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DOMINATING MEASURES FOR SPACES OF ANALYTIC FUNCTIONS

BY

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The mixed norm space $H(p, q, \alpha)$ is the collection of functions f analytic in the unit disk with finite norm

$$\|f\|_{p,q,\alpha} = \left[\int_0^1 (1-r)^{\alpha q - 1}\left(\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{q/p} dr\right]^{1/q}.$$

Sufficient conditions on a family of measures $\{\mu_r: 0 < r < 1\}$ on U and a measure ν on $[0, 1]$ are given to obtain an inequality

$$\|f\|_{p,q,\alpha}^q \le C \int \left(\int |f|^p \, d\mu_r\right)^{q/p} d\nu(r), \quad f \in H(p, q, \alpha)$$

with C independent of f. Similar results are obtained for spaces of "slow mean growth" ($q = \infty$) and the Hardy spaces ($q = \infty$, $\alpha = 0$). In the case of the Bergman spaces ($p = q$) these conditions are an improvement over those obtained in [5] and [6].

1. Introduction

Let U be the open unit disk $\{|z| < 1\}$ in the complex plane **C**. For $0 < p$, $q \le +\infty$ and $\alpha \ge 0$, the mixed norm spaces $H(p, q, \alpha)$ are defined as follows. For any Borel measurable function f on U define

$$M_p(f, r) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} \quad \text{if } p < +\infty.$$

Define

$$M_\infty(f, r) = \sup\left\{|f(re^{i\theta})|: 0 \le \theta < 2\pi\right\}.$$

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If $q < +\infty$ define

$$\|f\|_{p,q,\alpha} = \left(\int_0^1 (1-r)q^{\alpha-1} M_p(f,r)^q \, dr\right)^{1/q}$$

and define

$$\|f\|_{p,\infty,\alpha} = \sup\left\{(1-r)^\alpha M_p(f,r) : 0 < r < 1\right\}.$$

Let $B(p,q,\alpha)$ denote the class of functions f with $\|f\|_{p,q,\alpha} < +\infty$ and let $H(p,q,\alpha)$ denote the subspace of $B(p,q,\alpha)$ consisting of analytic functions. The space $H(p,\infty,\alpha)$ has been called the space of functions of slow mean growth [8]. $H(p,\infty,0)$ is the Hardy space H^p, and $H(p,p,(\beta+1)/p)$ is the weighted Bergman space commonly denoted $A^{p,\beta}$. If $q < +\infty$ then $H(p,q,0) = \{0\}$.

Let dm_β denote the measure $(1-r)^\beta r\,dr\,d\theta$ on U. In [3] necessary and sufficient conditions were obtained in order that a set $G \subset U$ satisfy

$$\int_U |f|^p \, dm_\beta \le C \int_G |f|^p \, dm_\beta \quad \text{for all} \quad f \in A^{p,\beta}.$$

In [6] this result was used to obtain conditions on a measure μ in order that

$$\int |f|^p \, dm_\beta \le C \int |f|^p \, d\mu.$$

In [4] the first result was extended to harmonic functions f and in [7] a special case of the second was generalized to include Hardy space norms. The methods used for harmonic functions differed from those used originally for $A^{p,\beta}$. Here, rather than deduce an inequality like $\int |f|^p \, dm_\beta \le C \int |f|^p \, d\mu$ for a relatively general class of measures from a like inequality for the measures $\chi_G \, dm_\beta$, we use the method developed in [5] for harmonic functions and adapt it slightly to apply directly to the more general measures μ. Moreover, the same basic method may be used for all the mixed norm spaces $H(p,q,\alpha)$. The main result (Theorem 2) is a sufficient condition on a family of measures $\{\mu_r : 0 \le r < 1\}$ on U and a measure ν on $[0,1)$ in order that $\|f\|_{p,q,\alpha}^q \le C\int(\int |f|^p \, d\mu_r)^{q/p} \, d\nu(r)$ for all $f \in H(p,q,\alpha)$. If either q or p is $+\infty$ the corresponding integral is replaced by an essential supremum. Our sufficient condition is somewhat difficult to state at this stage, but amounts to a condition that μ_r cannot, locally, be concentrated too near the zeros of a locally defined class of analytic functions. The condition is very easy to apply and provides a variety of examples which could not be obtained by previous methods.

In Section 2 we obtain some elementary estimates on functions in the spaces $H(p,q,\alpha)$ and prove two lemmas, the purpose of which is to establish for each $f \in H(p, q, \alpha)$ the existence of a "large" set F on which $|f(z)|^p$ is comparable to the mean of $|f|^p$ over a neighborhood of z. This will restrict the oscillation of $|f|$ enough that pointwise estimates can be made on F. The lemmas show that F is large enough that estimates made on F dominate those on U.

In Section 3, we investigate the zero sets of a certain class of analytic functions. As previously mentioned, the condition we obtain on the measures μ_r is that they not be concentrated too close to these zero sets. To obtain quantitative results we need some estimates on these zero sets.

In Section 4 we complete the special case $H(p, p, 1/p)$ which is the Bergman space $A^{p,0}$. We separate this special case for several reasons. The argument is somewhat simpler and the theorem is easier to state. The special case may be of independent interest to readers not concerned with mixed norm spaces. And, finally, the Bergman space has been the main focus of my previous work and I wish to show that even in this case stronger results have been obtained.

