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INVARIANT SUBSPACE LATTICES THAT COMPLEMENT EVERY SUBSPACE

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1. Introduction and the main result

In the theory of factorizations of operator polynomials and analytic operator functions (see [4] [5] [8] [9]) it is necessary to study invariant subspaces of a given operator (acting on a complex Hilbert space) that are direct complements to a given subspace. Of course, such invariant subspaces do not always exist. (Easy finite dimensional examples bear this out.) The following fact (due to D. Gurarie [6]) proved to be a very useful tool in the factorization theory (see [9] [10]): Let A be a (bounded linear) operator acting on a Hilbert space $\mathscr{H}$, which is similar to a normal operator with finite spectrum. Then for every (closed) subspace $\mathscr{M}$ of $\mathscr{H}$ there is an A-invariant subspace $\mathscr{R}$ such that $\mathscr{M} \cap \mathscr{R} = \{0\}$ and $\mathscr{M} + \mathscr{R} = \mathscr{H}$.

The main goal of this article is to prove that the operators similar to normals with finite spectrum are the only ones whose lattice of invariant subspaces complements every subspace.

It is convenient to introduce some definitions and notation. A *subspace* of a (complex) Hilbert space $\mathscr{H}$ is a closed linear manifold. Two subspaces, $\mathscr{M}$ and $\mathscr{R}$, are called *complementary* (denoted $\mathscr{M} \dot{+} \mathscr{R} = \mathscr{H}$) if $\mathscr{M} \cap \mathscr{R} = \{0\}$, $\mathscr{M} + \mathscr{R} = \mathscr{H}$. An *operator* $A : \mathscr{H} \to \mathscr{H}$ is a bounded linear transformation from $\mathscr{H}$ into itself, and we denote by $\mathscr{L}(\mathscr{H})$ the algebra of all operators acting on $\mathscr{H}$. The lattice of all invariant subspaces of $T \in \mathscr{L}(\mathscr{H})$ is denoted by Lat T. We shall say that Lat T has the *complement property* if for every subspace $\mathscr{M}$ and $\mathscr{H}$ there exists $\mathscr{R} \in \text{Lat}\, T$ such that $\mathscr{M} \dot{+} \mathscr{R} = \mathscr{H}$. The lattice Lat T is said to have the *chain complement property* if for every finite chain of subspaces

$$\mathscr{M}_1 \subset \mathscr{M}_2 \subset \cdots \subset \mathscr{M}_r \subset \mathscr{H}$$

there exist a chain

$$\mathscr{R}_1 \supset \mathscr{R}_2 \supset \cdots \supset \mathscr{R}_r$$

of T-invariant subspaces such that $\mathscr{M}_i \dot{+} \mathscr{R}_i = \mathscr{H}$ for all $i = 1, 2, \ldots, r$.

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The main result of this article is the following characterization of operators whose lattices of invariant subspaces have these properties.

THEOREM 1. *Let $T \in \mathscr{L}(\mathscr{H})$, where $\mathscr{H}$ is a complex Hilbert space. The following statements are equivalent*:
(i) Lat T *has the complement property*;
(ii) Lat T *has the chain complement property*;
(iii) T *is similar to a normal operator with finite spectrum*.

If $\mathscr{H}$ is finite dimensional, then it is not difficult to verify the theorem by using the Jordan form of an operator. (Observe that the complement property is a similarity invariant.) It is remarkable that exactly the same result remains true for infinite dimensional Hilbert spaces.

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2. Proof of the main result

Obviously, (ii) implies (i). As mentioned at the Introduction, (iii) implies (i). (A proof of this implication can be found in [9]). A simple argument (see Proposition 3 in [10]) shows that (i) and (iii) together imply (ii). It only remains to prove that (i) implies (iii).

