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QUANTIZATION AND AN INVARIANT FOR UNITARY REPRESENTATIONS OF NILPOTENT LIE GROUPS

BY

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1. Introduction

Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{G}$. Given a co-adjoint orbit $\mathfrak{O} \subset \mathfrak{G}^*$, the dual of $\mathfrak{G}$, the authors have defined a cohomology invariant $i(\mathfrak{O}) \in H^{2q+1}(\mathfrak{G})$, where $\dim \mathfrak{O} = 2q$ [1] (see Section 2 for details).

We now provide an interpretation of this invariant via the machinery of geometric quantization [6]. There is an Hermitian line bundle L over the orbit $\mathfrak{O}$ with a natural connection and G-action. A specific model for this "prequantization bundle" L is developed in Section 3. Let $T(L)$ be the unit circle bundle in L with respect to the Hermitian structure. The action of G on L yields a G-invariant map $\tilde{\pi}\colon G \to T(L)$, and hence a map

$$\tilde{\pi}^*\colon H_G^*(T(L)) \to H^*(\mathfrak{G}).$$

(Here $(H_G^*(T(L))$ denotes the G-invariant (real) cohomology of $T(L)$.) The connection in L yields a distinguished element $[V]$ in $H_G^{2q+1}(T(L))$. In Section 4 we show that $i(\mathfrak{O}) = \tilde{\pi}^*([V])$.

The prequantization model used allows us to relate $H_G^*(T(L))$ to Lie algebra cohomology. In particular, we show that $H_G^{2q+1}(T(L))$ is one-dimensional, generated by $[V]$, so that $i(\mathfrak{O}) = 0$ if and only if

$$\tilde{\pi}^*\colon H_G^{2q+1}(T(L)) \to H^{2q+1}(\mathfrak{G})$$

is the zero map.

We remark that geometric quantization uses the Hermitian structure, G-action and connection in L to determine the representation $\sigma_{\mathfrak{O}}$ of G corresponding to the coadjoint orbit $\mathfrak{O}$. The invariant $i(\mathfrak{O})$ is a composite derived from this geometric data, and can be regarded as an invariant for the representation $\sigma_{\mathfrak{O}}$. As such, it should also detect properties of the representation theory of G. We discuss some of these properties in Section 5.

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2. Lie algebra cohomology and the invariant for an orbit

Let G be a connected Lie group with Lie algebra $\mathfrak{G}$. Let $\mathfrak{G}^*$ be the linear dual of $\mathfrak{G}$. G acts on $\mathfrak{G}^*$ by the co-adjoint representation Ad^*. If G is nilpotent, by the theory of Kirillov [5], the orbits in $\mathfrak{G}^*$ under this action are in one-one correspondence with the irreducible unitary representations of G.

The left G-invariant forms ${}^G\Omega(G)$ on G yield a subcomplex of the de Rham complex $\Omega(G)$ which can be identified with the exterior algebra $\bigwedge(\mathfrak{G}^*)$. The cohomology of this complex is denoted by $H^*(\mathfrak{G})$.

Let $\mathfrak{O} \subset \mathfrak{G}^*$ be a co-adjoint orbit. As $\mathfrak{O}$ is a symplectic manifold, it has dimension $2q$ for some q [5]. Given $f \in \mathfrak{O}$, we have shown in [1] that $f \wedge (df)^q$ in $\bigwedge^{2q+1}(\mathfrak{G})$ is a closed form, and that $[f \wedge (df)^q]$ is independent of the choice of f. We define $i(\mathfrak{O})$ in $H^{2q+1}(\mathfrak{G})$ by

$$i(\mathfrak{O}) = \left[f \wedge (df)^q\right]. \tag{2.1}$$

See [1] for examples where $i(\mathfrak{O})$ is non-trivial.

Relative Lie algebra cohomology Let Ad^* and ad^* be the co-adjoint actions on $\bigwedge(\mathfrak{G}^*)$ of G and $\mathfrak{G}$ respectively. For $X \in \mathfrak{G}$, the substitution operator

$$i(X)\colon \textstyle\bigwedge^k(\mathfrak{G}^*) \to \bigwedge^{k-1}(\mathfrak{G}^*)$$

is given by $(i(X)\alpha)(Y_1, \dots, Y_{k-1}) = \alpha(X, Y_1, \dots, Y_{k-1})$ for $Y_i \in \mathfrak{G}$, $i = 1, \dots, k-1$.

