ILLINOIS JOURNAL OF MATHEMATICS
Volume 32, Number 1, Spring 1988

A MIXED BOUNDARY VALUE PROBLEM FOR THE LAPLACIAN

BY

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0. Introduction

The purpose of this paper is to analyse the problem of Zaremba concerning the Laplacian on planar simply connected domains. Consider a region $\mathring{M}$ which is the interior of a compact simply connected domain $M \subset R^2$. Assume Γ_- is a non-empty contractible open submanifold of ∂M, and define $\Gamma_+ = \partial M - \overline{\Gamma}_-$. Let $H_\Delta^{s,r}(M)$ denote the usual Sobolev space $H^s(\mathring{M})$ if $s > r$, or

$$H^s(\mathring{M}) \cap \{u \in D'(M) \colon \Delta u \in L^2(M)\}$$

provided with the graph norm if $s \le r$. Let $H^{(s),r}(M)$ denote the space $H^{s-2}(\mathring{M})$ if $s > r$, or $L^2(M)$ otherwise. We consider the map

$$\begin{aligned} H_\Delta^{s,3/2}(M) &\xrightarrow{\Delta_s} H^{(s),3/2}(M) \oplus H^{s-1/2}(\Gamma_+) \oplus H^{s-3/2}(\Gamma_-), \\ u &\longrightarrow \Delta u \oplus u|_{\Gamma_+} \oplus \frac{1}{i}\partial_\nu u|_{\Gamma_-}, \end{aligned} \tag{0.1}$$

where ∂_ν is the normal vector field with respect to ∂M.

It is clear that (0.1) defines a continuous linear operator for $s > \frac{3}{2}$. For $s \le \frac{3}{2}$ this is still true, but the result is less obvious and requires one to know that the restriction map

$$u \to u|_{\Gamma_+} \oplus \frac{1}{i}\partial_\nu u|_{\Gamma_-}$$

is continuous when defined on $H_\Delta^{s,3/2}(M)$ (see [1]). In any case, (0.1) defines a one parameter family of continuous linear operators between Hilbert spaces. Inspired by the works of Melrose and Melrose-Mendoza (see [2] and [3]), we examine the geometric properties of distributions in the kernel and cokernel as well as Fredholm properties of each operator. In some cases, we just outline

Received July 21, 1986
[1] This research was partially supported by CONICIT.

the basic points in the arguments required to prove our results, referring the reader to [9] for more complete information. The technique we use seems to be new, contrasting with the approaches in [4], [5], [6], [7] and [8]. Its power is symbolized by formula (3.7) below. Needless to say, this technique applies to a larger category of mixed elliptic boundary value problems, subject to be discussed in a forthcoming article.

The organization of the paper is as follows: in Section 1 we discuss those properties of the standard elliptic boundary value problems which are relevant in the analysis of the kernel and cokernel of (0.1). In Section 2, we discuss the structure of elements in the kernel of (0.1). As a byproduct, we also describe the form of elements in the cokernel. Finally, in Section 3, we use these results to prove that both the kernel and cokernel of Δ_s are finite dimensional spaces. Further, using a result of [9] showing that for $s \not\equiv \frac{1}{2}$ (mod Z) the range of Δ_s is closed, we conclude that Δ_s is a Fredholm map for those values of s. In Theorem 3.6 we compute the index of Δ_s, and, using (3.7), we show that for $s \equiv \frac{1}{2}$ the map Δ_s fails to be Fredholm precisely because its range is not closed.

1. Dirichlet and Neumann problems

Let X and S be manifolds with S closed and embedded in X. Given a vector bundle E over X, denote by $I(X, S; E)$ the space of distributional sections whose local Sobolev regularity remains stable under the action of vector fields which are tangent to S. If M is contained in a smooth manifold $\tilde{M}$ of the same dimension, then $\dot{A}(M, E)$ is the subspace of $I(\tilde{M}, \partial M; E)$ consisting of those distributions supported on M. This space has a natural topology, namely the topology of smooth functions for points in $\mathring{M}$, plus the symbol topology near ∂M. If Ω is the bundle of one densities on M, we set

$$A'(M, E) = \left(\dot{A}_c(M, E^* \otimes \Omega)\right)'.$$

The restriction to the boundary defined on $C^\infty(M, E)$ extends to $A'(M, E)$. However, since $A'(M, E)$ is not closed under differentiation, we work with $B(M, E)$, the minimal extension of $A'(M, E)$ which is closed under the action of differential operators of any order. When E is a trivial vector bundle of rank one, we shall simply write $B(M)$ instead of $B(M, E)$. The properties of this space of distributions are studied in [2], and for details we refer the reader to that article.

Suppose M is realized as a submanifold with boundary of a compact, boundaryless manifold $\tilde{M}$ of the same dimension, and assume that $P \in \mathit{Diff}^2(\tilde{M})$ is elliptic. Let V be a vector field transversal to ∂M and consider the map

$$\begin{aligned} B(M) &\xrightarrow{B_V^2} D'(\partial M) \oplus D'(\partial M), \\ u &\longrightarrow u|_{\partial M} \oplus Vu|_{\partial M}. \end{aligned} \tag{1.1}$$

If u is a distribution solving $Pu = 0$ in $\mathring{M}$, then the distribution u belongs to $B(M)$ (see [2]) and its boundary values, $U = B_V^2 u$, are not arbitrary, but satisfy certain constraints. To see this, observe that if u_c denotes the extension by zero outside M, then

$$Pu_c = (Pu)_c + \tilde{P}_V U. \tag{1.2}$$

If $Pu = 0$, then by applying a parametrix Q of P to this equation, we obtain

$$u|_{\mathring{M}} \equiv Q\tilde{P}_V U|_{\mathring{M}}. \tag{1.3}$$

Since $u \in B(M)$, we may apply B_V^2 to both sides of (1.3), so that

$$U \equiv B_V^2\left(Q\tilde{P}_V U|_{\mathring{M}}\right), \tag{1.4}$$

which shows that, modulo smooth errors, U is an eigenvector of distributions of the operator

$$\begin{aligned} C_{V,P}\colon (D'(\partial M))^2 &\to (D'(\partial M))^2, \\ W &\to B_V^2\left(Q\tilde{P}_V W|_{\mathring{M}}\right). \end{aligned} \tag{1.5}$$