Section 5 presents the main theorem, Theorem 2, of which Theorem 1 in Section 4 is a special case. By Section 5, enough preliminary work has been done so that the statement of the theorem takes more effort than the proof.

Section 6 presents some examples of applications of Theorem 2.

2. Preliminary estimates with mixed norms

The "norms" $\|\cdot\|_{p,q,\alpha}$ are true norms only if $p, q \geq 1$. If we let $s = \min(p, q, 1)$ then $\beta(f, g) = \|f - g\|^s_{p,q,\alpha}$ is a metric under which $H(p, q, \alpha)$ is complete. Unless $q < +\infty$ and $\alpha = 0$ (a case we henceforth formally exclude) each $H(p, q, \alpha)$ contains H^∞ and the point evaluations $f \to f(a)$ are continuous functionals on each space.

The pseudohyperbolic metric ρ on U is defined by

$$\rho(z, w) = |z - w|/|1 - \bar{w}z|.$$

As usual, for a set E, $\rho(z, E) = \inf\{\rho(z, w) \colon w \in E\}$ is the distance from z to E. Let C_r denote the circle of radius r centered at 0 and let

$$K_r = \{z \colon \rho(z, C_r) < \varepsilon\}$$

where ε is any conveniently chosen number with $0 < \varepsilon < 1$. If $r < \varepsilon$ then K_r is the disk of radius $(r + \varepsilon)/(1 + \varepsilon r)$ and center 0. Otherwise, K_r is the annulus centered at 0 with inner radius

$$r_* = (r - \varepsilon)/(1 - \varepsilon r)$$

and outer radius

$$r^* = (r + \varepsilon)/(1 + \varepsilon r).$$

It is clear that

$$(1+\varepsilon)(1-r) < 1 - r_* < \frac{1+\varepsilon}{1-\varepsilon}(1-r),$$
$$\frac{1-\varepsilon}{1+\varepsilon}(1-r) < 1 - r^* < (1-\varepsilon)(1-r),$$

and

$$2\varepsilon(1-r) < r^* - r_* < \frac{4\varepsilon}{1-\varepsilon^2}(1-r).$$

Because $M_p(f, r)$ increases with r, there is a constant $C = C(\varepsilon)$ such that

$$C^{-1}M_p^p(f, r_*) \le \frac{1}{1-r}\int_{K_r} |f|^p\, dm \le CM_p^p(f, r^*) \tag{2.1}$$

for all f analytic in U, $r > \varepsilon$. Here $m = m_0$ is just Lebesgue area measure $r\,dr\,d\theta$. Define $B_p(f, r) = [(1-r)^{-1}\int_{K_r}|f|^p\, dm]^{1/p}$. It is then easy to verify that for $p < +\infty$ the norm on $H(p, q, \alpha)$ is equivalent to

$$\left(\int_0^1 B_p(f, r)^q (1-r)^{\alpha q - 1}\, dr\right)^{1/q} \quad \text{if } q < +\infty,$$
$$\sup(1-r)^\alpha B_p(f, r) \quad \text{if } q = +\infty.$$

The next lemma generalizes an observation in [3]. Our method for proving the main theorem is to integrate a certain pointwise estimate. This estimate is not valid at all points of U and the following lemma will be used to show that the set of points where it is valid is large enough. For the ε fixed earlier define the sequence $\{r_n\}$ by $r_0 = 0$, $r_{n+1} = r_n^* = (r_n + \varepsilon)/(1 + r_n\varepsilon)$. Also, let

$$D(z) = \{w \in U : \rho(z, w) < \varepsilon\} \quad \text{for } z \in U.$$

LEMMA 1. *Let ν be a measure on $[0,1)$ such that*

$$0 < \nu[r_{n-1}, r_n) < C\nu[r_n, r_{n+1}).$$

There is a number $\delta > 0$ and a constant K such that if f is analytic in U and

$$F = \left\{z \in U : |f(z)|^p > \delta\int_{D(z)} |f|^p\, dm/m(D(z))\right\}$$

then

$$\int M_p(f, r)^q\, d\nu(r) \le K\int M_p(\chi_F f, r)^q\, d\nu(r),$$

provided both sides of the inequality are finite. The constant δ depends only on ε, p/q, and

$$\sup_n \frac{\nu[r_{n-1}, r_n)}{\nu[r_n, r_{n+1})}.$$

Proof. Let δ be a positive number (to be specified later); let F be defined as in the statement of the lemma, and let $E = U \setminus F$ so that we have

$$\begin{aligned} |\chi_E(z)f(z)|^p &\le \delta \int_{D(z)} |f|^p \, dm/m(D(z)) \\ &= \delta \int_{K_r} \frac{\chi_{D(z)}(w)|f(w)|^p}{m(D(z))} \, dm(w) \end{aligned}$$

where $r = |z|$. (We use the fact that $K_r \cap D(z) = D(z)$.) Putting $z = re^{i\theta}$ and integrating with respect to θ gives

$$\begin{aligned} M_p^p(\chi_E f, r) &\le \delta \int_{K_r} |f(w)|^p \int_0^{2\pi} \frac{\chi_{D(w)}(re^{i\theta})}{m(D(re^{i\theta}))} \, d\theta \, dm(w) \\ &\le \delta C B_p^p(f, r) \end{aligned} \tag{2.2}$$