Let H be a closed subgroup of G with Lie algebra $\mathfrak{H}$. If

$$\mathfrak{K} = \{\alpha \in \textstyle\bigwedge(\mathfrak{G}^*)\colon i(X)\alpha = 0 \text{ for all } X \in \mathfrak{H}\},$$

the subcomplexes of H-basic and $\mathfrak{H}$-basic elements of $\bigwedge(\mathfrak{G}^*)$ are defined by

$$(\textstyle\bigwedge\mathfrak{G}^*)_H = \mathfrak{K} \cap \{\alpha \in \bigwedge(\mathfrak{G}^*)\colon \mathrm{Ad}^*h(\alpha) = \alpha \text{ for all } h \in H\} \tag{2.2}$$

and

$$(\textstyle\bigwedge\mathfrak{G}^*)_{\mathfrak{H}} = \mathfrak{K} \cap \{\alpha \in \bigwedge(\mathfrak{G}^*)\colon \mathrm{ad}^*X(\alpha) = 0 \text{ for all } X \in \mathfrak{H}\}. \tag{2.3}$$

These complexes yield the relative cohomology theories $H^*(\mathfrak{G}, H)$ and $H^*(\mathfrak{G}, \mathfrak{H})$. If π is the projection $\pi\colon G \to G/H$, then $(\bigwedge\mathfrak{G}^*)_H = \pi^*({}^G\Omega(G/H))$, so that $H^*(\mathfrak{G}, H)$ corresponds to the cohomology of G-invariant forms on G/H. When H is connected, $H^*(\mathfrak{G}, H) = H^*(\mathfrak{G}, \mathfrak{H})$, but this is not true in general. If $\mathfrak{H}$ is an *ideal* in $\mathfrak{G}$, then $H^*(\mathfrak{G}, \mathfrak{H}) = H^*(\mathfrak{G}/\mathfrak{H})$.

The three cohomology algebras $H^*(\mathfrak{H})$, $H^*(\mathfrak{G})$ and $H^*(\mathfrak{G}, \mathfrak{H})$ are related by the Hochschild-Serre spectral sequence [4]. This is a first quadrant spectral sequence that provides an algebraic analogue of the Serre spectral sequence for the fibration $H \hookrightarrow G \twoheadrightarrow G/H$. The E_0 and E_1 terms of the spectral sequence can be identified as

$$E_0^{i,j} \cong \Lambda^j(\mathfrak{H}, \Lambda^i((\mathfrak{G}/\mathfrak{H})^*)) \quad \text{and} \quad E_1^{i,j} \cong H^j(\mathfrak{H}, \Lambda^i((\mathfrak{G}/\mathfrak{H})^*)).$$

The latter cohomology involves coefficients in the $\mathfrak{H}$-module $\Lambda^i((\mathfrak{G}/\mathfrak{H})^*)$ (under the ad*-action). We refer the reader to [4]. The E_2 term can be difficult to compute, however $E_2^{i,0} \cong H^i(\mathfrak{G}, \mathfrak{H})$ and $E_2^{0,j}$ can be identified with a submodule of $H^i(\mathfrak{G})$. Moreover, when $\mathfrak{H}$ is an ideal in $\mathfrak{G}$, one has

$$E_2^{i,j} \cong H^i(\mathfrak{G}/\mathfrak{H}, H^i(\mathfrak{H})).$$

If, in addition, $\mathfrak{G}/\mathfrak{H}$ acts trivially on $H^j(\mathfrak{H})$ via ad*, then

$$E_2^{i,j} \cong H^i(\mathfrak{G}/\mathfrak{H}) \otimes H^j(\mathfrak{H}).$$

The spectral sequence converges to E_∞, which is related to $H^*(\mathfrak{G})$ by a filtration in the usual manner. In particular, $H^n(\mathfrak{G}) \cong \oplus_{i+j=n} E_\infty^{i,j}$.

Suppose now that G is simply connected and nilpotent. It is known that if G has a co-compact discrete subgroup Γ then $H^*(\mathfrak{G}) \cong H^*(\Gamma \setminus G)$, the real cohomology of a compact manifold [8]. In particular, such subgroups exist whenever $\mathfrak{G}$ has rational structure constants. In general, one always has the following result.

2.4 LEMMA. *If $\mathfrak{G}$ is nilpotent, then $H^*(\mathfrak{G})$ satisfies Poincaré duality. In particular, $H^n(\mathfrak{G}) \cong \mathbf{R}$ where $n = \dim(\mathfrak{G})$.*

This is known more generally for any unimodular Lie algebra [3]. Using Lemma 2.4 together with the spectral sequence one obtains a relative version.

2.5 LEMMA. *Let $\mathfrak{G}$ be nilpotent and $\mathfrak{H}$ a subalgebra of $\mathfrak{G}$. Then*

$$H^s(\mathfrak{G}, \mathfrak{H}) \cong \mathbf{R} \text{ where } s = \dim(\mathfrak{G}/\mathfrak{H}).$$

Proof. We must have either $H^s(\mathfrak{G}, \mathfrak{H}) = 0$ or $H^s(\mathfrak{G}, \mathfrak{H}) \cong \mathbf{R}$ since $\Lambda^s(\mathfrak{G}^*)_{\mathfrak{H}}$ is one dimensional. Let $\nu \in \Lambda^s(\mathfrak{G}^*)_{\mathfrak{H}}$ be non-zero (a left-invariant volume form). We need only show that ν is not exact in the complex $\Lambda(\mathfrak{G}^*)_{\mathfrak{H}}$.

