we can assume that it is a_{m_1+1}. Replace $f_{p,1}$ by its quotient

a_{m_1+1}. Repeating this idea we get a matrix of coefficients for the f's like

$$\begin{array}{c|c|c|c|l}
\begin{matrix}1\\01\\001\end{matrix} & 0 & 0 & 0 & \left.\vphantom{\begin{matrix}1\\01\\001\end{matrix}}\right\} m_1 \\ \hline
0 & \begin{matrix}1\\01\\001\end{matrix} & \begin{matrix}*********\end{matrix} & \begin{matrix}************\end{matrix} & \left.\vphantom{\begin{matrix}1\\01\\001\end{matrix}}\right\} m_{p^1} \\ \hline
0 & 0 & \begin{matrix}1\\01\\001\end{matrix} & \begin{matrix}************\end{matrix} & \left.\vphantom{\begin{matrix}1\\01\\001\end{matrix}}\right\} m_{p^2} \\
\underbrace{\qquad}_{m_1} & \underbrace{\qquad}_{m_p} & \underbrace{\qquad}_{m_{p^2}} & &
\end{array}$$

where this gives the coefficients a_i of $x_i^{p^j}$ in $f_{p^j,\cdot}$. The empty boxes can be made zero by subtracting lower rows in the same square. (An example should clarify this procedure.) At last we have natural equations: $f = (x_1, \ldots, x_{m_1})$ and for $j \geq 1$,

$$f_{p,i} = x^{p^j}_{\Sigma_{h<j}m_k+i} + \sum_{k>\Sigma_{h\leq j}m_h}^{n} a_k x_k^{p^j}.$$

Once we have such a natural system of equations we can explain the structure of H. For $0 \leq j \leq n$, let V_j be the coordinate subspace of $\mathbf{A}^n$ where all the coordinates are zero except for the first $\Sigma_{h\leq j} m_k$. Then $V_0 \subset V_1 \subset \cdots \subset V_d$ is an increasing sequence. Consider the induced sequence

$$H_0 \subset H_1 \subset \cdots \subset H_d \subset H$$

of Hironaka subgroup scheme where $H_i = H \cap V_d$.

The successive quotients $S_i \equiv H_i/H_{i-1} \subset V_i/V_{i-1} \equiv W_i$ are Hironaka subgroup schemes. We also have a quotient $S_{d+1} \equiv H/H_d$ in $\mathbf{A}^n/V_d \equiv W_{d+1}$. Now we will explain the remarkable facts about this filtration.

PROPOSITION (a) *$H_0 = \{0\}$ with reduced structure,*
(b) *For $1 \leq j \leq d$, H_i has numerical invariants $(m_1, \ldots, m_j, 0)$.*
(c) *For $1 \leq j \leq d$, S_i is the kernel of the j*th *Frobenius in W_i.*
(d) $S_{d+1} = W_{d+1}$.

Proof. For (a), H_0 is given by the vanishing of the coordinates in V_0. For (b), the equations of H_j are given by the first $\Sigma_{h\leq j} m_h$ column in the matrix. For (c), by (b) we know rank $(H_i) = \Sigma_{1\leq k\leq i} m_i p^i$. Thus

$$\operatorname{rank}(S_i) = m_i p^i = (\dim W_i) p^i.$$

Now H_i and, hence, S_i are killed by the ith Frobeneous. So

$$S_i \subset \operatorname{Ker}\left(\operatorname{Frob}^i \text{ in } W_i\right) \equiv K_i,$$

but rank (K_i) $= (\dim W_i)p^i$. Thus $S = K_i$. For (d), $\dim S_{d+1} = \dim H = \dim W_{d+1}$. Thus $S_{d+1} = W_{d+1}$. Q.E.D.

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