because $m(D(re^{i\theta})) \sim (1-r)^2$ and $\int_0^{2\pi} \chi_{D(w)}(re^{i\theta}) \, d\theta \le C(1-r)$. Raise to the q/p power and integrate with respect to $d\nu(r)$ on the interval $[r_{n-1}, r_n)$:

$$\begin{aligned} \int_{r_{n-1}}^{r_n} M_p^q(\chi_E f, r) \, d\nu(r) &\le (\delta C)^{q/p} \int_{r_{n-1}}^{r_n} B_p^q(f, r) \, d\nu(r) \\ &\le C'(\delta C)^{q/p} M_p^q(f, r_{n+1}) \nu[r_{n-1}, r_n) \\ &\le C'(\delta C)^{q/p} \int_{r_{n-1}}^{r_{n+2}} M_p^q(f, r) \, d\nu(r) \end{aligned}$$

where C depends only on ε and C' may differ from one line to the next, but it depends only on ε and ν. Sum on n and raise to the power s/q to obtain

$$\beta_\nu(0, \chi_E f) \le \gamma \beta_\nu(0, f)$$

where β_ν is defined in a manner similar to β at the beginning of this section:

$$\beta_\nu(f, g) = \left[\int_0^1 M_p(f - g, r)^q \, d\nu(r) \right]^{s/q}$$

(with $s = \min(p, q, 1)$ as before) and $\gamma = (3C')^{s/q}(\delta C)^{s/p}$. Since β_ν is a metric we get

$$\beta_\nu(0, \chi_F f) = \beta_\nu(\chi_E f, f) \geq \beta_\nu(0, f) - \beta_\nu(0, \chi_E f) \geq (1-\gamma)\beta_\nu(0, f).$$

Now choose $\delta = (2C)^{-1}(3C')^{-p/q}$ so $\gamma = 2^{-s/p}$, and let $K = (1-\gamma)^{-q/s}$. Note that δ depends only on ε, ν and the ratio p/q. □

There is a formulation of Lemma 1 for $q = +\infty$, which requires considerably weaker hypotheses on ν:

LEMMA 2. *Suppose $\nu[r_{n-1}, r_n] > 0$ for every n. There is a number $\delta > 0$ and a constant K such that if f is analytic and F is as in Lemma 1, then*

$$\nu\text{-ess sup}(1-r)^\alpha M_p(f, r) \leq K\ \nu\text{-ess sup}(1-r)^\alpha M_p(\chi_F f, r),$$

provided both sides are finite. In case $\alpha = 0$ we need only require $\nu[r, 1) > 0$ for every $r < 1$.

Proof. From (2.1) and (2.2) it follows that

$$(1-r)^\alpha M_p(\chi_E f, r) \leq \delta^{1/p} C(1-r')^\alpha M_p(f, r')$$

whenever $r_{n-1} \leq r < r_n$ and $r_{n+1} \leq r' < r_{n+2}$. Thus a similar inequality holds between the ν-essential suprema. If $\alpha = 0$ the weaker hypothesis is sufficient because $M_p(f, r)$ is increasing. Now we can use the same argument as in Lemma 1 on an appropriate distance function. □

3. Zero sets of analytic functions

The type of condition we are heading for is that the measures $\mu_r|_{D(z)}$ do not "lie too close to" the zero set of any function in a certain class of analytic functions on $D(z)$ when $|z| = r$. In this section we will discuss the class of analytic functions and its zero sets. The analytic functions are just those suggested by the definition of F in Lemma 1, namely those satisfying

$$|f(z)|^p \geq \delta \int_{D(z)} |f|^p\, dm/m(D(z)).$$

By taking appropriate conformal transformations from $D(z)$ to U we may concentrate on the collection $\mathcal{G}(\delta)$ defined by

$$\mathcal{G}(\delta) = \left\{ f \text{ analytic in } U\colon |f(0)|^p \geq \delta,\ \pi^{-1}\int_U |f(z)|^p\, dm \leq 1 \right\}.$$

Our next lemma will provide some estimates on the possible zero sets of $\mathcal{G}(\delta)$. This lemma is not needed in the proof and is used only in later examples of applications of the main theorems. It may be omitted on first reading.

LEMMA 3. *If f is analytic in U with zero sequence $\{a_k\}$ and $f(0) \neq 0$, then*

$$\log|f(0)| + \sum_{|a_k|<R}\left[\log\frac{R}{|a_k|} - \frac{1}{2}\left(1 - \frac{|a_k|^2}{R^2}\right)\right] = \frac{1}{\pi R^2}\int_{|z|<R}\log|f|\,dm \tag{3.1}$$

for any positive $R \leq 1$. In particular if $f \in \mathcal{G}(\delta)$ then the number of zeros in $|z| < R$ is at most $(4/p)\log(1/\delta)/(1 - R^2)^2$. Moreover, all the zeros satisfy

$$|a_k| \geq \delta^{1/p}e^{-1/2}.$$

Proof. Take Jensen's formula

$$\log|f(0)| + \sum_{|a_k|\leq r}\log\frac{r}{|a_k|} = \int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta/2\pi,$$

multiply it by $2r\,dr$ and integrate from 0 to R to obtain (3.1). By Jensen's inequality the right hand side of (3.1) is at most

$$p^{-1}\log\left[\int_{|z|<R}|f|^p\,dm/(\pi R^2)\right].$$

This will be nonpositive when $R = 1$. Thus for f in $\mathcal{G}(\delta)$

$$\frac{1}{2}\sum_k\left[\log\frac{1}{|a_k|^2} - \left(1 - |a_k|^2\right)\right] \leq p^{-1}\log(1/\delta). \tag{3.2}$$