The operator $C_{V,P}$ is the Calderón projector associated to P (depending on V). It is a matrix of classical pseudo-differential operators, with matrix of principal symbols having rank 1. It can be computed by

$$\sigma_{p-q}\left((C_{V,P})_{p,q}\right)(y,\eta) = \frac{1}{2\pi i}\sum_{r=0}^{1-q}\int_{\gamma_+}\frac{\xi^{p+q}b_{r+q+1}(y,\eta)}{p_2(0,y,\xi,\eta)}\,d\xi, \quad 0 \le p, q \le 1. \tag{1.6}$$

Here, (x, y, ξ, η) are coordinates in T^*M near ∂M, x vanishes simply on ∂M, $V = -i\partial_x$, γ_+ is a simple contour encircling all the zeros with positive imaginary part of the principal symbol of P frozen at $x = 0$, $p_2(0, y, \xi, \eta)$, and

$$p_2(0,y,\xi,\eta) = \sum_{r=0}^{2}\xi^2 b_r(y,\eta).$$

For the Laplacian, the matrix of symbols of the Calderón projector is given by

$$\sigma(C_{V,\Delta})(y,\eta) = \frac{1}{2}\begin{pmatrix} 1 & (i|\eta|)^{-1} \\ i|\eta| & 1 \end{pmatrix}. \tag{1.7}$$

From now on, we shall use a coordinate system (x, y) for which $V = -i\partial_x$ is the unit normal to ∂M, and $\overline{\Gamma}_{\pm} = \{y: \mp \geq 0\}$. Following conventions, we shall call u_0 and u_1 the restriction to the boundary of u and Vu, respectively. Thus, if $u \in B(M)$, then $U = B_V^2 u$ is just (u_0, u_1). Consider the set of boundary conditions

$$B^0 U = B^0 \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = (1,0) \cdot \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = u_0,$$

$$B^1 U = B^1 \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = (0,1) \cdot \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = u_1.$$

The kernel of $(I - \sigma(C_{V,\Delta}))(y, \eta)$ is generated by $(1, i|\eta|)$. On the other hand, the kernel of $\sigma(B^0)(y, \eta)$ is generated by $(0,1)$ and the kernel of $\sigma(B^1)(y, \eta)$ is generated by $(1,0)$. Hence both the Dirichlet problem (Δ, B^0) and the Neumann problem (Δ, B^1) satisfy the Shapiro-Lopatinski condition $\ker(I - \sigma(C_{V,\Delta})) \cap \ker \sigma(B^i) = (0,0)$. Thus, the solutions to the problems

$$\begin{cases} \Delta u = 0 \text{ in } \mathring{M} \\ B^i U = u_i \in D'(\partial M) \end{cases}$$

are determined, up to smooth errors, by the boundary condition u_i (in this particular case, more is known, but this is all we need).

We first attempt to understand (0.1) by searching for elements in its kernel, that is to say, for solutions to the problem

$$\begin{cases} \Delta u = 0 \text{ in } \mathring{M} \\ u_0|_{\Gamma_+} = 0 \\ u_1|_{\Gamma_-} = 0. \end{cases} \tag{1.8}$$

Since $\Delta u = 0$ in $\mathring{M}$, a distribution solving (1.8) belongs to $B(M)$ and, therefore, it makes sense to restrict the distributions u and Vu to ∂M. By (1.4) and (1.5), the distributional vector (u_0, u_1) must be, modulo smooth errors, an eigenvector of the operator $C_{V,\Delta}$ with eigenvalue one. Using the form of $\sigma(C_{V,\Delta})$ shown in (1.7), we conclude that u_0 and u_1 are related by a classical pseudo-differential operator N, whose symbol is $i|\eta|$. Hence, in order to describe the kernel of Δ_s, we need to characterize the set of distributions u_0 in $D'(\partial M)$ such that

$$\begin{cases} u_0|_{\Gamma_+} = 0 \\ N u_0|_{\Gamma_-} = 0. \end{cases} \tag{1.9}$$

Let us call the intersection points $\overline{\Gamma}_+ \cap \overline{\Gamma}_- = \{p_n, p_s\}$ the north and south pole, respectively. In the next section we shall see that if u_0 is a distribution satisfying (1.9), then, near the north (resp. south) pole, u_0 has an asymptotic expansion of the form

$$u_0(y) \sim \sum_{j=0}^{\infty} c_j y_+^{m+1/2+j}, \tag{1.10}$$

meaning that the difference between u_0 and a suitable truncation of the sum is in any preassigned space $C^k(\partial M)$, near the pole in question. The number m in (1.10) is a fixed integer. This result will allow us to understand all the relevant properties of the kernel of Δ_s.

We finish this section by proving that finding the elements in the cokernel of Δ_s precisely involves the analysis of distributions with property (1.10) (this feature is particular of the case here considered, and for a general result we refer the reader to [9]).