Let $r = \dim(\mathfrak{H})$ and $\mu \in \Lambda^r(\mathfrak{H}^*)$ be a volume form. The element

$$\mu \otimes \nu \in E_0^{s,r} = \Lambda^r(\mathfrak{H}, \Lambda^s((\mathfrak{G}/\mathfrak{H})^*))$$

generates $E_0^{s,r} \cong \mathbf{R}$. Denoting the differential in E_k by d_k one has $d_0(\mu \otimes \nu) = d\mu \otimes \nu = 0$. Thus we obtain a class $[\mu \otimes \nu] \in E_1^{s,r}$.

Assume that ν is exact in $\Lambda(\mathfrak{G}^*)_{\mathfrak{H}}$; $\nu = d\beta$ where $\beta \in \Lambda^{s-1}(\mathfrak{G}^*)_{\mathfrak{H}}$. We obtain elements $\mu \otimes \beta \in E_0^{s-1,r}$ and $[\mu \otimes \beta] \in E_1^{s-1,r}$ as before. One computes

$$d_1([\mu \otimes \beta]) = [\mu \otimes d\beta] = [\mu \otimes \nu].$$

It follows that $E_2^{s,r} = 0$ and hence $E_\infty^{s,r} = 0$. Let $n = s + r = \dim(\mathfrak{G})$. One sees trivially that $E_0^{a,b} = 0$ for $a + b = n$ if either $a > s$ or $b > r$. Hence

$$H^n(\mathfrak{G}) \cong \sum_{a+b=n} E_\infty^{a,b} = E_\infty^{s,r} = 0.$$

This contradicts Lemma 2.4. □

2.6 Corollary. *Let G be nilpotent with $H \subset G$ a closed subgroup. Then $H^s(\mathfrak{G}, H) \cong \mathbf{R}$ where $s = \dim(G/H)$.*

Proof. A generator ν for $\Lambda^s(\mathfrak{G}^*)_H$ is given by a left invariant volume form on G/H. We have an inclusion of complexes, $\Lambda(\mathfrak{G}^*)_H \subset \Lambda(\mathfrak{G}^*)_{\mathfrak{H}}$. Since ν is not exact in $\Lambda(\mathfrak{G}^*)_{\mathfrak{H}}$, it is certainly not exact in $\Lambda(\mathfrak{G}^*)_H$. □

3. Prequantization

Let G be simply connected and nilpotent and $\mathfrak{O} \subset \mathfrak{G}^*$ a coadjoint orbit with canonical symplectic form $\omega \in \Omega^2(\mathfrak{O})$. Geometric quantization on the symplectic manifold $(\mathfrak{O}, \omega)$ produces an irreducible unitary representation of G [6]. The first step involves constructing a complex line bundle L over $\mathfrak{O}$ with an Hermitian structure $\langle\ ,\ \rangle$ and a compatible connection α with curvature ω. In our setting, such a prequantization bundle exists, and is unique up to a strong notion of equivalence [6]. Moreover, there is an action of G on L that preserves $\langle\ ,\ \rangle$ and α, and coincides with the coadjoint action of G on $\mathfrak{O}$.

Let L^* be the bundle of non-zero vectors in L and

$$T(L) = \{v \in L: \langle v, v\rangle = 1\}.$$

L^* is the principal bundle for L with fibre $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$, and $T(L)$ is a circle bundle over $\mathfrak{O}$ which completely determines $\langle\ ,\ \rangle$ on L. The connection α, a complex-valued one-form on L^*, is compatible with $\langle\ ,\ \rangle$ in the sense that α is the extension of a real-valued connection form on $T(L)$.

To describe an explicit model for $(L, \langle\ ,\ \rangle, \alpha)$, we need only construct a circle bundle $T(L)$ over $\mathfrak{O}$ with connection form $\alpha \in \Omega^1(T(L))$. If $\rho: T(L) \to \mathfrak{O}$ is the projection, then α has curvature form ω. (That is, $d\alpha = \rho^*(\omega)$.)

Choose $f \in \mathfrak{O}$ and let $G_f = \{g \in G: \mathrm{Ad}^*g(f) = f\}$. $\mathfrak{O}$ is identified with G/G_f, so that the coadjoint action of G on $\mathfrak{O}$ becomes the usual action of G on G/G_f. For G simply connected and nilpotent, G_f is connected and $G_f = \exp(\mathfrak{G}_f)$, where $\mathfrak{G}_f = \{X \in \mathfrak{G}: \mathrm{ad}^*X(f) = 0\}$ [5].

Assume $f \neq 0$. One obtains a character $\chi_f\colon G_f \to T$ defined by $\chi_f(\exp X) = e^{2\pi i f(X)}$ for $X \in \mathfrak{G}_f$. Let $K_f = \operatorname{Ker} \chi_f$. K_f is a normal subgroup of G_f and, as χ_f is surjective, $G_f/K_f \cong T$. The Lie algebra of K_f is $\mathfrak{K}_f = \operatorname{Ker}(f|\mathfrak{G}_f)$ and $\exp(\mathfrak{K}_f)$ is the identity component of K_f. The following fact is easily verified.