A power series expansion shows that

$$\log\frac{1}{1-x} - x \geq x^2/2.$$

We apply this to $x = 1 - |a_k|^2$ in (3.2) and sum only over $|a_k| \leq R$ to obtain

$$N(1 - R^2)^2/4 \leq p^{-1}\log(1/\delta)$$

where N is the number of a_k with $|a_k| \leq R$. Finally, taking only a single term in (3.2) we get

$$[\log(1/|a_k|^2) - 1]/2 \le p^{-1}\log(1/\delta)$$

which leads to $|a_k| \ge \delta^{1/p}e^{-1/2}$. □

The next lemma, while quite simple, is the key to the main theorem.

LEMMA 4. *Let $0 < R < 1$ and let $\mathscr{M}$ be a family of probability measures on $|z| \le R$ such that no measure μ in the weak* closure of $\mathscr{M}$ is supported on a zero set of $\mathscr{G}(\delta)$, then there is a positive constant η depending only on δ, p and $\mathscr{M}$ such that $\int |f|^p\, d\mu \ge \eta |f(0)|^p$ for all $f \in \mathscr{G}(\delta)$.*

Proof. Let $\overline{\mathscr{M}}$ denote the weak* closure of $\mathscr{M}$ so that $\overline{\mathscr{M}}$ is compact in the weak* topology. $\mathscr{G}(\delta)$ is a normal family in U and so it is a compact set in the topology of uniform convergence on compacta. It is routine to verify that the function $(\mu, f) \to \int |f|^p\, d\mu$ is jointly continuous on $\overline{\mathscr{M}} \times \mathscr{G}(\delta)$ and the hypotheses imply it is never zero on this compact set. Since its infimum is attained we obtain $\int |f|^p\, d\mu \ge \eta > 0$ for all $\mu \in \overline{\mathscr{M}}$ (and so in particular for all $\mu \in \mathscr{M}$) and all $f \in \mathscr{G}(\delta)$. Since $|f(0)|^p \le \pi^{-1}\int |f|^p\, dm \le 1$, this is what was wanted. □

4. The Bergman spaces

In this section we consider the special case $A^p = H(p, p, 1/p)$. The problem here is to find conditions on the measure μ on U such that

$$\int |f|^p\, dm \le C \int |f|^p\, d\mu \quad \text{for all} \quad f \in A^p.$$

This is apparently more general than the Bergman space case of the main theorem, in that we do not here require the measure μ to have the form $d\mu_r(z)d\nu(r)$. It would certainly be more general if we required μ_r to be supported on $|z| = r$. However, if we put $d\mu_r = C\chi_{K_r}\, d\mu/(1 - r)$ and $d\nu = C^{-1}\Sigma(1 - r_n)\delta_{r_n}$ for appropriate constants and with r_n chosen so that $\{K_{r_n}\}$ partitions U, then $d\mu = d\mu_r\, d\nu(r)$ and Theorem 1 (to follow) is a consequence of Theorem 2. Nevertheless, for clarity we treat this special case separately.

The method of proof will be to make estimates of the form

$$|f(z)|^p \le \int_{D(z)} |f|^p\, d\mu / m(D(z))$$

and then to integrate these pointwise estimates with respect to $dm(z)$. These pointwise estimates are much too strong, however; so we make them only at the set of points F described by Lemma 1, a set varying with f. The pointwise estimates themselves are obtained by applying Lemma 4 to a class $\mathcal{M}$ consisting of normalized Mobius translates of localizations of μ.

DEFINITION 1. $\mathcal{M} = \mathcal{M}_\mu$ is the collection of all measures formed by the following process: Let

$$\psi_a(z) = \frac{1}{\varepsilon}\frac{z-a}{1-\bar{a}z}.$$

Then ψ_a maps $D(a)$ onto U. Form $\mu^a = \chi_{\{|z| \le R\}}(\mu \circ \psi_a^{-1})$ and let $\mathcal{M}_\mu$ be the collection of all normalizations $\mu^a/\|\mu^a\|$ as a varies over U.

What we have done is considered the restrictions of μ to each of the disks $\tilde{D}(a) \subseteq D(a)$, where $\tilde{D}(a) = \{z: \rho(z, a) \le R\varepsilon\}$, then transferred these restrictions to the disk $|z| \le R$ via ψ_a, then normalized them. We are now requiring that ε and R both be fixed numbers in (0, 1). The derivative

$$\psi_a'(z) = \frac{1}{\varepsilon}\frac{1-|a|^2}{(1-\bar{a}z)^2}$$

satisfies $\psi_a'(z)/\psi_a'(a) = (1-|a|^2)^2/(1-\bar{a}z)^2 = (1+\bar{a}\varepsilon w)^2$ where $w = \psi_a(z)$. If $z \in \tilde{D}(a)$ then $|w| < R$. Consequently,

$$(1-\varepsilon R)^2 < |\psi_a'(z)/\psi_a'(a)| < (1+\varepsilon R)^2$$

and $|\arg(\psi_a'(z)/\psi_a'(a))|$ is bounded away from π. Thus the distortion produced in $\mu|_{\tilde{D}(a)}$ by ψ_a is under control (independent of a) and the reader may think of μ^a as obtained by translating and dilating $\mu|_{\tilde{D}(a)}$. (In fact all of the following could be done with this as a definition of μ^a, but that would complicate Lemmas 1 and 2 and some of the manipulations of integrals to follow.)