Throughout the remainder of this section, $T \in \mathscr{L}(\mathscr{H})$ will always denote an operator whose lattice of invariant subspaces has the complement property. For S in $\mathscr{L}(\mathscr{H})$, we denote by ran S the range (or image) of S (ran $S = \{Sx: x \in \mathscr{H}\}$), and $\ker S = \{x \in \mathscr{H}: Sx = 0\}$. The closed linear span of a collection of $\{\mathscr{X}_i\}_{i \in \Gamma}$ of subsets of $\mathscr{H}$ is denoted by

$$\bigvee\{\mathscr{X}_i: i \in \Gamma\}.$$

Finally, $\mathscr{X}^-$ stands for the closure of a subset $\mathscr{X}$ of $\mathscr{H}$.

We begin with two simple observations.

LEMMA 1. (i) *Every $\mathscr{M}$ in* Lat T *has a complement in* Lat T.
(ii) $\mathscr{H} = \bigvee\{\ker(T - \lambda): \lambda \in \mathbf{C}\}$.

Proof. (i) This is a trivial consequence of the definition, applied to the subspaces in Lat T.

(ii) Let $\mathscr{H}_0 = \bigvee\{\ker(T - \lambda): \lambda \in \mathbf{C}\}$. If $\mathscr{H}_0 \neq \mathscr{H}$, then we can find a subspace $\mathscr{M}$ of $\mathscr{H}$ such that $\mathscr{H}_0 \subset \mathscr{M}$, and $\dim \mathscr{M}^\perp = 1$. Let $\mathscr{R} \in$ Lat T be a complement of $\mathcal{M}$. Clearly, $\mathcal{R} = \bigvee\{e\}$ for some unit vector e such that $Te = \mu e$ for some $\mu \in \mathbf{C}$. Therefore, $\mathcal{R} \subset \mathcal{H}_0 \subset \mathcal{M}$, a contradiction. ■

LEMMA 2. *For each* $\mu \in \mathbf{C}$,

$$[\operatorname{ran}(T-\mu)]^- = \bigvee\{\ker(T-\lambda)\colon \lambda \neq \mu\}$$

and

$$\mathcal{H} = \ker(T-\mu) \dotplus [\operatorname{ran}(T-\mu)]^-.$$

Moreover, $[\operatorname{ran}(T-\mu)]^-$ *is the only complement of* $\ker(T-\mu)$ *in* $\operatorname{Lat} T$. *Conversely*, $\ker(T-\mu)$ *is the only complement of* $[\operatorname{ran}(T-\mu)]^-$ *in* $\operatorname{Lat} T$.

Proof. We can assume $\mu = 0$. Let $\mathcal{M} = \ker T$, and let $\mathcal{R} \in \operatorname{Lat} T$ be some complement of $\mathcal{M}$. Suppose y is a unit vector in $\ker(T-\lambda)$, for some $\lambda \neq 0$, and let $y = m + r$, where $m \in \mathcal{M}$, $r \in \mathcal{R}$; then

$$y = \frac{1}{\lambda}Ty = \frac{1}{\lambda}(Tm + Tr) = \frac{1}{\lambda}Tr \in \mathcal{R},$$

because $r \in \mathcal{R}$ and $T\mathcal{R} \subset \mathcal{R}$. It follows that

$$\mathcal{R} \supset \bigvee\{\ker(T-\lambda)\colon \lambda \neq 0\}.$$

Since, by Lemma 1 (ii),

$$\mathcal{H} = \ker T \vee \left(\bigvee\{\ker(T-\lambda)\colon \lambda \neq 0\}\right) = \mathcal{M} \dotplus \mathcal{R},$$

it follows from the above inclusion that

$$\mathcal{R} = \bigvee\{\ker(T-\lambda)\colon \lambda \neq 0\}.$$

Clearly, $\operatorname{ran} T = T\mathcal{R}$, and $(\operatorname{ran} T)^- = (T\mathcal{R})^- = \mathcal{R}$ is the only complement of $\ker T$ in $\operatorname{Lat} T$.