Let us consider an element (f, g, h) in the cokernel of Δ_s. Then (f, g, h) is an element in the dual space of $H_\Delta^{(s),3/2}(M) \oplus H^{s-1/2}(\Gamma_+) \oplus H^{s-3/2}(\Gamma_-)$, that is to say, if $s > \frac{3}{2}$ then $f \in \mathring{H}^{2-s}(M)$, or if $s \leq \frac{3}{2}$, $f \in L^2(M)$, and the distributions g and h are elements of $\mathring{H}^{1/2-s}(\Gamma_+)$ and $\mathring{H}^{3/2-s}(\Gamma_-)$, respectively. Moreover, we have the relation

$$0 = \langle \Delta_s u, (f, g, h)\rangle, \quad \forall u \in C^\infty(M). \tag{1.11}$$

The right-hand side of (1.11) is

$$\langle \Delta_s u, (f, g, h)\rangle = \langle \Delta u, f\rangle + \langle u|_{\Gamma_+}, g\rangle + \left\langle \frac{1}{i}\partial_x u|_{\Gamma_-}, h\right\rangle, \tag{1.12}$$

where the first pairing in the right-hand side is the usual pairing between the Sobolev space $H^r(\mathring{M})$ and its dual $\mathring{H}^{-r}(\mathring{M})$, and the last two pairings are the usual pairings between $H^{s-1/2}(\Gamma_+)$ and $\mathring{H}^{1/2-s}(\Gamma_+)$ and $H^{s-3/2}(\Gamma_-)$ and $\mathring{H}^{3/2-s}(\Gamma_-)$, respectively.

We recall here that the dual of $H^{-r}(\Gamma_\pm)$ is identified with the normed space $\mathring{H}^{-r}(\Gamma_\pm)$ by the pairing $\langle u, v\rangle = \langle \tilde{u}, v\rangle$, where $\tilde{u}$ is any extension of $u \in H^r(\Gamma_\pm)$ to an element in $H^r(\partial M)$. Hence, we can write the last two pairings in the right-hand side of (1.12) as

$$\langle u|_{\Gamma_+}, g\rangle = \langle u_0, g\rangle \quad \text{and} \quad \left\langle \frac{1}{i}\partial_x u|_{\Gamma_-}, h\right\rangle = \langle u_1, h\rangle,$$

where each bracket denotes the pairing between $H^r(\partial M)$ and $H^{-r}(\partial M)$. Inserting these into (1.12), we obtain

$$\begin{aligned} 0 = \langle \Delta_s u, (f, g, h)\rangle &= \langle \Delta u, f\rangle + \langle u_0, g\rangle + \langle u_1, h\rangle \\ &= \langle \Delta u, f\rangle + \langle u \otimes \delta(x), g\rangle + \left\langle u \otimes \frac{1}{i}\partial_x \delta(x), h\right\rangle \\ &= \left\langle u, \Delta f + g \otimes \delta(x) + h \otimes \frac{1}{i}\partial_x \delta(x)\right\rangle. \end{aligned} \tag{1.13}$$

Since the equation above holds for any function $u \in C^\infty(M)$, by density we conclude that

$$\Delta f + g \otimes \delta(x) + h \otimes \frac{1}{i}\partial_x \delta(x) = 0. \tag{1.14}$$

Using the jump formula (1.2) for the Laplacian, the equation (1.14) implies that

$$f|_{\partial M} = f_0 = h \quad \text{and} \quad \frac{1}{i}\partial_x f|_{\partial M} = f_1 = g.$$

Moreover, from this expression we also conclude that the distribution f is harmonic in $\mathring{M}$. Thus, elements (f, g, h) in the cokernel of Δ_s satisfy the relations

$$\begin{cases} \Delta u = 0 \text{ in } \mathring{M} \\ f_0|_{\Gamma_+} = h|_{\Gamma_+} = 0 \\ f_1|_{\Gamma_-} = g|_{\Gamma_-} = 0. \end{cases}$$

Therefore, since its elements are solutions of the same type of equations, as stated above, the structure of the cokernel of Δ_s is the same as that of the kernel.

2. The structure of the kernel and cokernel

We now start the proof of (1.10), the form of the restriction to the boundary of an element u in the kernel of Δ_s. Using the coordinate y on ∂M described in Section 1, we can assume that near the north (resp. south) pole the distribution u_0 is just a compactly supported distribution on the real line, with support contained in $\overline{R}_+$, and such that the support of Nu_0 is contained in $\overline{R}_-$, where N is a classical pseudo-differential operator whose principal symbol is $i|\eta|$.

Given manifolds X and S as in Section 1, let us call $I(X, S)$ the space of conormal distributions in X with respect to S. If M is a manifold with boundary, the space $\dot{A}(M, \partial M)$ is, as before, the set of distributions on an extension $\tilde{M}$ which are conormal with respect to ∂M and supported on M. The space $A(M, \partial M)$ is the set of restrictions to $\mathring{M}$ of distributions in $I(\tilde{M}, \partial M)$. When X and M are one dimensional, then given a point p in S or ∂M which corresponds to 0 in the coordinate y, the local definition of the above spaces of distributions is simply that for each element u there exists a real number s such that u will remain in $H^s_{loc}(X)$, $\mathring{H}^s_{loc}(\mathring{M})$ and $H^s_{loc}(\mathring{M})$, respectively, when acted on by any of the operators $(yD_y)^k$.

Let ν be a compactly supported distribution in R, whose support is contained in $\overline{R}_+$. Consider a pseudo-differential operator N whose action over

ν is given by

$$N\nu(y) = \frac{1}{2\pi}\int\int e^{i(y-y')\eta} n(y,\eta)\nu(y')\,dy'\,d\eta. \tag{2.1}$$

THEOREM 2.2. *Let ν be a compactly supported distribution with* $\operatorname{supp}\nu \in \overline{R}_+$. *Suppose N is an elliptic pseudo-differential operator such that $N\nu|_{R_+}$ can be extended to a smooth function on R. Then $\nu(x) \in I(R, \{y = 0\})$.*

Proof. We prove the theorem by stages, first showing that with the hypothesis on ν, the distribution $w = N\nu|_{R_-}$ belongs to $A(\overline{R}_-, \{y = 0\})$. For that, it certainly will be enough to show that, for some fixed s and m,

$$(yD_y)^k y^m w(y) \in H^s_{loc}(R_-) \quad \forall k.$$

Using the representation (2.1), and restricting y to R_-, we have

$$\begin{aligned}(yD_y)^k y^m w(y) &= \frac{1}{2\pi}(yD_y)^k \int e^{iy\eta} y^m n(y,\eta)\hat{\nu}(\eta)\,d\eta \\ &= \frac{1}{2\pi}\sum_{j=0}^{k}\int\int e^{i(y-y')\eta} y^{m+j} n_j(y,\eta)\nu(y')\,dy'\,d\eta,\end{aligned} \tag{2.3}$$

where the functions n_j are symbols with $\deg n_j = \deg n + j$.