Let δ and K be as in Lemma 1 with $d\nu(r) = r\,dr$ and $p = q$. Thus if $f \in A^p$ and

$$F = \left\{z \in U: |f(z)|^p > \delta\int_{D(z)}|f|^p\,dm/m(D(z))\right\},$$

then

$$\int_U |f|^p\,dm \le K\int_F |f|^p\,dm.$$

Now let $a \in F$ and let

$$\phi_a(z) = \psi_a^{-1}(z) = \frac{\varepsilon z + a}{1+\bar{a}\varepsilon z}.$$

Then

$$|f(\phi_a(0))|^p = |f(a)|^p > \delta \int_{D(a)} |f|^p \, dm/m(D(a))$$
$$= \delta \int_U |f \circ \phi_a|^p |\phi_a'|^2 \, dm/m(D(a)) > \delta' \int_U |f \circ \phi_a|^p \, dm.$$

The last inequality is from the estimate $|\phi_a'(z)|^2/m(D(a)) \geq c > 0$ with c independent of a and z (depending only on ε). Thus $f \circ \phi_a$ will belong to $\mathcal{G}(\delta')$ when appropriately normalized. If we assume that the family $\mathcal{M}_\mu$ satisfies the hypotheses of Lemma 4, then we immediately obtain

$$\eta |f(a)|^p = \eta |f(\phi_a(0))|^p$$
$$\leq \int_U |f \circ \phi_a|^p \, d\mu^a/\|\mu^a\|$$
$$= \int_{|z| \leq R} |f \circ \phi_a|^p \, d\mu \circ \phi_a/\mu(\tilde{D}(a))$$
$$= \int_{\tilde{D}(a)} |f|^p \, d\mu/\mu(\tilde{D}(a)).$$

If we assume additionally that $\mu(\tilde{D}(a)) \geq cm(D(a))$ we may integrate this inequality over F with respect to $dm(a)$, obtaining

$$\int_F |f|^p \, dm \leq C \int_U |f(z)|^p \int_F [\chi_{D(a)}(z)/m(D(a))] \, dm(a) \, d\mu(z)$$
$$\leq C' \int_U |f|^p \, d\mu$$

where we have used $\int_{D(z)} m(D(a))^{-1} \, dm(a) \leq$ constant. Finally, Lemma 1 gives us

$$\int_U |f|^p \, dm \leq K \int_F |f|^p \, dm \leq KC' \int_U |f|^p \, d\mu.$$

The following theorem reflects all of the above considerations:

THEOREM 1. *Let* $0 < \varepsilon < 1$ *and* $0 < R < 1$. *For* $z \in U$, *define* $D(z)$ *and* $\tilde{D}(z)$ *to be the* ρ-*disks about* z *with radii* ε *and* εR, *respectively. Let* $p > 0$. *There is a constant* $\delta' > 0$, *depending only on* ε, *such that the following holds. Let* μ *be a positive measure on* U, *and* $\mathcal{M}_\mu$ *the family of measures defined in Definition* 1. *If* $\inf\{\mu(\tilde{D}(a))/m(D(a)) \colon a \in U\} > 0$ *and if no weak* limit* *point of $\mathcal{M}_\mu$ is concentrated on a zero set of $\mathcal{G}(\delta')$, then there is a constant C such that for all $f \in A^p$, $\int |f|^p\, dm \leq C \int |f|^p\, d\mu$.*

Proof. Most of the argument was given prior to the statement of the theorem. The δ' in the theorem is $c\delta$ where δ is from Lemma 1 and depends only on ε, and

$$c = \inf\left\{ |\phi_a'(z)|^2 / m(D(a)) \colon z \in U, a \in U \right\}$$

$$= \inf\left\{ \pi \frac{(1 - \varepsilon^2 |a|^2)^2}{|1 - \bar{a}\varepsilon z|^4} \colon z \in U, a \in U \right\}$$

$$= \pi \left(\frac{1 - \varepsilon}{1 + \varepsilon} \right)^2.$$

Thus δ' depends only on ε. □

It will be instructive to consider examples where Theorem 1 may be applied.

Example 1. Let $\mu = \chi_G\, dm$. This is the case solved in [3]. The condition on G which guaranteed $\int |f|^p\, dm \leq C \int |f|^p\, d\mu$ was that $\mu(S) \geq cm(S)$ for all Carleson "squares" S. In [3] this was shown (for this particular form of μ) to be equivalent to $\mu(\tilde{D}(a)) \geq cm(D(a))$ for some value of the radius εR of $\tilde{D}(a)$ and all $a \in U$. Let us see how this implies the relevant condition on $\mathcal{M}_\mu$. It is not hard to see that $\mu^a / \|\mu^a\| = h_a \chi_{G^a}\, dm$ where $G^a = \psi_a(G \cap D(a))$ satisfies $m(G^a) \geq c > 0$ and h_a is bounded above and below. Thus weak* limits cannot be concentrated in a finite number of points and so cannot sit on a zero set of any analytic function. (This argument could be applied almost without change for harmonic functions and is essentially the same as the proof in [4]).

