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WEIGHTED SOBOLEV INEQUALITIES AND UNIQUE CONTINUATION FOR THE LAPLACIAN PLUS LOWER ORDER TERMS

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I. Introduction

We are going to consider Sobolev inequalities in L^p-spaces with the weights $e^{\tau\phi(x)}$, where $\phi(x) = x_n + \frac{1}{2}x_n^2$ is the functions used in Hormander [1] to prove unique continuation properties. Here $x \in (x', x_n) \in \mathbf{R}^n$. The first type of inequality concerns to the gradient

$$(1) \quad \left\| e^{\tau\phi(x)} \nabla u \right\|_2 \le c\tau^{\alpha(p_1, n)} \left\| e^{\tau\phi(x)} \Delta u \right\|_{p_1}, \quad \text{uniformly in } \tau \in (\tau_0, \infty),$$

for the Sobolev range $1/p_1 - 1/2 \le 1/n$. The point in this inequality is to control the dependence of the exponent α and constants on the weight parameter τ. These exponents happen to be non-positive for $1/p_1 - 1/2 \le 2/(3n-2)$; hence in this range (1) is a Carleman estimate. The second type of inequality

$$(2) \qquad \left\| e^{\tau\phi(x)} u \right\|_q \le c \left\| e^{\tau\phi(x)} \Delta u \right\|_{p_0}$$

holds for $(1/p_0, 1/q)$ in the open triangle ABC in Figure 1.

Our motivation to study inequality (2) for this range of p's and q's is the following unique continuation result for the Laplacian (corollary), which put together first and zero order perturbations:

Assume $v \in L^r_{\mathrm{loc}}(\mathbf{R}^n)$, $w \in L^s_{\mathrm{loc}}(\mathbf{R}^n)$, $r = (3n-2)/2$, $s > n/2$ and let $u \in H^{2,t}_{\mathrm{loc}}$ for $t = 2(3n-2)/(3n+2)$ be a solution of the inequality

$$(3) \qquad |\Delta u(x)| \le |v(x) \cdot \nabla u(x)| + |w(x) u(x)|.$$

Then if u vanishes in an open non-empty set, it must be zero everywhere.

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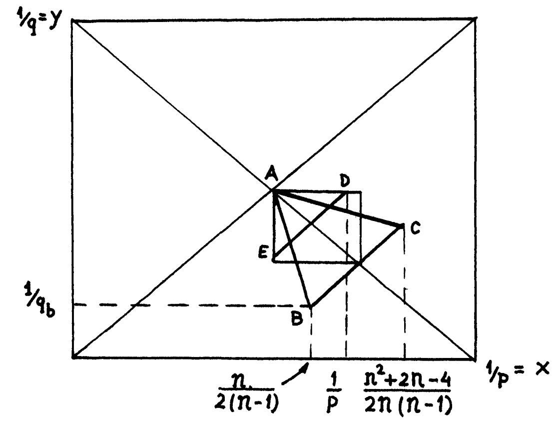

$$D = \left(\frac{3n+2}{6n\text{-}4}, 1/2\right)$$

$$\overline{BC} \subset \left\{x - y = \frac{2}{n}\right\}$$

$$\overline{ED} \subset \left\{x - y = \frac{2}{3n-2}\right\}$$

FIG. 1

This unique continuation result was proved by Hormander [1] for elliptic equations with Lipschitz coefficients and $r > (3n-2)/2$, $s = (4n-2)/7$, so our corollary is an extension for the Laplace operator of this result.

Jerison and Kenig [4] also proved strong unique continuation for the solution of (3) without the gradient term and with $s = n/2$. Jerison [3] gave a new proof of Jerison-Kenig's result using a discrete restriction theorem for the Fourier transform.

Kenig, Ruiz and Sogge [5] obtained the weaker unique continuation for the same range of exponents as a consequence of the uniform Sobolev inequality

$$\|u\|_{L^q} \leq c\|(P(D) + \bar{a} \cdot \nabla + b)u\|_{L^p} \tag{4}$$

which holds for any lower order perturbation with constant coefficients of the second order constant coefficient differential operator $p(D)$; (4) implies Carleman inequality (2) for the weight $e^{\tau x_n}$ and it suffices to obtain the unique continuation property.