Let us call w_j the jth term appearing in the right hand side of (2.3). Assume that $\nu \in H^1_c(R) \subset C(R)$, and fix m so that $m - \deg n > 1$. For $y \in R_-$ and y' in the support of ν, $y - y' < 0$. Hence,

$$\begin{aligned} w_j(y) &= \frac{1}{2\pi}\int\int_{\overline{R}_+} e^{i(y-y')\eta} y^{m+j} n_j(y,\eta)\nu(y')\,dy'\,d\eta \\ &= \frac{1}{2\pi}\int\int_{\overline{R}_+}\left[\frac{D_\eta^{m-j} e^{i(y-y')\eta}}{(y-y')^{m+j}}\right] y^{m+j} n_j(y,\eta)\nu(y')\,dy'\,d\eta \\ &= \frac{1}{2\pi}\int\int_{\overline{R}_+} e^{i(y-y')\eta}\left(\frac{y}{y-y'}\right)^{m+j}\bar{n}_j(y,\eta)\nu(y')\,dy'\,d\eta, \end{aligned} \tag{2.4}$$

where $\bar{n}_j$ is a symbol with $\deg \bar{n}_j = \deg n - m < -1$, independent of k. The term

$$\left(\frac{y}{y-y'}\right)^{m+j}\nu(y')$$

is bounded when $y < 0$ and $y' \geq 0$. Hence, if we restrict y to a bounded set, we see that the supremum norm of w_j is bounded by a multiple of $\int(1 + |\eta|)^{\deg \tilde{n}_j}\, d\eta$, which is finite since the exponent of the integrand is less than -1. This holds for any j. Thus

$$\sup_{-1 \leq x < 0}\left|(yD_y)^k y^m w(y)\right| \leq C.$$

Away from $y = 0$ the distribution $w(y)$ is smooth. We conclude that $w(y)$ belongs to $A(\overline{R}_-, \{y = 0\})$. The assumption $v \in H_c^1(R)$ is not essential to our arguments. Details are left to the reader.

Therefore, the distribution $w(y)$ belongs to $A(R_-, \{y = 0\})$. But the restriction map

$$\dot{A}(\overline{R}_-, \{0\}) \to A(\overline{R}_-, \{0\})$$

is surjective. Let $\tilde{w}$ be a distribution in $\dot{A}(\overline{R}_-, \{0\})$ such that $\tilde{w}|_{R_-} = w$. Let $g(x)$ be a smooth extension of $Nv|_{R_+}$. Then,

$$h(y) = \begin{cases} g(y), & y > 0, \\ \tilde{w}(y), & y \leq 0, \end{cases}$$

defines an element of $I(R, \{y = 0\})$ such that $Nv - h$ is supported on $y = 0$. It follows that

$$Nv(y) = h(y) + \sum_{j \leq k} c_j \delta^j(y),$$

proving that $Nv \in I(R, \{y = 0\})$.

For the final step, use a parametrix of N and recall that pseudo-differential operators preserve $I(R, \{y = 0\})$. We conclude that $v \in I(R, \{y = 0\})$. ■

In the coordinates (x, y) near p_n (resp. p_s) described in Section 1, the restriction to the boundary, u_0, of an element u in the kernel of Δ_s is a distribution satisfying the hypothesis of Theorem 2.2. Also, if (f, g, h) is an element in the cokernel of Δ_s, the restriction to the boundary of f, f_0, coincides with h and satisfies the conditions of the theorem. We obtain:

COROLLARY 2.5. *Assume that u_0 is the restriction to the boundary of an element in the kernel of Δ_s. Then $u_0 \in I(\partial M, \{p_n, p_s\})$.* ■

COROLLARY 2.6. *Assume that (f, g, h) is an element in the cokernel of Δ_s. Then the restriction to the boundary of f, f_0, belongs to $I(\partial M, \{p_n, p_s\})$.* ■

Let us return to the previous setting. We have a distribution v and a pseudo-differential operator N on the real line satisfying the conditions of

Theorem 2.2. Both distributions, ν and $N\nu$, are conormal with respect to $\{y = 0\}$ and, consequently, modulo smooth functions, they can be represented as the inverse Fourier transform of a symbol in the fiber of T^*R. For $N\nu$ we write

$$N\nu(y) = \frac{1}{2\pi}\int e^{iy\eta}f(\eta)\,d\eta, \tag{2.7}$$

where $f(\eta)$ is a symbol in T^*R, y-independent.

Suppose N is classical, with symbol

$$\sigma(N)(y,\eta) \sim \sum_{j=0}^{\infty} \sigma_j^N(y,\eta), \tag{2.8}$$

where $\sigma_j^N(y,\eta)$ is a homogeneous symbol of degree $n-j$, n the order of N. Since N is elliptic, the principal symbol (that abusing notation we call $\sigma(N)$) can be written as

$$\begin{aligned}\sigma(N)(y,\eta)\big|_{y=0} = \sigma_0^N(y,\eta)\big|_{y=0} &= a_+\eta_+^n + a_-\eta_-^n \\ &= a_+^1\left(\eta_+^n + a_+^{-1}a_-\eta_-^n\right).\end{aligned} \tag{2.9}$$

where $a_+ = \sigma(N)(0,1)$ and $a_- = \sigma(N)(0,-1)$. In the expression above, $\eta_\pm^n$ stands for the distribution that coincides with η^n on the set $\{\eta\colon \pm\eta \geq 0\}$, and is identically zero otherwise. This distribution is homogeneous when n is not a negative integer. If n is a negative integer, homogeneity of σ_0^N implies that $a_- = (-1)^n a_+$ and the decomposition (2.9) is still valid.