Conversely, let $\mathcal{S} \in \operatorname{Lat} T$ be some complement of $\mathcal{N} = (\operatorname{ran} T)^-$. Then a unit vector $m \in \ker T$ can be (uniquely) written as $m = n + s$, where $n \in \mathcal{N}$, $s \in \mathcal{S}$. It follows that $0 = Tm = Tn + Ts$, so that

$$Ts = -Tn \in \operatorname{ran} T,$$

and therefore

$$Ts \in (\operatorname{ran} T)^- \cap \mathcal{S} = \mathcal{N} \cap \mathcal{S} = \{0\}.$$

Thus $n = m - s \in \ker T \cap \mathcal{N} = \{0\}$, whence we conclude that $s \in \ker T$; that is, $\mathcal{S} \supset \ker T$ and $\mathcal{S}$ is a complement of $(\operatorname{ran} T)^-$. Since $\ker T$ is also a complement of $(\operatorname{ran} T)^-$, we deduce that $\mathcal{S} = \ker T$. ■

More generally, we have:

LEMMA 3. *Let* Γ *be a subset of* $\mathbf{C}$, *and let*

$$\mathcal{M}(\Gamma) = \vee\{\ker(T-\lambda) \colon \lambda \in \Gamma\};$$

then

$$\mathcal{H} = \mathcal{M}(\Gamma) \dotplus \mathcal{M}(\mathbf{C} \setminus \Gamma).$$

Furthermore, $\mathcal{M}(\mathbf{C} \setminus \Gamma)$ *is the only complement of* $\mathcal{M}(\Gamma)$ *in* Lat T.

Proof. Clearly, $\mathcal{M}(\Gamma), \mathcal{M}(\mathbf{C} \setminus \Gamma) \in \operatorname{Lat} T$. Let $\mathcal{R} \in \operatorname{Lat} T$ be some complement of $\mathcal{M}(\Gamma)$, and let y be a unit vector in $\ker(T-\mu)$, for some $\mu \notin \Gamma$; then $y = m + r$, where $m \in \mathcal{M}(\Gamma)$ and $r \in \mathcal{R}$, and we have

$$\mu y = Ty = Tm + Tr = \mu m + \mu r,$$

so that

$$(T-\mu)r = -(T-\mu)m \in \mathcal{R} \cap \mathcal{M}(\Gamma) = \{0\}.$$

By Lemma 2, $\ker(T-\mu)$ is the only complement of $[\operatorname{ran}(T-\mu)]^-$. Since (by Lemma 2) $[\operatorname{ran}(T-\mu)]^- = \vee\{\ker(T-\lambda)\colon \lambda \neq \mu\}$ *includes* $\mathcal{M}(\Gamma)$, we deduce from $m \in \mathcal{M}(\Gamma)$ and $(T-\mu)m = 0$, that $m = 0$.

It readily follows that $y = r \in \mathcal{R}$. Hence,

$$\mathcal{R} \supset \mathcal{M}(\mathbf{C} \setminus \Gamma) = \vee\{\ker(T-\lambda)\colon \lambda \notin \Gamma\}.$$

Since $\mathcal{H} = \mathcal{M}(\Gamma) \vee \mathcal{M}(\mathbf{C} \setminus \Gamma) = \mathcal{M}(\Gamma) \dotplus \mathcal{R}$, it follows from the above inclusion that $\mathcal{R} = \mathcal{M}(\mathbf{C} \setminus \Gamma)$ is the only complement of $\mathcal{M}(\Gamma)$ in Lat T. ∎

COROLLARY 4. *If T has only finitely many distinct eigenvalues, then T is similar to a normal operator with finite spectrum.*

The proof follows immediately from Lemma 3.

Let $P(\Gamma)$ denote the projection of $\mathcal{H}$ onto $\mathcal{M}(\Gamma)$ along $\mathcal{M}(\mathbf{C} \setminus \Gamma)$. It follows from Lemma 3 that $\mathcal{B} = \{P(\Gamma)\colon \Gamma \subset \mathbf{C}\}$ is a *complete Boolean algebra* of idempotents. According to [2, Lemma XVII.3.3, p. 2196], $\mathcal{B}$ is bounded; that is, there exists a constant $C \geq 1$ such that $\|P(\Gamma)\| \leq C$ for all $\Gamma \subset \mathbf{C}$.