3.1 LEMMA. *f is in $\Lambda^1(\mathfrak{G}^*)_{K_f}$, and hence f yields a G-invariant 1-form α on G/K_f.*

Consider the bundle

$$\begin{array}{ccc} G_f/K_f & \hookrightarrow & G/K_f \\ & & \downarrow \rho \\ & & G/G_f. \end{array}$$

This is a circle bundle in view of the identification $G_f/K_f \cong T$. The right action of T on G/K_f is given by $(gK_f, g_0K_f) \mapsto gg_0K_f$. Note that this is well defined since K_f is normal in G_f. It is not hard to show that α is invariant under this right T-action. Given $t \in \mathbf{R}$, regarded as the Lie algebra of T, choose $X_0 \in \mathfrak{G}_f$ with $f(X_0) = t$. Then the vertical vector field V_t on G/G_f is the left invariant vector field $X_0 + \mathfrak{K}_f$, and $\alpha(V_t) = t$. These remarks prove the following lemma.

3.2 LEMMA. *The form α from Lemma 3.1 is a connection form in the circle bundle*

$$G/K_f \underset{\rho}{\twoheadrightarrow} G/G_f.$$

3.3 LEMMA. *Let ω be the symplectic form on $\mathfrak{O} \cong G/G_f$. Then* $\operatorname{curv}(\alpha) = \omega$.

Proof. Let $\pi_f\colon G \twoheadrightarrow G/G_f$ and $\tilde{\pi}_f\colon G \twoheadrightarrow G/K_f$ be the usual projection maps. Then ω is uniquely determined by the identity $\pi_f^*(\omega) = df$ in $\Lambda^2(\mathfrak{G}^*) \subset \Omega^2(G)$. On the other hand, $\operatorname{curv}(\alpha)$ is characterized by $\rho^*(\operatorname{curv}\alpha) = d\alpha$.

We see that

$$\begin{aligned} \pi_f^*(\operatorname{curv}\alpha) &= \tilde{\pi}_f^*\rho^*(\operatorname{curv}\alpha) \\ &= \tilde{\pi}_f^*(d\alpha) \\ &= d\tilde{\pi}_f^*(\alpha) \\ &= df \quad (\text{by definition of } \alpha) \\ &= \pi_f^*(\omega). \end{aligned}$$

Since π_f is a submersion, $\operatorname{curv}\alpha = \omega$. □

Together, Lemmas 3.2 and 3.3 show the following.

3.4 THEOREM. *The bundle* $G_f/K_f \hookrightarrow G/K_f \twoheadrightarrow G/G_f$ *together with the one-form* f *in* $\bigwedge(\mathfrak{G}^*)_{K_f}$ *is a model for* $(T(L), \alpha)$*—the circle bundle with connection given by prequantization.*

Our model for L is the associated complex line bundle

$$G/K_f \times_{G_f/K_f} \mathbf{C} \cong G \times_{G_f} \mathbf{C},$$

whose elements are equivalence classes $[g, c]$ where $(gg_0, c) \sim (g, \chi_f(g_0)c)$ for all $g_0 \in G_f$, $g \in G$, and $c \in \mathbf{C}$. The Hermitian structure on $G \times_{G_f} \mathbf{C}$ is just $\langle [g, c], [g, c'] \rangle = c\bar{c}'$. A model for L^* is given by $G \times_{G_f} \mathbf{C}^*$. The connection α in $T(L)$ gives a complex-valued connection in L^* by prolongation. This is the unique form on L^* whose lift to $G \times \mathbf{C}^*$ is

$$\tilde{\alpha} = f + \frac{1}{2\pi i}\, dz/z.$$

This is the model for the prequantization bundle that can be found in [6].

Notice that the structures $\langle\ ,\ \rangle$ and α are invariant under the obvious left G-actions on $G \times_{G_f} \mathbf{C}$ and $G \times_{G_f} \mathbf{C}^*$. These actions extend the left action of G on G/K_f and are compatible with the G-action on $G/G_f \cong \mathfrak{O}$ in the sense that the projection maps are all G-equivariant.

4. The invariant via prequantization

Let $(L, \langle\ ,\ \rangle, \alpha)$ be a prequantization bundle over a co-adjoint orbit $\mathfrak{O} \subset \mathfrak{G}^*$ of dimension $2q$. The G-action on L preserves $\langle\ ,\ \rangle$ and hence $T(L)$. Writing L_g: $T(L) \to T(L)$ for the action of $g \in G$ on $T(L)$, one has

$${}^G\Omega(T(L)) = \left\{ \beta \in \Omega(T(L)) \colon L_g^*\beta = \beta \quad \text{for all } g \in G \right\},$$

the complex of left G-invariant forms on $T(L)$. We will denote the cohomology of this complex by $H_G^*(T(L))$.