Example 2. Let $\mu = \sum_{a \in E} (1 - |a|)^2 \delta_a$ where δ_a denotes a unit mass at a. Suppose the set E satisfies $\operatorname{card}(E \cap \tilde{D}(a)) > N$ where N is an integer independent of a. (One condition that will guarantee this is that E be an η-lattice in the sense of [1] for small η. Another is that E consist of concentrated clusters of points, each cluster containing N points and the set consisting of one point from each cluster forming an εR-lattice). It is clear that a typical measure $\mu^a / \|\mu^a\|$ in $\mathcal{M}_\mu$ has more than N point masses. If we further suppose that E is separated (i.e., $\inf\{\rho(a, b) \colon a, b \in E, a \neq b\} \neq 0$) then the amount of mass placed by the measure $\mu^a / \|\mu^a\|$ at each point of its support is bounded away from zero. Furthermore, the distances between these masses are bounded from zero. Thus weak* limits consist of at least N point masses. If

$$N > (4/p) \log(1/\delta') / (1 - R^2)^2$$

then, by Lemma 3, $\mathscr{M}_\mu$ satisfies the hypotheses of Theorem 1. Thus E separated plus

$$\operatorname{card}(E \cap \tilde{D}(a)) > (4/p)\log(1/\delta')/(1 - R^2)^2$$

is sufficient for

$$\int |f|^p \, dm \leq C \sum_{a \in E} (1 - |a|)^2 |f(a)|^p, \quad f \in A^p.$$

Another condition sufficient for this inequality is that E be separated and

$$\sup_{z \in U} \inf_{a \in E} \rho(z, a) < \varepsilon(\delta')^{1/p} e^{-1/2}.$$

This comes from consideration of the last part of Lemma 3.

The previous two examples (except for the specific quantitative estimates in Example 2) were known (e.g., see [5], [6], [7]). The following example illustrates the advantage of Theorem 1 over previous methods. It could not be obtained from previous results.

Example 3. We build a set L by including in L the two radii $[0, 1)$ and $[0, -1)$ and the two half radii $[\frac{1}{2}i, i)$ and $[-\frac{1}{2}i, -i)$, and, at the n^{th} stage, the outermost segment of 2^{n-1} radii of length 2^{1-n} which bisect the 2^{n-1} arcs determined by all previous segments of radii. Thus, if $I_n = [1 - 2^{-n+1}, 1)$, then

$$L = \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{2^n} I_n \exp[ik2^{-n+1}\pi].$$

Let $d\mu$ be $(1 - |z|)\, ds$ where ds is arclength on L. A moment's consideration (or tedious calculations) reveals that there are choices for ε and R and a positive number c, such that $\tilde{D}(a)$ will contain a segment from L of length $\geq c(1 - |a|)$, for any choice of $a \in U$. Then $\mu^a/\|\mu^a\|$ will be boundedly absolutely continuous with respect to arclength on a circular arc with length bounded away from zero and with Radon-Nikodym derivative bounded away from zero. Weak* limits will then be non-zero functions times arclength on non-degenerate circular arcs. These clearly cannot be concentrated on zero sets of any $\mathscr{G}(\delta)$. Thus $\int |f|^p \, dm \leq C \int |f|^p \, d\mu$ for this particular μ.

In general, examples can be created at will by putting conditions on $\mu|_{\tilde{D}(a)}$ that prevent μ^a from being concentrated on a zero set and making sure the conditions are uniform in a and preserved under weak* limits of $\mu^a/\|\mu^a\|$.

The additional condition $\mu(\tilde{D}(a)) \geq cm(D(a))$ was shown in [6] to be necessary (for some $\varepsilon R < 1$) provided $\int |f|^p \, d\mu \leq C \int |f|^p \, dm$ was satisfied.

(This is a so-called Carleson inequality, the reverse of that considered by Theorem 1). Thus, if it is assumed that $A^p \subseteq L^p(\mu)$, a portion of the hypotheses of Theorem 1 is necessary for the conclusion and the conclusion amounts to saying that A^p is closed in $L^p(\mu)$.

The arguments in Example 1 apply almost without change to measures of the form $h\,dm$ when h is bounded and positive. The conclusion is that the necessary condition $\int_{D(a)} h\,dm \geq cm(D(a))$ is sufficient. It is easy to show that this is equivalent to the necessary and sufficient condition given in Corollary 1 of [3].

5. Mixed norm spaces: The main result

We assume that $\{\mu_r: 0 < r < 1\}$ is a family of positive finite measures on U. The particular case where μ_r is concentrated on $|z| = r$ is more or less typical of the more general case and may be assumed on first reading. This is the case considered in [2], though there the setting was the unit ball in $\mathbf{C}^N$, the μ_r were always discrete, and only the Bergman and Hardy spaces were studied. The generalization of the present results to $\mathbf{C}^N$ would be routine and is left to the reader. (See [2] for certain special cases.) In this section we study inequalities of the form

$$\|f\|_{p,q,\alpha}^q \leq C\left[\int |f(z)|^p \, d\mu_r(z)\right]^{q/p} d\nu(r),$$

where ν is a measure on $[0,1)$. As in the previous section we need a family of measures on $|z| \leq R$.