Our next observation is that

$$(*) \qquad \sigma(T|\mathcal{M}(\Gamma)) = \left[\Gamma \cap \sigma_p(T)\right]^-$$

for every $\Gamma \subset \mathbf{C}$, where $\sigma_p(T)$ denotes the point spectrum of T. To prove $(*)$, we can directly assume that $\Gamma \subset \sigma_p(T)$. Let $\Gamma_1 = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ be a finite subset of Γ, and let

$$x = \sum_{j=1}^{m} x_j \in \mathscr{M}(\Gamma_1), \quad \|x\| = 1,$$

where $x_j \in \ker(T - \lambda_j)$ $(j = 1,2,\dots, m)$. If $\lambda \notin \Gamma^-$, then

$$\|(T-\lambda)x\| = \left\| \sum_{j=1}^{m} (\lambda_j - \lambda)x_j \right\| \geq C' \min\{|\lambda - \lambda_j| : 1 \leq j \leq m\},$$

where C' is a constant depending only on C. Thus,

$$\|(T-\lambda)x\| \geq C' \operatorname{dist}[\lambda, \Gamma],$$

and since $\mathscr{M}(\Gamma) = \bigvee\{\mathscr{M}(\Gamma_1)\colon \Gamma_1 \subset \Gamma, \Gamma_1 \text{ finite}\}$, it follows that

$$\|(T-\lambda)y\| \geq C' \operatorname{dist}[\lambda, \Gamma]\|y\|$$

for every $y \in \mathscr{M}(\Gamma)$ and every $\lambda \notin \Gamma^-$. On the other hand,

$$x = \sum_{j=1}^{m} x_j = \sum_{j=1}^{m} (\lambda_j - \lambda)^{-1}(T-\lambda)x_j$$
$$= (T-\lambda)\left[\sum_{j=1}^{m} (\lambda_j - \lambda)^{-1} x_j\right].$$

belongs to $(T-\lambda)\mathscr{M}(\Gamma)$, so that $[(T-\lambda)\mathscr{M}(\Gamma)]^- = \mathscr{M}(\Gamma)$.

It follows that $(T-\lambda)|\mathscr{M}(\Gamma)$ is invertible for all $\lambda \notin \Gamma^-$. So we have

$$\sigma(T|\mathscr{M}(\Gamma)) \subset \Gamma^-.$$

Since we obviously have $\Gamma \subset \sigma(T|\mathscr{M}(\Gamma))$, we obtain $(*)$.

It follows from [2, Chapter XV] that T is a spectral operator. Furthermore, since $\mathscr{H} = \bigvee\{\ker(T-\lambda)\colon \lambda \in \sigma_p(T)\}$, it is not difficult to deduce that the spectral integral $\int \lambda P(d\lambda)$ coincides with T, and hence T is a *scalar type* spectral operator (see [2, XV.4] for details). By using a well-known result of J. Wermer [11] (Theorem XV.6.4 in [2]) we have the following conclusion.

COROLLARY 5. *T is similar to a normal operator with purely atomic spectral measure.*

Le coup de grâce.

LEMMA 6. *If $N = \oplus\{\lambda I_\lambda\colon \lambda \in \sigma_p(N)\}$ (I_λ the identity on* $\ker(N-\lambda)$*) is a normal operator with infinite spectrum, then* Lat N *does not have the complement property.*

Proof. $\sigma(N) = [\sigma_p(N)]^-$. Thus, if $\sigma(N)$ is not finite, then $\sigma_p(N)$ includes a sequence $\{\lambda_n\}_{n=1}^\infty$ of distinct eigenvalues. By passing, if necessary, to a subsequence of $\{\lambda_n\}_{n=1}^\infty$, and replacing N by $N - \alpha I$ for a suitable $\alpha \in \mathbf{C}$, we can directly assume that $\lambda_n \neq 0$ for all n, and $|\lambda_{n+1}/\lambda_n| \leq 1/2$, $n = 1, 2, \dots$.