Choose $p_0 \in T(L)$ and define $\tilde{\pi}$: $G \to T(L)$ by $\tilde{\pi}(g) = L_g(p_0)$. This is a lifting of π: $G \to \mathfrak{O}$ to $T(L)$, where $\pi(g) = \mathrm{Ad}^*g(\rho(p_0))$. (Recall that ρ is the projection ρ: $T(L) \to \mathfrak{O}$.) Since $\tilde{\pi}$ is G-equivariant, we obtain a map

$$\tilde{\pi}^* \colon {}^G\Omega(T(L)) \to {}^G\Omega(G) = \bigwedge(\mathfrak{G}^*)$$

and hence a map π^*: $H_G^*(T(L)) \to H^*(\mathfrak{G})$.

4.1 LEMMA. $\tilde{\pi}^*$: $H_G^*(T(L)) \to H^*(\mathfrak{G})$ *does not depend on the choice of* $p_0 \in T(L)$.

Proof. Let $p_0, p_1 \in T(L)$ be two chosen points used to construct $\tilde{\pi}_0$ and $\tilde{\pi}_1$: $G \to T(L)$. As can be seen from the explicit model of $T(L)$ given in Section 3, G acts transitively on $T(L)$ so that we must have $p_1 = \bar{g}p_0$ for some $\bar{g} \in G$. We thus have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\tilde{\pi}_0} & \\ {\scriptstyle R_{\bar{g}}}\downarrow & & T(L) \\ G & \xrightarrow{\tilde{\pi}_1} & \end{array}$$

where $R_{\bar{g}}$: $G \to G$ is right multiplication. This dualizes to a diagram of complexes

$$\begin{array}{ccc} & \xrightarrow{\tilde{\pi}_1^*} & \Lambda(\mathfrak{G}^*) \\ {}^{G}\Omega(T(L)) & & \downarrow{\scriptstyle \mathrm{Ad}^*(\bar{g})} \\ & \xrightarrow{\tilde{\pi}_0^*} & \Lambda(\mathfrak{G}^*) \end{array}$$

This shows that the maps $\tilde{\pi}_0^*$ and $\tilde{\pi}_1^*$ in cohomology $H_G^*(T(L)) \to H^*(\mathfrak{G})$ differ by $\mathrm{Ad}^*(\bar{g})$: $H^*(\mathfrak{G}) \to H^*(\mathfrak{G})$. It is well known that for G connected, the co-adjoint representation on $H^*(\mathfrak{G})$ is trivial [2]. □

Moreover, the entire construction $\tilde{\pi}^*$: $H_G^*(T(L)) \to H^*(\mathfrak{G})$ is unique up to isomorphism.

4.2 LEMMA. *If $(L_1, \langle\ ,\ \rangle_1, \alpha_1)$ and $(L_2, \langle\ ,\ \rangle_2, \alpha_2)$ are two prequantization bundles for $\mathfrak{O}$, then there is an isomorphism τ^*: $H_G^*(T(L_1)) \to H_G^*(T(L_2))$ such that the diagram*

$$\begin{array}{ccc} H_G^*(T(L_1)) & \xrightarrow{\tau^*} & H_G^*(T(L_2)) \\ {\scriptstyle \tilde{\pi}_1^*}\searrow & & \swarrow{\scriptstyle \tilde{\pi}_2^*} \\ & H^*(\mathfrak{G}) & \end{array}$$

commutes.

Proof. The prequantization bundle is unique in a strong sense [6]. There is a vector bundle isomorphism τ: $L_2 \to L_1$ such that:

(i) $\langle \tau(v), \tau(w) \rangle_1 = \langle v, w \rangle_2$; $v, w \in L_2$;

(ii) $\tau^*(\alpha_1) = \alpha_2$; (4.3)

(iii) $\tau(L_g v) = L_g \tau(v)$, $v \in L_2$, $g \in G$.

In view of (i), τ yields an isomorphism of T-bundles τ: $T(L_2) \to T(L_1)$ which is G-equivariant by (iii). If $p \in L_2$ is any chosen point then we use p and

$\tau(p)$ to construct the maps

$$\tilde{\pi}_2: G \to T(L_2) \quad \text{and} \quad \tilde{\pi}_1: G \to T(L_1).$$

Clearly $\tilde{\pi}_1 = \tau \circ \tilde{\pi}_2$ and $\tau^*: H_G^*(T(L_1)) \to H_G^*(T(L_2))$ is a suitable isomorphism. □

We remark that the isomorphism τ^* in Lemma 4.2 is essentially canonical. If $\tau_0, \tau_1: L_2 \to L_1$ both satisfy conditions 4.3, then

$$\bar{\tau} = \tau_1 \tau_0^{-1}: L_1 \to L_1$$

preserves $\langle \ , \ \rangle_1, \alpha_1$ and the G-action on L_1. It follows that $\bar{\tau}$ comes from the right action R_t of some fixed element $t \in T$ on $T(L_1)$ [6]. This shows that τ_0^* and τ_1^* can only differ by the right action of T on $H_G^*(T(L))$.