One can wonder if (1) holds for the same weight function $\phi(x) = x_n$, that would be a particular case of a uniform Sobolev inequality

$$\|\nabla u\|_2 \leq c\left\|\left(P(D) + \Sigma a_j D_j + b\right)u\right\|_p. \tag{4'}$$

Unfortunately the answer is negative; it can be shown that (1) for $\phi(x) = x_n$ is true uniformly in $\tau \in \{\tau_k\} \to \infty$ only for $p = q = 2$. The counterexamples are similar to those used in Fourier transform restriction theorems.

We approach inequalities (1) and (2) in this direction; we give a reinterpretation of them as uniform Sobolev inequalities similar to (4), but for the one-parameter family of variable coefficients perturbation of Laplace operator

$$|D + i\tau\phi'(x)(0,0,\ldots,1)|^2.$$

In this direction, Theorem 2 shows that for any regular function $\phi(x)$, the best range of p's and q's which gives a uniform inequality

$$\|e^{\tau\phi(x)}\nabla u\|_q \leq c(n,p)\|e^{\tau\phi(x)}\Delta u\|_p$$

must reduce to $1/p - 1/q \leq 2/(3n-2)$. In this sense, inequality (1) is sharp and consequently unique continuation property for solution of

$$|\Delta u(x)| \leq |v(x)\cdot\nabla u(x)|,\ v \in L^r_{\text{loc}},\ r < \frac{3n-2}{2},$$

cannot be obtained by using Carleman's method, the classical tool to prove uniqueness.

Inequality (1) involves the same geometry as Carleman estimates for the Dirac operator in Jerison [3]. In any case, restriction theorem are the corner stone, either Sogge's version or Stein-Tomas' one (see [7] and [10]).

Finally we remark, as can be seen in the proof of (2), that we could state this inequality with c replaced by $c\tau^{\beta(n,p)}$, where β is a negative number which decreases with the difference $1/p_0 - 1/q$ and becomes $-3/2$ for $p_0 = q = 2$.

II. Statement of theorems and consequences

Let $x = (x', x_n) \in \mathbf{R}^n$, $x_n \in \mathbf{R}$; eventually we will write $y = x_n$. Let $H^{2,s}_{\text{loc}}$ denote the space of functions with two derivatives locally in L^s.

We denote by $c(n,p)$ any constant depending only on n or p, which may change at any occurrency.

D_j will be $\partial/i\,\partial x_j$, $D = (D_1, \dots, D_n)$, and $D' = (D_1, \dots, D_{n-1})$.

$\mathscr{S}^l$ is the class of symbols of pseudodifferential operators $p(x,D)$, with estimates

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial \xi^\partial}p(x,\xi)\right| \leq c_{\beta,\alpha}(1+|\xi|)^{l-|\beta|}.$$

$S^{n-1}(t)$ will denote the sphere of radius t in $\mathbf{R}^n$.

THEOREM 1. *Let $\phi(x) = x_n + \frac{1}{2}x_n^2$, $n > 2$, and $\mathscr{U} = \mathbf{R}^{n-1}x[-1/2, 1/2] \subset \mathbf{R}^n$. Then there exist constants $c_i(n, p_i, q)$ and τ_0 such that for $\tau > \tau_0$ and $u \in C_0^\infty(\mathscr{U})$,*

$$\|\exp(\tau\phi(x))\nabla u\|_{L^2(\mathscr{U})} \leq C_1\tau^{\alpha(\eta,\gamma)}\|\exp(\tau\phi(x)\,\Delta u\|_{L^{p_1}(\mathscr{U})} \tag{1}$$

for

$$\gamma = \frac{1}{p_1} - \frac{1}{2} \leq \frac{1}{n},\ \alpha(\eta,\gamma) = \frac{(3n-2)\gamma - 2}{4}$$

and

$$\|\exp(\tau\phi(x))u\|_{L^q(\mathscr{U})} \leq c_2\|\exp(\tau\phi(x))\,\Delta u\|_{L^{p_0}(\mathscr{U})} \tag{2}$$

for $(1/p_0, 1/q)$ *in the open triangle ABC of Figure* 1 *with vertices*

$$A = (1/2, 1/2), \quad B = \left(\frac{n}{2(n-1)}, \frac{1}{q_b}\right), \quad C = \left(\frac{n^2+2n-4}{2n(n-1)}, \frac{1}{q_c}\right)$$

where

$$\frac{n}{2(n-1)} - \frac{1}{q_b} = \frac{n^2+2n-4}{2n(n-1)} - \frac{1}{q_c} = \frac{2}{n}.$$

The proof will be postponed until Section III.