Using the principal branch of log, with $\arg z = -\pi$ for $z \in R_-$, set

$$\gamma = \frac{1}{2\pi i}\log a_+^{-1}a_-. \tag{2.10}$$

THEOREM 2.11. *Assume that ν is a distribution and N is a pseudo-differential operator satisfying the hypothesis of Theorem 2.2. Suppose N is classical. Then ν is classical and its Fourier transform can be expanded as*

$$\hat{\nu}(\eta) \sim \sum_{j=0}^{\infty} v_{k-n\gamma/2-j}(\eta), \tag{2.12}$$

where v_r is a homogeneous symbol of degree r and k is a fixed integer.

Proof. Consider the representation (2.7) of $N\nu$. That representation is rapidly decreasing at $\pm\infty$. Moreover, modulo rapidly decreasing functions, we can assume that it is supported in $\overline{R}_-$. Indeed, if g is any smooth extension of

$N\nu|_{R_+}$ with support contained in $\{y: y \geq -1\}$, it is a Schwartz function and $N\nu - g$ coincides with $N\nu$ modulo that space of functions. Hence, under this assumption, the Fourier transform of $N\nu$ is a symbol with analytic symbolic extension to the upper half-plane. On the other hand, the Fourier transform of ν is a symbol with an analytic symbolic extension to the lower half-plane. Let us call f and v the symbols of $\widehat{N\nu}$ and $\hat{\nu}$, respectively. Let r be the order of v.

By Taylor expansion about $y = 0$, we obtain

$$\sigma(N)(y, \eta) = \sigma(N)(0, \eta) + y\tilde{\sigma}(N)(y, \eta),$$

where $\tilde{\sigma}(N)$ is a symbol of the same order as $\sigma(N)$. Since ν is conormal and the operator "multiplication by y" increases the regularity of cornormal distributions by one, we have that

$$f(\eta) \equiv \sigma(N)(0, \eta)v(\eta) \quad \operatorname{mod} S^{n+r-1}(T^*). \tag{2.13}$$

Consider the distributions

$$\beta_+(\eta) = \eta + i0, \qquad \beta_-(\eta) = \eta - i0.$$

They have analytic extensions to the upper and lower half-plane, respectively. For $z \neq 0$ define

$$w(z) = \begin{cases} a_+^{-1}f(z)(\beta_+(z))^{-n/2+\gamma}, & \arg \in [0, \pi), \\ v(z)(\beta_-(z))^{n/2-\gamma}, & \arg \in [-\pi, 0), \end{cases}$$

Then, $w(z)$ is analytic in each open half-plane and continuous in the closures. Since $e^{2\pi i\gamma} = a_+^{-1}a_-$, using (2.9) and (2.13) we conclude that $w(z)$ has a jump on the real axis which is a symbol of order $n/2 + r - 1 - \gamma_1$, where γ_1 is the real part of γ. We can find a symbol $d(z)$, of order $n/2 + r - 1 - \gamma_1$, such that for $z \neq 0$

$$w_0(z) = \begin{cases} a_+^{-1}f(z)(\beta_+(z))^{-n/2+\gamma}, & \arg \in [0, \pi), \\ v(z)(\beta_-(z))^{n/2-\gamma} + d(z), & \arg \in [-\pi, 0), \end{cases} \tag{2.14}$$

is smooth in the punctured plane $C - \{0\} = R^2 - \{0\}$ and has as $\partial_{\bar{z}}$ derivative, $\partial_{\bar{z}}w_0(z)$, a symbol $g_0(z)$ of order $n/2 + r - \gamma_1 - 2$. Consider a smooth function φ such that $\varphi \equiv 0$ in the region $\|z\| < 1/2$ and $\varphi \equiv 1$ in the region $\|z\| > 1$, and set

$$w_0^s(z) = \begin{cases} \varphi(sz)w_0(z), & \|z\| > 1/2s, \\ 0, & \|z\| \leq 1/2s. \end{cases}$$

Then, $w_0^s(z)$ is a smooth function in C such that $\partial_{\bar{z}} w_0^s(z) = g_s(z)$ is a symbol of order $n/2 + r - \gamma_1 - 2$. We can find a symbol $u(z)$ with the following properties.

1. $\partial_{\bar{z}} u = -g_s$.
2. If either $t = n/2 + r - \gamma_1 - 2 < -1$ or $t \geq -1$, we have:
   2.1 For any K such that $t + 1 + K < -1$, the function u admits an expansion of the type
   $$u(z) = \sum_{q=0}^{K} c_q z^{-q-1} + u_K(z),$$
   where $u_K(z)$ is a symbol of order $-K - 1 - \delta$, $0 \leq \delta < [t] - t + 1$.
   2.2. If $t \geq -1$, then $u(z)$ can be chosen to be a symbol of order $t + 1 + \delta$ with t and δ as above.

Hence, $w_0^s(z) + u(z)$ is analytic, and

$$w_0^s(z) + u(z) = \sum_{j \leq M} c_j z^j, \tag{2.15}$$

the summation being finite since both, v and u, have finite order.