For each $n = 1, 2, \dots$, pick a unit vector e_n in $\ker(N - \lambda_n)$. Note that $\{e_n\}_{n=1}^\infty$ is an orthonormal system. Now observe that if $\mathrm{Lat}(N|\mathscr{X})$ does not have the complement property for some $\mathscr{X} \in \mathrm{Lat}\, N$, then $\mathrm{Lat}\, N$ itself does not have that property. (Let $\mathscr{M}$ be a subspace of $\mathscr{X}$, let $\mathscr{R} \in \mathrm{Lat}\, N$ be a complement of $\mathscr{M}$, and let P denote the projection of $\mathscr{H}$ onto $\mathscr{R}$ along $\mathscr{M}$; then $\mathscr{X} = \{m + s\colon m \in \mathscr{M},\ s \in \mathscr{S} := P\mathscr{X}\}$. It is not difficult to check that $\mathscr{S} \in \mathrm{Lat}(N|\mathscr{X})$ is a complement of $\mathscr{M}$ in $\mathscr{X}$.)

By using this observation, we can directly assume that

$$\mathscr{H} = \bigvee\{e_n\}_{n=1}^\infty.$$

Since $\sigma(N) = \{0\} \cup \{\lambda_n\}_{n=1}^\infty$ (where $\{\lambda_n\}_{n=1}^\infty$ is a sequence converging to 0), it is straightforward to check that every invariant subspace of N is also invariant under the orthogonal projection of $\mathscr{H}$ onto $\ker(N - \lambda_n)$, and therefore reduces N.

Now define

$$f_p = (2p - 1)^{-1/2}(e_{(p-1)^2+1} + e_{(p-1)^2+2} + \cdots + e_{p^2})$$

($p = 1, 2, \dots$; $\{f_p\}_{p=1}^\infty$ an orthogonal system) and

$$\mathscr{M} = \left(\bigvee\{f_p\}_{p=1}^\infty\right)^\perp.$$

If $\mathscr{R} \in \mathrm{Lat}\, N$ is a complement of $\mathscr{M}$, then $\mathscr{R}$ is spanned by a subset of the orthonormal basis $\{e_n\}_{n=1}^\infty$ (because $\mathscr{R}$ reduces N), including $e_{s(p)}$ for some $s(p)$, $(p-1)^2 + 1 \leq s(p) \leq p^2$ ($p = 1, 2, \dots$; because otherwise it is impossible to write $f_p = m_p + r_p$, with $m_p \in \mathscr{M}$ and $r_p \in \mathscr{R}$).

Observe that

$$h_p = e_{s(p)} - (2p-2)^{-1} \sum_{j=(p-1)^2+1;\ j \neq s(p)}^{p^2} e_j \in \mathscr{M} \quad (p = 1, 2, \dots).$$

Therefore,

$$\mathrm{dist}\left[e_{s(p)}, \mathscr{M}\right] \leq \|e_{s(p)} - h_p\| = (2p-2)^{-1/2} \to 0 \quad (p \to \infty),$$

whence it follows that $\mathscr{M}$ and $\mathscr{R}$ *cannot be complementary subspaces*.

Hence, $\mathrm{Lat}\, N$ does not have the complement property. ■

From Corollary 5 and Lemma 6, it easily follows that T is similar to a normal operator with finite spectrum. Indeed, by Corollary 5, T is similar to a normal operator N with purely atomic spectral measure. If $\sigma(N)$ is not finite, then Lemma 6 shows that Lat N does not have the complement property; since the complement property is a similarity invariant, we conclude that Lat T does not have the complement property, contradicting our assumption. Hence, $\sigma(N)$ must be a finite set.

The proof of the Theorem 1 is now complete. ∎

3. Concluding remarks

(a) By combining the main result with [7, Theorem 4 (iii)], we have the following result.

THEOREM 2. *The following are equivalent for T in $\mathscr{L}(\mathscr{H})$:*
(i) Lat T *has the complement property;*
(ii) Lat T *has the chain complement property;*
(iii) *T is similar to a normal operator with finite spectrum;*
(iv) *the similarity orbit T,*

$$\mathscr{S}(T) = \{WTW^{-1}\colon W \in \mathscr{L}(\mathscr{H}) \text{ is invertible}\}.$$

is a closed subset of $\mathscr{L}(\mathscr{H})$.