The differential form $V = \alpha \wedge (d\alpha)^q \in \Omega^{2q+1}(T(L))$ is G-invariant since α is invariant, and closed since $\dim(T(L)) = 2q + 1$. We obtain a cohomology class

$$[V] \in H_G^{2q+1}(T(L)). \tag{4.4}$$

Notice that we can also write $V = \alpha \wedge \rho^*(\omega^q)$ where $\omega \in \Omega^2(\mathfrak{O})$ is the symplectic form. Since ω^q is a volume form on $\mathfrak{O}$ and α is non-zero on vectors tangent to the fibres of $T(L)$, we see that V is a volume form on $T(L)$. On can regard α as a contact structure that gives rise to the volume form V.

4.5 Lemma. *The class $[V] \in H_G^{2q+1}(T(L))$ is well defined up to the isomorphism τ^* in Lemma* 4.2.

Proof. If $(L_1, \langle \ , \ \rangle_1, \alpha_1), (L_2, \langle \ , \ \rangle_2, \alpha_2)$ are two prequantization bundles then the isomorphism $\tau^*: H_G^*(T(L_1)) \to H_G^*(T(L_2))$ is induced by a G-equivariant map $\tau: T(L_2) \to T(L_1)$ with the property that $\tau^*(\alpha_1) = \alpha_2$. □

Lemmas 4.1, 4.2 and 4.5 show that the class $\tilde{\pi}^*([V]) \in H^{2q+1}(\mathfrak{G})$ does not depend on the choice of prequantization bundle $(L, \langle \ , \ \rangle, \alpha)$ or on the choice of $p_0 \in T(L)$ used to define $\tilde{\pi}: G \to T(L)$. We now return to the specific model for $(L, \langle \ , \ \rangle, \alpha)$ described in Section 3. In particular, we take $T(L) = G/K_f$ for some $f \in \mathfrak{O}$. Using eK_f (where $e \in G$ is the identity element) as the point p_0, $\tilde{\pi}: G \to G/K_f$ becomes the usual projection $\tilde{\pi}(g) = gK_f$. Since α is characterized by $\tilde{\pi}^*(\alpha) = f$, one also has $\tilde{\pi}^*(V) = f \wedge (df)^q$. This proves the following theorem.

4.6 Theorem. *Let $\mathfrak{O}$ be any co-adjoint orbit and $(L, \langle \ , \ \rangle, \alpha)$ a prequantization bundle for $\mathfrak{O}$. Then $i(\mathfrak{O}) = \tilde{\pi}^*([V])$.*

The model G/K_f for $T(L)$ also gives us a way of computing $H_G^*(T(L))$. Indeed, ${}^G\Omega(G/K_f)$ is a model for ${}^G\Omega(T(L))$ and the former can be identified with $\bigwedge(\mathfrak{G}^*)_{K_f}$ via $\tilde{\pi}^*$. We see that $H_G^*(T(L)) \cong H^*(\mathfrak{G}, K_f)$. In particular

$$H_G^{2q+1}(T(L)) \cong \mathbf{R}$$

in view of Corollary 2.6.

4.7 THEOREM. *There is an isomorphism* $H_G^*(T(L)) \cong H^*(\mathfrak{G}, K_f)$. *Moreover,* $H_G^{2q+1}(T(L)) \cong \mathbf{R}$, *generated by* $[V]$.

4.8 COROLLARY. *The following are equivalent*:
(a) $i(\mathfrak{O}) = 0$.
(b) $\tilde{\pi}^*\colon H_G^{2q+1}(T(L)) \to H^{2q+1}(\mathfrak{G})$ *is the zero map.*
(c) *The map* $H^{2q+1}(\mathfrak{G}, K_f) \to H^{2q+1}(\mathfrak{G})$ *induced by the inclusion* $\bigwedge(\mathfrak{G}^*)_{K_f} \hookrightarrow \bigwedge(\mathfrak{G}^*)$ *is the zero map.*

We remark that for computational purposes, it is often easier to work with $H^*(\mathfrak{G}, \mathfrak{K}_f)$. The map $H^*(\mathfrak{G}, K_f) \to H^*(\mathfrak{G}, \mathfrak{K}_f)$ arising from $\bigwedge(\mathfrak{G}^*)_{K_f} \hookrightarrow \bigwedge(\mathfrak{G}^*)_{\mathfrak{K}_f}$ need not be an isomorphism, since K_f need not be connected. However, in the top dimension $2q+1$, we do have $H^{2q+1}(\mathfrak{G}, K_f) \cong H^{2q+1}(\mathfrak{G}, \mathfrak{K}_f) \cong \mathbf{R}$ as was shown in Lemma 2.5 and Corollary 2.6. In particular, $i(\mathfrak{O}) = 0$ if and only if $H^{2q+1}(\mathfrak{G}, \mathfrak{K}_f) \to H^{2q+1}(\mathfrak{G})$ is the zero map. This observation allows one to use the Hochschild-Serre spectral sequence E_r for the pair $(\mathfrak{G}, \mathfrak{K}_f)$ to study vanishing of the invariant. The map $H^i(\mathfrak{G}, \mathfrak{K}_f) \to H^i(\mathfrak{G})$ can be written in terms of the spectral sequence as a composition

$$H^i(\mathfrak{G}, \mathfrak{K}_f) \cong E_2^{i,0} \twoheadrightarrow E_\infty^{i,0} \hookrightarrow H^i(\mathfrak{G}).$$

This shows that $i(\mathfrak{O}) = 0$ if and only if $E_\infty^{2q+1,0} = \{0\}$.