Corollary. *Let* $X \subset \mathbf{R}^n$ *be an open set,* $u \in H^{2,t}_{\text{loc}}(X)$, $t = 6n - 4/(3n + 2)$. *Suppose u satisfies the inequality*

$$|\Delta u(x)| \leq |v(x) \cdot \nabla u(x)| + |w(x) u(x)| \tag{3}$$

where $v = (v_1, \ldots, v_j)$, $v_j \in L^r_{\text{loc}}(\mathbf{R}^n)$ *for* $r = (3n-2)/2$, *and* $w \in L^s_{\text{loc}}(\mathbf{R}^n)$ *for* $s > n/2$. *Then* $u \equiv 0$ *if u vanishes in an open non-empty set contained in* $\overline{X}$.

Proof. Let us write $1/s = 2/n - \varepsilon$. From (1) we obtain

$$\|e^{\tau\phi(x)} \nabla u\|_{L^2(U)} \leq c \|e^{\tau\phi(x)} \Delta u\|_{L^p(U)} \quad \text{for } p = \frac{6n-4}{3n+2} \tag{4}$$

(i.e. $1/p - 1/2 = 2/(3n-2)$).
From (2),

$$\|e^{\tau\phi(x)} u\|_{L^2(U)} \leq c \|e^{\tau\phi(x)} \Delta u\|_{L^p(U)} \tag{5}$$

with the same p and $1/p - 1/q = 2/n - \varepsilon$ (in fact

$$q = \frac{6n-4}{3n^2 - 10n + 8} - O(\varepsilon)$$

in the range of (2)′ in Theorem 1).
As in [5] the corollary is an easy consequence of the following lemma:

Lemma 1. *Let X, U, v w be as in the corollary, such that inequality* (3) *is satisfied in a neighborhood of* S^{n-1}. *Then* $u \equiv 0$ *in a neighborhood of* S^{n-1} *if this is true on one side.*

Proof. We are going to take first the case where u vanishes in an exterior neighborhood of S_1^{n-1}. We may suppose S^{n-1} is centered in $-\mathbf{1} = (0, \ldots, -1)$ and $0 \in X$; it suffices to prove that u is zero in a neighborhood of 0.

Let $\varepsilon > 0$ be small enough such that $u(x) = 0$ for $|(x + \mathbf{1})| > 1$ and $|x| < \varepsilon$, and ϕ is an increasing function for $|x| < \varepsilon$, $\varepsilon < 1/2$. Take $g(x) = \eta(|x|)u(x)$ where $\eta \in C_0^\infty([-\varepsilon, \varepsilon])$ is such that $\eta(x) = 1$ if $s < \varepsilon/2$.

Let us take $S_\rho = \{x; |x| < \rho\}$, $\rho < \varepsilon/2$ to be fixed later.

From (4) and (5) we get

$$\begin{aligned}
&\|e^{\tau\phi(x)}g\|_{L^q(S_\rho)} + \|e^{\tau\phi(x)}\nabla g\|_{L^2(S_\rho)} \\
&\quad\le c\|e^{\tau\phi}\Delta g\|_{L^p(\mathbf{R}^n)} \\
&\quad\le c\|e^{\tau\phi}\Delta g\|_{L^p(\mathbf{R}^n\setminus S_\rho)} + c\|e^{\tau\phi}v\cdot\nabla g\|_{L^p(S_\rho)} + c\|e^{\tau\phi}w\cdot g\|_{L^p(S_\rho)} \\
&\quad\le c\|e^{\tau\phi}\Delta g\|_{L^p(\mathbf{R}^n\setminus S_\rho)} + c\|v\|_{L^r(S_\rho)}\|\nabla g\|_{L^2(S_\rho)} \\
&\qquad + c\|w\|_{L^s(S_\rho)}\|g\|_{L^q(S_\rho)}.
\end{aligned}$$

Since $v \in L^r_{\mathrm{loc}}$, $w \in L^s_{\mathrm{loc}}$ we can choose ρ small enough to get

$$\|v\|_{L^r(s_\rho)} < \frac{1}{2c}, \quad \|w\|_{L^s(s_\rho)} < \frac{1}{2c}.$$

Insert this in the above inequality; the corresponding terms can be absorbed by the left hand side and give

$$\|e^{\tau\phi(x)}g\|_{L^q(S_\rho)} + \|e^{\tau\phi(x)}\nabla g\|_{L^2(S_\rho)} \le 2c\|e^{\tau\phi}\Delta g\|_{L^p(\mathbf{R}^n\setminus S_\rho)}.$$