Definition 2. Let S be a subset of $[0,1)$. Let $\mathscr{M}(S)$ be the collection of measures formed as follows. Let ψ_a be as in Definition 1. Form all

$$\mu^a = \chi_{\{|z| \leq R\}}\left(\mu_{|a|} \circ \psi_a^{-1}\right)$$

as a varies with $|a| \in S$. Let $\mathscr{M}(S)$ denote the collection of all $\mu^a/\|\mu^a\|$ thus obtained.

We wish to allow for the possibility that many μ_r may be zero. In that case we would take S to consist only of r for which μ_r is not zero. What follows is the main result and is the mixed norm version of Theorem 1.

Theorem 2. *Let $0 < p$, $q < +\infty$ and $\alpha > 0$. Let $\{\mu_r: 0 \leq r < 1\}$ be as above and let ν be a positive measure on* $[0,1)$. *Suppose*

$$\inf_\theta \mu_r\left(\tilde{D}(re^{i\theta})\right) \geq (1-r)\chi_S(r)$$

where S is some subset of $[0,1)$. Define $d\nu_S = \chi_S\, d\nu$ and assume

$$\nu_S([r, r^*)) > c_1 \nu_S([r_*, r)) > c_1 c_2 (1-r)^{q\alpha}$$

for some $c_i > 0$ and all $r \in [0,1)$. There is a $\delta' = \delta'(c_1, p/q, \varepsilon)$ such that if $\mathscr{M}(S)$ and $\mathscr{G}(\delta')$ satisfy the hypotheses of Lemma 4 then

$$\|f\|_{p,q,\alpha}^q \le C \int \left(\int |f|^p \, d\mu_r \right)^{q/p} d\nu(r)$$

for all $f \in H(p, q, \alpha)$, with C independent of f.

Note. The hypothesis

$$\inf_\theta \mu_r(\tilde{D}(re^{i\theta})) \ge (1-r)\chi_S(r)$$

may easily be changed to

$$\inf_\theta \left[\mu_r(\tilde{D}(re^{i\theta}))\right]^{q/p} \ge h(r)\chi_S(r)$$

provided the definition of ν_S is changed to

$$d\nu_S(r) = [h(r)/(1-r)]^{q/p} \chi_S(r)\, d\nu(r).$$

The form stated in the theorem is no loss of generality because we can multiply μ_r by any function of r on any set S and divide ν by the q/p power of that same function on S without altering any of the integrals in the conclusion.

Proof. We have, from Lemma 4, the definition of $\mathscr{M}(S)$ and the argument preceeding Theorem 1, that

$$|f(re^{i\theta})|^p \le C \int_{\tilde{D}(re^{i\theta})} |f|^p \, d\mu_r (1-r)^{-1},$$

provided $r \in S$ and $re^{i\theta} \in F$ where F is as defined in Lemma 1 relative to ν_S. (Note: $\delta = \delta(c_1, p/q, \varepsilon)$ and δ' is obtained from δ just as in §4.) Multiply this inequality by $\chi_F(re^{i\theta})$ to get an inequality holding whenever $r \in S$. Integrate it with respect to θ to obtain

$$\int_0^{2\pi} \chi_F(re^{i\theta}) |f(re^{i\theta})|^p \, d\theta \le C \int_U |f(z)|^p g(z) \, d\mu_r(z) (1-r)^{-1},$$

where $g(z) = \int_0^{2\pi} \chi_{F \cap \tilde{D}(z)}(re^{i\theta})\, d\theta \le C(1-r)$. So, for $r \in S$,

$$\int_0^{2\pi} (\chi_F |f|^p)(re^{i\theta})\, d\theta \le C \int_U |f|^p \, d\mu_r.$$

Raise this to the q/p power and integrate with respect to ν_S. If we observe that the hypotheses on ν_S are just those of Lemma 1 we obtain

$$\int M_p^q(f,r)\, d\nu_S(r) \le C \int \left(\int |f|^p \, d\mu_r \right)^{q/p} d\nu(r).$$

It remains only to observe that

$$\begin{aligned} \int_{r_n}^{r_{n+1}} M_p^q(f,r)\, d\nu_S(r) &\ge \nu_S([r_n, r_{n+1}))\, M_p^q(f, r_n) \\ &\ge c_2 (1 - r_{n+1})^{q\alpha} M_p^q(f, r_n) \\ &\ge c \int_{r_{n-1}}^{r_n} M_p^q(f,r)(1-r)^{q\alpha - 1}\, dr, \quad n \ge 1. \end{aligned}$$

Summing on n gives $c\|f\|_{p,q,\alpha}^q \le \int M_p^q(f,r)\, d\nu_S(r)$, completing the proof. □

Remarks. (1) The choice of δ in Lemma 1 depends only on the constant in the hypothesis on ν, the original ε, and the ratio p/q. This determines the set F and a δ' to define $\mathcal{G}(\delta')$. $\mathcal{M}(S)$ is required to satisfy Lemma 4 for this δ' and some choice of R.