5. Square integrable representations

Suppose that ρ is an irreducible unitary representation of G corresponding to a co-adjoint orbit $\mathfrak{O} \subset \mathfrak{G}^*$. Then ρ is square-integrable modulo the center $Z(G)$ of G if and only if $G_f = Z(G)$ for $f \in \mathfrak{O}$ [7].

5.1 THEOREM. *If G has one-dimensional center and ρ is square integrable modulo the center, then* $i(\mathfrak{O}_\rho) \neq 0$.

Proof. We need only show that $H^{2q+1}(\mathfrak{G}, \mathfrak{K}_f) \to H^{2q+1}(\mathfrak{G})$ is not the zero map. In the present case, $\mathfrak{K}_f = \{0\}$ since it is a codimension-one subalgebra of $\mathfrak{G}_f$. □

Theorem 5.1 was also proved in [1] using different methods. When $\dim(Z(G)) > 0$, one can obtain useful information by studying the spectral sequence for $(\mathfrak{G}, \mathfrak{R}_f)$. Since $\mathfrak{R}_f$ is a subalgebra of $Z(\mathfrak{G})$, $\mathfrak{R}_f$ is an ideal in $\mathfrak{G}$ and $\mathfrak{G}$ acts trivially (via ad*) on $H^*(\mathfrak{R}_f)$. As noted in Section 2, the E_2-term in the spectral sequence is thus tame, $E_2^{i,j} \cong H^i(\mathfrak{G}/\mathfrak{R}_f) \otimes H^j(\mathfrak{R}_f)$. In fact, $H^j(\mathfrak{R}_f) = \Lambda^j(\mathfrak{R}_f^*)$ since $\mathfrak{R}_f$ is abelian. Note that $E_2^{i,j} = \{0\}$ for $j \geq \dim(Z(G))$ and hence $E_\infty = E_{\dim(Z(G))+1}$. In particular, the invariant vanishes if and only if $E_{\dim(Z(G))+1}^{2q+1,0} = \{0\}$.

The differential d_2: $E_2^{i,1} \to E_2^{i+2,0}$ is given by

$$d_2([\alpha] \otimes [h]) = (-1)^i[\alpha \wedge d\tilde{h}] \quad \text{for } h \in \mathfrak{R}_f^* \text{ and } [\alpha] \in H^i(\mathfrak{G}/\mathfrak{R}_f)$$

where $\tilde{h} \in \mathfrak{G}^*$ is any linear functional extending h to $\mathfrak{G}$ (that is, $\tilde{h}|\mathfrak{R}_f = h$). This can be written as $d_2([\alpha] \otimes [h]) = \tau([h]) \cdot [\alpha]$, where τ: $H^1(\mathfrak{R}_f) \to H^2(\mathfrak{G}/\mathfrak{R}_f)$ is given by $\tau([h]) = [d\tilde{h}]$.

5.2 THEOREM. *Let ρ be square integrable modulo the center $Z(G)$ of G where* $\dim(Z(G)) > 1$. *Let $\mathfrak{O}$ be the corresponding orbit and $f \in \mathfrak{O}$. If τ: $H^1(\mathfrak{R}_f) \to H^2(\mathfrak{G}/\mathfrak{R}_f)$ is not the zero map, then $i(\rho) = 0$. Moreover, if* $\dim(Z(G)) = 2$, *then this condition is also necessary for the vanishing of $i(\rho)$.*

Proof. $E_3^{2q+1,0} = \{0\}$ if and only if $[V] \in H^{2q+1}(\mathfrak{G}/\mathfrak{R}_f) \cong E_2^{2q+1,0}$ is in the image of d_2. Equivalently, we must be able to write $[V]$ in the form $\tau([h]) \cdot [\alpha]$ for some $[h] \in H^1(\mathfrak{R}_f)$, $[\alpha] \in H^{2q}(\mathfrak{G}/\mathfrak{R}_f)$. Since $\mathfrak{G}/\mathfrak{R}_f$ is a nilpotent Lie algebra of dimension $2q+1$, $H^*(\mathfrak{G}/\mathfrak{R}_f)$ satisfies Poincaré duality and $E_3^{2q+1,0} = \{0\}$ if and only if $\tau([h]) \neq 0$ for some $[h] \in H^1(\mathfrak{R}_f)$. The condition $E_3^{2q+1,0} = \{0\}$ implies $E_\infty^{2q+1,0} = \{0\}$ and thus $i(\rho) = 0$.