(b) Theorem 1 suggests the following:

Problem. Characterize those operators A in $\mathscr{L}(\mathscr{H})$ such that Lat A has the quasi-complement property: for each subspace $\mathscr{M}$ of $\mathscr{H}$, there exists $\mathscr{R}$ in Lat A such that

$$\mathscr{M} \cap \mathscr{R} = \{0\}, (\mathscr{M} + \mathscr{R})^- = \mathscr{H}.$$

Observe that if either $\mathscr{M}$ or $\mathscr{M}^\perp$ is *finite dimensional*, then a quasi-complement of $\mathscr{M}$ is actually a complement. Therefore, Lemma 1 (ii) remains true in this more general setting; more precisely:

$$\mathscr{H} = \vee\{\ker(A-\lambda)\colon \lambda \in \mathbf{C}\} = \vee\{\ker(A-\lambda)^*\colon \lambda \in \mathbf{C}\},$$

where A^* denotes the adjoint of A.

Moreover, each $\mathscr{M}$ in Lat A has a quasi-complement in Lat A, and we can mimic part of the proof of Lemma 2 in order to show that

$$[\operatorname{ran}(A-\mu)]^- = \vee\{\ker(A-\lambda)\colon \lambda \neq \mu\}$$

is the only quasi-complement of $\ker(A-\mu)$ in Lat A. Indeed, if $\mu = 0$, $\mathscr{M} = \ker A$, and $\mathscr{R} \in$ Lat A is a quasi-complement of $\mathscr{M}$, then given a unit vector

$$y \in \ker(A-\lambda) \qquad (\lambda \neq 0)$$

and $\varepsilon > 0$, we can write $y = m_\varepsilon + r_\varepsilon + t_\varepsilon$, where $m_\varepsilon \in \mathscr{M}$, $r_\varepsilon \in \mathscr{R}$, and $\|t_\varepsilon\| < \varepsilon$. It follows that

$$y = \frac{1}{\lambda}Ty = \frac{1}{\lambda}Tr_\varepsilon + \frac{1}{\lambda}Tt_\varepsilon,$$

so that

$$\operatorname{dist}[y, \mathscr{R}] \leq \frac{1}{|\lambda|}\|T\|\varepsilon.$$

Since ε can be chosen arbitrarily small, and $\mathscr{R}$ is closed, we deduce that $y \in \mathscr{R}$. Therefore,

$$\mathscr{R} \supset \bigvee\{\ker(A - \lambda) : \lambda \neq 0\} = (\operatorname{ran} A)^{-},$$

whence we conclude as in the proof of Lemma 2 that $\mathscr{R} = (\operatorname{ran} A)^{-}$. Unfortunately, the remaining results of the article do not have simple "translations" to this new setting. The results of [3] can be relevant in connection with this problem.

References

1. N. Dunford, *Spectral operators*, Pacific J. Math., vol. 4 (1954), pp. 321–354.
2. N. Dunford and J.T. Schwartz, *Linear operators*, *Part III*, Interscience, New York, 1971.
3. C.-K. Fong, *Operators with complemented invariant subspace lattices*, Indiana Univ. Math. J., vol. 26 (1977), pp. 1045–1056.
4. I. Gohberg, P. Lancaster and L. Rodman, *Matrix polynomials*, Academic Press, New York, 1982.
5. ______, *Representations and divisivility of operator polynomials*, Canad. J. Math., vol. 30 (1978), pp. 1045–1069.
6. D. Gurarie, *On a geometric problem in Hilbert space*, preprint.
7. D.A. Herrero, *Closure of similarity orbits of Hilbert space operators*, *III*, Math. Ann., vol. 232 (1978), pp. 195–204.
8. H. Langer, *Factorization of operator pencils*, Acta Sci. Math. (Szeged), vol. 38 (1976), pp. 83–96.
9. A.S. Marcus and V.I. Matsaev, *On spectral factorization of holomorphic operator-valued functions*, Mat. Issled., vol. 47 (1978), pp. 71–100 (Russian).
10. L. Rodman, *On factorization of operator polynomials and analytic operator functions*, Rocky Mountain J. Math., vol. 16 (1986), pp. 153–162.
11. J. Wermer, *Commuting spectral measures on Hilbert space*, Pacific J. Math., vol. 4 (1954), 355–361.

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