The importance of (2.15) is that, from the properties of u, it isolates the contribution of order $n/2 + r - \gamma_1$ in w_0^s, and shows that this contribution is a homogeneous symbol of integer degree. Hence, using this result on the real line, we conclude that the functions f and v decompose as

$$f = f_0 + f_1, \qquad v = v_0 + v_1,$$

where, for some integer k, f_0 and v_0 are homogeneous symbols of degree $k + n/2 + \gamma$ and $k - n/2 + \gamma$, respectively, and f_1 and v_1 are symbols of order at most $k + n/2 + \gamma_1 - 1$ and $k - n/2 + \gamma_1 - 1$, respectively. The functions f_1 and v_1 have analytic symbolic extensions to the half-planes where f and v enjoy similar properties and, on the real line, they satisfy the relation

$$f_1(\eta) \equiv \sigma(N)(0, \eta) v_1(\eta) + h_1(\eta),$$

where $h_1(\eta)$ is a homogeneous symbol of degree $k + n/2 + \gamma_1 - 1$, coming from the interaction of the principal contribution of v with the lower order terms in the expansion of the symbol of N.

The rest of the induction is just a mere repetition of the arguments above, in order to take care of the interactions of the terms of v previously found with the lower order terms in the asymptotic expansion of the symbol of N. Details are left to the reader (see [9]). ∎

With this result we are now ready to prove the expansion (1.10) for the Dirichlet condition of an element u in the kernel of Δ_s.

THEOREM 2.16 (*Structure of the kernel*). *Assume that u is an element in the kernel of* Δ_s. *Let* (x, y) *be coordinates near* p_n (*resp.* p_s) *such that* $V = (1/i)\partial_x$ *and* $\overline{\Gamma}_{\pm} = \{y\colon \mp y \geq 0\}$. *Then* u_0, *the restriction of u to the boundary, has an asymptotic expansion of the form*

$$u_0(y) \sim \sum_{j=0}^{\infty} c_j y_{+}^{m+1/2+j},$$

for some integer m.

Proof. In the coordinate y, if u_1 is the restriction of Vu to the boundary, then u_0 is supported in $\overline{R}_+$, $u_1 \equiv Nu_0$ with u_1 supported in $\overline{R}_-$, and the operator N is classical and elliptic with principal symbol $i|\eta|$. In this case, the number γ defined from $\sigma(N)$ via (2.10) is 0. Applying theorem 2.11, near the pole in question the Fourier transform of u_0 can be expanded as in (2.12),

$$\hat{u}_0(\eta) \sim \sum_{j=0}^{\infty} u_{0,j}(\eta),$$

where $u_{0,j}(\eta)$ is a homogeneous symbol of degree $k - \frac{1}{2} - j$, for some integer k. The inverse Fourier transform of $u_{0,j}$ will then be certain linear combination of $y_{+}^{-k-1/2+j}$ and $y_{-}^{-k-1/2+j}$. Since u_0 is supported on $\overline{R}_+$, the coefficient of $y_{-}^{-k-1/2+j}$ has to vanish for all j. The desired result follows by setting $m = -k - 1$. ∎

By the discussion at the end of Section 1, we also obtain the following:

THEOREM 2.17 (*Structure of the cokernel*). *Assume that* (f, g, h) *is an element in the cokernel of* Δ_s *and consider the coordinates* (x, y) *of the previous theorem. Then,* $f_0 = f|_{\partial M} = h$, *and*

$$f_0(y) \sim \sum_{j=0}^{\infty} c_j y_{-}^{m+1/2+j},$$

for some integer m. ∎

3. Fredholm properties of Δ_s

Using the structural theorems proven in Section 2, we want to analyse Fredholm properties of the map Δ_s defined in (0.1).
We begin proving the following:

THEOREM 3.1. *For all* $s \in R$, *the kernel of* Δ_s *is a finite dimensional space. Moreover, if* $s \geq \frac{1}{2}$, $\ker \Delta_s = \{0\}$.

Proof. For $u \in \ker \Delta_s$, let $(u_0, u_1) = B_V^2 u$ be its vector of boundary values. By Theorem 2.11, when localized near the poles, the distribution u_0 has an expansion of the form

$$u_0(y) \sim \sum_{j=0}^{\infty} c_j y_+^{m+1/2+j}, \tag{3.2}$$

for some integer m. Since $u_0 \in H^{s-1/2}(\partial M)$, we must have $\frac{3}{2} + m = -s - \varepsilon$ for some positive ε. This shows that the set

$$\mathscr{K}_s(\Delta) = \left\{ u_0 \in H^{s-1/2}(\partial M) \colon u_0 = u|_{\partial M},\, u \in \ker \Delta_s \right\}$$

is contained in $H^{s'-1/2}(\partial M)$ for some s' strictly bigger than s. Thus

$$\mathscr{K}_s(\Delta) \to H^{s'-1/2}(\partial M) \to H^{s-1/2}(\partial M). \tag{3.3}$$

By Rellich's theorem, the last inclusion in (3.3) is compact. This shows that $\mathscr{K}_s(\Delta)$ is a finite dimensional space. But the Dirichlet data, u_0, of a harmonic distribution u in $\mathring{M}$ determines the distribution itself. Hence, $\ker \Delta_s$ is also a finite dimensional space.

Let $d\lambda$ be the usual measure on M, and let $d\sigma$ be the induced measure on ∂M. Green's formula

$$\int_M \Delta u \bar{v}\, d\lambda = -\int_M \nabla u \overline{\nabla v}\, d\lambda + \frac{1}{i}\int_{\partial M} u_1 \overline{v_0}\, d\sigma$$

can be extended by continuity to $H_\Delta^1(\mathring{M}) \oplus H^1(\mathring{M})$, where $H_\Delta^r(\mathring{M})$ is the space of elements in $H^r(\mathring{M})$ with Laplacian in $L^2(M)$.