(2) In case $q = +\infty$, $\alpha > 0$ we have the functions of slow mean growth. In that case we use Lemma 2 and obtain

$$\nu_S\text{-ess}\sup (1-r)^\alpha M_p(f,r) \le K\ \nu\text{-ess}\sup \left(\int |f|^p \, d\mu_r \right)^{1/p},$$

provided ν_S satisfies the hypotheses of Lemma 2, namely $\nu_S([r, r^*)) > 0$, all r. It is easy to see that this alone implies

$$\sup_r (1-r)^\alpha M_p(f,r) \le C\ \nu_S\text{-ess}\sup (1-r)^\alpha M^p(f,r).$$

In case $\alpha = 0$ (the Hardy spaces, $H^p = H(p, \infty, 0)$) we require only $\nu_S([r,1)) > 0$, all r. This gives the following.

SCHOLIUM. *Theorem 2 remains valid when $q = +\infty$ if we replace the condition on ν_S with the weaker conditions above and replace the conclusion with*

$$\|f\|_{p,\infty,\alpha} \le C\ \nu\text{-ess}\sup \left(\int |f|^p \, d\mu_r \right)^{1/p}.$$

(3) The case $p = +\infty$ will be left to the reader or a later work. The corresponding result would have a conclusion such as

$$\|f\|^q_{\infty, q, \alpha} \leq C \int (\mu_r\text{-ess sup}|f|)^q \, d\nu(r).$$

The most straightforward theorem could be obtained by replacing L^p norms by L^∞ norms in all definitions, lemmas, and theorems, and little good would be served by playing out the details here.

6. Some examples in H^p

In [7] it was shown that if $r_n \to 1^-$ and if points a_{nk}, $k = 1, 2, \ldots, k_n$ were chosen appropriately (see page 331 of [7]) with $|a_{nk}| = r_n$, then

$$\|f\|^p_{H^p} \leq C \sup_n \sum_{k=1}^{k_n} |f(a_{nk})|^p (1 - r_n).$$

This follows from the $q = +\infty$, $\alpha = 0$ case of Theorem 2. (See the remarks following its proof.) We would take $\mu_r = 0$ if $r \notin \{r_1, r_2, \ldots\}$ and

$$\mu_{r_n} = (1 - r_n) \sum_k \delta_{a_{nk}}.$$

The measure ν could be any measure with positive mass at each of the r_n and S could be $\{r_1, r_2, \ldots\}$. The a_n need to be chosen separated (i.e., $\inf\{\rho(a_{nj}, a_{nk}) \colon j \neq k, n = 1, 2, \ldots\} > 0$) and such that $\tilde{D}(r_n e^{i\theta})$ always contains more than N of the points $\{a_{nk}\}$ where N is as in Lemma 3:

$$N \geq (4/p)\log(1/\delta)/(1 - R^2)^2.$$

Or chosen separated but closer together than $\varepsilon \delta^{1/p} e^{-1/2}$ in the metric ρ. (The $H(p, q, \alpha)$ analogue of this example was observed in [2] to follow from the methods of [7].)

As a second example let $r_n \to 1^-$ again, and let G_n be measurable subsets of $|z| = 1$. Then a sufficient condition that there exist a constant C such that

$$\sup_r \int |f(re^{i\theta})|^p \, d\theta \leq C \sup_n \int_{G_n} |f(r_n e^{i\theta})|^p \, d\theta \tag{6.1}$$

for all $f \in H^p$, is that there exist positive constants B and b such that

$$\text{arclength}(G_n \cap [\theta, \theta + B(1 - r_n)]) > b(1 - r_n)$$

for all n and all θ. This follows from Theorem 2 if we define measures μ_{r_n} supported on $|z| = r_n$ to be arclength on $r_n G_n$. Notice that

$$\mu_{r_n}\left(\tilde{D}\left(r_n e^{i\theta}\right)\right) = \text{arclength}(G_n \cap I_n)$$

where I_n is an interval about θ whose length is proportional to $1 - r_n$. A sequence G_n can be constructed based on the example constructed in [5, page 158], which shows that (6.1) may fail for $f \notin H^p$. There is an $H(p, q, \alpha)$ analogue for this example as well.

Finally, consider the set L constructed in Example 3, Section 4. Let K_n be the annulus with radii $1 - 2^{-n+1}$ and $1 - 2^{-n}$. Let $L_n = K_n \cap L$. Thus L_n consists of 2^n equally spaced segments of radii that lie in A_n. Let $\mu_r = ds$ on L_n if $r = 1 - 2^{-n}$, $\mu_r = 0$ otherwise. Then there is a constant C such that

$$\sup_r \int \left|f\left(re^{i\theta}\right)\right|^p d\theta \leq C \sup \int_{L_n} |f|^p \, ds$$

for all $f \in H^p$. Moreover, there exist constants C' and C'' such that

$$\sup_r (1 - r)^\alpha M_p(f, r) \leq C' \sup_n (1 - r_n)^\alpha \left(\int_{L_n} |f|^p \, ds \right)^{1/p}$$

for all $f \in H(p, \infty, \alpha)$, and

$$\int_0^1 (1 - r)^{q\alpha - 1} M_p^q(f, r) \, dr \leq C'' \sum_n (1 - r_n)^{q\alpha} \left(\int_{L_n} |f|^p \, ds \right)^{q/p}$$

for all $f \in H(p, q, \alpha)$. Any one of these inequalities may fail if the left member of the inequality is infinite (i.e., if f is not in the appropriate $H(p, q, \alpha)$ space).

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