If $\dim(Z(G)) = 2$ then one has $E_\infty = E_3$ so that $i(\rho) = 0$ if and only if $\tau \neq 0$. □

We remark that for ρ square integrable, $\mathfrak{R}_f$ is an ideal in $\mathfrak{G}$ and hence independent of $f \in \mathfrak{O}$ chosen. It follows that the condition in Theorem 5.2 makes reference to an invariant τ that depends only on (the equivalence class of) the representation ρ.

The content of Theorem 5.2 can be clarified by carrying out computations using explicit bases. Suppose that $\mathfrak{G}$ has basis $\{Z_1, Z_2, X_1, \ldots, X_n\}$ where $\{Z_1, Z_2\}$ is a basis for $Z(\mathfrak{G})$. Suppose that ρ is square integrable modulo $Z(G)$ and corresponds to an orbit $\mathfrak{O}$ with $f \in \mathfrak{O}$. Let $\{\lambda_1, \lambda_2, \alpha_1, \ldots, \alpha_n\}$ be the dual basis for $\mathfrak{G}^*$. We must have $f|Z(\mathfrak{G}) \neq 0$, so that $f|Z(\mathfrak{G}) = a\lambda_1 + b\lambda_2$ where $a \neq 0$ or $b \neq 0$. Hence, $\mathfrak{R}_f = \langle bZ_1 - aZ_2 \rangle$ and $\mathfrak{R}_f^*$ is generated by $b\lambda_1 - a\lambda_2$. According to Theorem 5.2, $i(\rho) = 0$ if and only if $[b\,d\lambda_1 - a\,d\lambda_2] \neq 0$ in $H^2(\mathfrak{G}/\langle bZ_1 - aZ_2 \rangle)$.

As an example, consider the Lie algebra $\mathfrak{G}$ with basis

$$\{Z_1, Z_2, X_1, X_2, Y_1, Y_2\} \quad \text{where } [X_1, Y_1] = Z_1 = [X_2, Y_2]$$

and all other brackets vanish (this is the Lie algebra for the direct product of a Heisenberg group with $\mathbf{R}$). Let $\{\lambda_1, \lambda_2, \nu_1, \nu_2, \mu_1, \mu_2\}$ be the dual basis and $f = \lambda_1$. Then $\mathfrak{G}_{\lambda_1} = Z(\mathfrak{G})$ so that $\mathfrak{O} = \mathfrak{O}_{\lambda_1}$ is square integrable. Since $d\lambda_2 = 0$, we must have $i(\mathfrak{O}) \neq 0$. Indeed, $i(\mathfrak{O})$ is represented by the form

$$2\lambda_1 \wedge \mu_1 \wedge \nu_1 \wedge \mu_2 \wedge \nu_2$$

which is not exact in $\bigwedge(\mathfrak{G}^*)$. Next consider $\mathfrak{G}'$, the Lie algebra obtained by introducing another non-zero bracket: $[X_1, Y_2] = Z_2$. As before, $\mathfrak{O} = \mathfrak{O}_{\lambda_1}$ is square integrable but now $d\lambda_2 = \mu_2 \wedge \nu_1 \neq 0$. In fact $[d\lambda_2] \neq 0$ in

$$H^2(\mathfrak{G}'/\langle Z_2\rangle) = H^2(\langle Z_1, X_1, X_2, Y_1, Y_2\rangle)$$

so that we now must have $i(\mathfrak{O}) = 0$. Indeed, one has

$$2\lambda_1 \wedge \mu_1 \wedge \nu_1 \wedge \mu_2 \wedge \nu_2 = d(2\lambda_1 \wedge \lambda_2 \wedge \mu_1 \wedge \nu_2) \quad \text{in } \bigwedge(\mathfrak{G}'^*).$$

In general, $\mathfrak{O}_{a\lambda_1 + b\lambda_2} \subset \mathfrak{G}'^*$ is square integrable for any $a, b \in \mathbf{R}$, with $a \neq 0$, and $i(\mathfrak{O}_{a\lambda_1 + b\lambda_2}) = 0$. These are the orbits of maximal dimension in $\mathfrak{G}'^*$. In addition, there are two dimensional (non-square integrable) orbits

$$\mathfrak{O}_{a\lambda_2 + b\nu_2 + c\mu_1} = \{a\lambda_2 + x\nu_1 + b\nu_2 + c\mu_1 + x\mu_2 \colon x, y \in \mathbf{R}\}, \quad a \neq 0,$$

with

$$i\left(\mathfrak{O}_{a\lambda_2 + b\nu_2 + c\mu_1}\right) = a^2[\lambda_2 \wedge \mu_2 \wedge \nu_1] \neq 0.$$

The remaining orbits in $\mathfrak{G}'^*$ are single points in the subspace $\langle \nu_1, \nu_2, \mu_1, \mu_2\rangle$ and correspond to characters.

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