Suppose u is in the kernel of Δ_s for $s \geq \frac{1}{2}$. Then, u_0 is at least an $L^2(\partial M)$ function. Since $y_+^{-1/2}$ is almost in L^2 but not quite, the integer m in (3.2) has to be greater than or equal to 0. Hence, the worst singularity of u_0 is of the form $y_+^{1/2}$, allowing us to conclude that $u_0 \in H^{1-\delta}(\partial M)$ for any positive δ. It follows that $u \in H^{3/2-\delta}(\mathring{M})$, showing that $\ker \Delta_s \subset H_\Delta^{3/2-\delta}(M)$ if $s \geq \frac{1}{2}$. For u in the kernel of Δ_s, set $v = u$ in the Green's formula above. We easily conclude that u has to be constant on M. But u is almost in $H^{3/2}(\mathring{M})$. Hence, it is continuous and vanishes on Γ_+. It must vanish everywhere. ■

If we use Theorem 2.17 combined with the discussion at the end of Section 1, by the same arguments we obtain the following:

THEOREM 3.4. *For all $s \in R$, the cokernel of Δ_s is a finite dimensional space. Moreover, if $s \leq \frac{3}{2}$,* coker $\Delta_s = \{0\}$. ■

Concerning Fredholm properties of Δ_s, the only unresolved point is the closedness of its range. This is a little delicate, and in fact the map Δ_s has closed range if, and only if, s is not congruent to $\frac{1}{2}$ modulo Z. In [9] we have proven a proposition that for this particular situation reads as follows:

PROPOSITION 3.5. *For $s \not\equiv \frac{1}{2} \bmod Z$, the range of Δ_s is closed.* ∎

Let us briefly outline a proof of this proposition. Recall that $s \geq 1/2$ the map

$$H^s(\mathring{M}) \to H^{s-2}(\mathring{M}) \oplus H^{s-1/2}(\partial M),$$
$$u \to \Delta u \oplus u|_{\partial M}$$

is bijective with continuous inverse. Using this, we can prove that closedness of the range of Δ_s is equivalent to closedness of the non-homogeneous version of (1.9):

$$\mathring{H}^{s-1/2}(\Gamma_-) \to H^{s-3/2}(\Gamma_-),$$
$$u \to Nu|_{\Gamma_-}.$$

Away from the poles, this map can be inverted without any restriction on s. At the poles, we can invert it modulo compact operators when $s \not\equiv \frac{1}{2}$ (mod Z). For that, we use the coordinates (x, y, ξ, η) of T^*M described in Section 1 to reduce our problem to a similar one on $L^2(R_+)$ for an operator $\tilde{N}$ whose principal symbol is $i(\eta + i0)^{s-3/2}|\eta|(\eta - i0)^{-s+1/2}$. If $g \in L^2(R_+)$, we can think of it as an element of $L^2(R)$. Then $\tilde{g} = \tilde{N}^{-1}g|_{R_-}$ is in $A(\overline{R}_-, \{y = 0\})$ (see proof of Theorem 2.2). We can find a conormal distribution v_g, supported on $\overline{R}_-$, such that $\tilde{N}^{-1}v_g|_{R_-} = \tilde{g}$. With this result, a parametrix modulo compact operators can be easily constructed. The compactness of the error is proven using Theorem 2.11. The details are technical and would make our exposition a little bit too long. We therefore skip it.

Note that Proposition 3.5 does not say anything about the behavior of the range when $s \equiv \frac{1}{2} \bmod Z$. The reason for this restriction is related to the integer m appearing in the expansion of Theorem 2.16. In fact, for those prohibited values of s, m is somewhat undetermined and, as a consequence, the range of Δ_s is not closed.

The combination of Theorems 3.1 and 3.4, and Proposition 3.5, shows that the map Δ_s is Fredholm for $s \not\equiv \frac{1}{2}$. Our final goal is to compute the index of our operators at these values of s, and in doing so, to understand the behavior of Δ_s at the exceptional values $s \equiv \frac{1}{2}$.

Let $[t]$ denote the integer part of $t \in R$.

THEOREM 3.6. *Let k be any non-negative integer. For*

$$s \in \left[\frac{1 - 2(k+1)}{2}, \frac{1 - 2k}{2}\right)$$

the kernel of Δ_s *is a* $2(k+1)$*-dimensional space. Similarly, for*

$$s \in \left(\frac{3+2k}{2}, \frac{5+2k}{2}\right]$$

the cokernel of Δ_s *is a* $2(k+1)$*-dimensional space. If* $s \not\equiv \frac{1}{2} \bmod Z$ *then*

$$\operatorname{Index} \Delta_s = -2\left[s - \frac{1}{2}\right]. \tag{3.7}$$

For $s \equiv \frac{1}{2} \bmod Z$*, the range of* Δ_s *is not closed.*

Proof. Putting together Theorems 3.1, 3.4, and Proposition 3.5, we conclude that Δ_s is an isomorphism for $s \in (\frac{1}{2}, \frac{3}{2})$.

We shall prove the result for the dimension of the kernel. We have seen that the cokernel obeys the same type of relations as the kernel. The result for the dimension of the cokernel will then follow from the same arguments, if we just replace the role of $s - \frac{1}{2}$ by that of $\frac{3}{2} - s$.

We use induction on k. First, we construct a two parameter family of solutions to (1.8) out of $H^s(\mathring{M})$ for $s \in [-\frac{1}{2}, \frac{1}{2})$ (this corresponds to $k = 0$). If there is any such solution u, its restriction to the boundary, u_0, has an asymptotic expansion as in (3.2) near the poles $\{p_n, p_s\}$. Consider a distribution u_0^n (resp. u_0^s) supported near p_n (resp. p_s) that agrees with $y_+^{-1/2}$ in a neighborhood of p_n (resp. p_s). Then, u_0^n (resp. u_0^s) lies in $\mathring{H}^{-\varepsilon}(\Gamma_-)$ for any positive ε. Let N be the operator relating the Dirichlet and Neumann conditions of a harmonic distribution in the interior of M. It is a classical pseudo-differential operator whose principal symbol is given by $i|\eta|$ in the coordinates (y, η) of $T^*\,\partial M$. The principal contribution of the Fourier transform of Nu_0^n (resp. Nu_0^s) will be of the form $c(\eta + i0)^{1/2}$, and hence, it produces a distribution supported on $y \leq 0$, which corresponds to $\overline{\Gamma}_-$ in the coordinates used. Thus, the distribution Nu_0^n (resp. Nu_0^s) has its principal contribution supported on $\overline{\Gamma}_+$, allowing us to conclude that $Nu_0^n|_{\Gamma_-}$ (resp. $Nu_)^s|_{\Gamma_-}0$ lies in $H^{-\varepsilon}(\Gamma_-)$ for any positive ε. Taking linear combinations of u_0^n and u_0^s we obtain a two parameter family of distributions ν_0 such that:

1. $\nu_0 \in \mathring{H}^{-\varepsilon}(\Gamma_-)$, $\forall \varepsilon > 0$.
2. $h_0 = N\nu_0|_{\Gamma_-} \in H^{-\varepsilon}(\Gamma_-)$, $\forall \varepsilon > 0$.

Let u_{ν_0} be the unique solution to the problem

$$\begin{cases} \Delta u = 0 \text{ in } \mathring{M} \\ u|_{\partial M} = \nu_0. \end{cases} \tag{3.8}$$

Consider the element u_{h_0} of $H_\Delta^{3/2-\varepsilon}(M)$ that corresponds to $0 + 0 + h_0$ via Δ_s when $s \in (\frac{1}{2}, \frac{3}{2})$ (recall that Δ_s is an isomorphism within that range). Then,

$$w = u_{\nu_0} - u_{h_0} \tag{3.9}$$

is a solution to (1.8), lying in $H^{1/2-\varepsilon}(\mathring{M})$ for any $\varepsilon > 0$. Note that w is not trivial since the Sobolev regularity if u_{ν_0} is almost one unit lower than the Sobolev regularity of u_{h_0}. Hence, (3.9) defines a two parameter family in the kernel of Δ_s for $[-\frac{1}{2}, \frac{1}{2})$, and consequently, for those values of s, the dimension of the kernel is at least 2.

Using the expansion (3.2) for the restriction to the boundary of an element in the kernel, we can easily show that in this range of s's, the dimension cannot be bigger than 2. Indeed, let t_n and t_s be the leading coefficient in the expansion of u_0 near p_n and p_s, respectively. Consider the map

$$\begin{aligned} \ker \Delta_s &\xrightarrow{T} C^2 \\ u &\longrightarrow (t_n, t_s). \end{aligned}$$

Suppose we have three linearly independent elements in the kernel of Δ_s for $s \in [-\frac{1}{2}, \frac{1}{2})$, say v_1, v_2, v_3. Then, for some constants c_1 and c_2, we have

$$T(v_3) = c_1 T(v_1) + c_2 T(v_2). \tag{3.10}$$

It follows that $v = v_3 - c_1 v_1 - c_2 v_2$ solves (1.8). Moreover, from (3.10) we see that the leading term in the expansion (3.2) for the restriction of v to the boundary, vanishes. Hence, the distribution v lies in $H^{3/2-\varepsilon}(\mathring{M})$ for any positive ε. By Theorem 3.1, it must vanish identically, contradicting the independency of v_1, v_2 and v_3. Thus, no such v exists, which shows that the dimension of the kernel in this range of values of s is less than or equal to 2. Hence, it is exactly 2.

To complete the induction, we assume the result is true up to $k - 1$. Starting with $y_+^{-1/2-k}$ instead of $y_+^{-1/2}$, we construct a two parameter family ν_k with the properties:

1. $\nu_k \in \mathring{H}^{-k-\varepsilon}(\Gamma_-)$, $\forall \varepsilon > 0$.
2. $h_k = N\nu_k|_{\Gamma_-} \in H^{-k-\varepsilon}(\Gamma_-)$, $\forall \varepsilon > 0$.

The solution u_{ν_k} to (3.8) is still unique and lies in $H^{-1/2-k-\varepsilon}(\mathring{M})$ for any positive ε. By Theorem 3.4 and the induction hypothesis, there exists a $2k$-parameter family of distributions u_{h_k} that corresponds to $0 + 0 + h_k$ via Δ_s for

$$s \in \left(\frac{1-2k}{2}, \frac{1-2(k-1)}{2}\right).$$

It follows that $u_{\nu_k} - u_{h_k}$ is a $2(k+1)$-parameter family of elements in the kernel of Δ_s for

$$s \in \left[\frac{1-2(k+1)}{2}, \frac{1-2k}{2}\right).$$

The same procedure will also prove that the dimension of

$$\frac{\left\{\ker \Delta_s \colon s \in \left[\frac{1-2(k+1)}{2}, \frac{1-2k}{2}\right)\right\}}{\left\{\ker \Delta_s \colon s \in \left[\frac{1-2k}{2}, \frac{1-2(k-1)}{2}\right)\right\}}$$

cannot be greater than 2, completing the induction.

As we pointed out at the beginning, the result for the cokernel follows from the same arguments.

Formula (3.7) is obtained counting the number of jumps in the dimension of the kernel and cokernel, respectively, and observing that the index of Δ_s is 0 on the interval $(\frac{1}{2}, \frac{3}{2})$.

Finally, when $s \equiv \frac{1}{2} \bmod Z$, the range of Δ_s cannot be closed. Otherwise, in virtue of Theorems 3.1, 3.4, and Proposition 3.5, Δ_s would be a continuous family of Fredholm maps in a neighborhood of those points. Therefore, it would have constant index across them, contradicting formula (3.7). ∎

Acknowledgements. The author would like to thank Professor Richard B. Melrose for invaluable conversations during the elaboration of this work, Rafe Mazzeo for a careful reading of the manuscript, and the organizers of the VII Latin American School of Mathematics, where part of this material was presented.

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