Since $g \equiv 0$ for $|x + \mathbf{1}| > 1$ there exists a $\rho' < 0$, such that $g(x) \equiv 0$ on $(\mathbf{R}^n \setminus S_\rho) \cap \{x : x_n > \rho'\}$. Then

$$\|e^{\tau\phi(x)}g\|_{L^q(T)} + \|e^{\tau\phi(x)}\nabla g\|_{L^2(T)} \le c\|e^{\tau\phi}\Delta g\|_{L^p(\mathbf{R}^n\setminus S_\rho)}$$

for $T = S_\rho \cap \{x : x_n > \rho'/2\}$.

If $x \in \operatorname{supp} g \setminus S_\rho$, $\phi(x) < \phi(\rho') = \rho' + (\frac{1}{2}\rho')^2$ then

$$\|e^{\tau\phi(x)}g\|_{L^q(T)} + \|e^{\tau\phi(x)}\nabla g\|_{L^2(T)} \le ce^{\tau\phi(\rho')}\|\Delta g\|_{L^p(\mathbf{R}^n)}$$

or

$$\|e^{\tau(\phi(x)-\phi(\rho'))}g\|_{L^q(T)} \le c\|\Delta g\|_{L^p(\mathbf{R}^n)}$$

for $\tau > \tau_0$. Since $\phi(x) = \phi(x_n) \ge \phi(\rho'/2)$ on T, this is possible only in case $g \equiv 0$ on T.

The case when u vanishes in an interior neighborhood is reduced to the above by reflection (see [5]).

The range of r in the first order terms cannot be improved via Carleman estimates, as we can see from:

THEOREM 2. *Let U be an open set in $\mathbf{R}^n$ and ϕ a regular real valued function not identically zero. If*

$$\|e^{\tau\phi}\nabla f\|_{L^q(U)} \leq c\|e^{\tau\phi}\Delta f\|_{L^p(U)}$$

holds for every $f \in C_0^\infty(U)$ uniformly for any $\tau \in \{\tau_k\} \to \infty$, then

$$\frac{1}{p} - \frac{1}{4} \leq \frac{2}{3n-2};$$

in particular if $q = 2$, then $p \geq (6n-4)/(3n+2)$.

Proof. We construct a counterexample as in [3].

We may assume $\nabla\phi(0) = (1, 0, \ldots, 0)$ and $0 \in U$. By writing $g(x) = e^{\tau\phi(x)}f(x)$ we have

$$\left\|\left(\frac{\partial}{\partial x_j} - \tau\frac{\partial\phi}{\partial x_j}\right)g\right\|_p \leq c\|\Delta g + \tau^2|\nabla\phi|^2 g - \tau\Delta\phi g - 2\tau\langle\nabla\phi, \nabla g\rangle\|_q. \tag{6}$$

Take

$$g(x) = e^{i\tau x_2}\phi(\sigma_\tau x)$$

where

$$\sigma_\tau x = \left(\tau^{1/2}x_1, \tau^{1/2}x_2, \tau^{3/4}x'\right) \quad \text{and} \quad \phi = \prod_{i=1}^{n}\psi(x_i), \quad \psi \in C_0^\infty(\mathbf{R}^n).$$

Then

$$\left\|\frac{\partial g}{\partial x_2} - \tau\frac{\partial\phi}{\partial x_2}g\right\|_p = \left\|i\tau g + \tau^{1/2}e^{i\tau x_2}\frac{\partial\phi}{\partial x_2}(\sigma_\tau x) - \tau O(|x|)g\right\|_p$$
$$\geq c\tau^{1-1/p(1+(3/4)(n-2))}$$

for τ big enough. The right hand side of (6) is

$$\Bigg\|\tau e^{i\tau x_2}\left(\sum_1^{n-1}\frac{\partial^2\phi}{\partial x_i^2}(\sigma_\tau x)\right) + \tau^{3/2}e^{i\tau x_2}\left(2i\frac{\partial\phi}{\partial x_2}(\sigma_\tau x) + \sum_{i=3}^{n-1}\frac{\partial^2\phi}{\partial x_i^2}(\sigma_\tau x)\right)$$
$$+\tau^2 O(|x|) - 2\tau g - 2\tau^{3/2}e^{i\tau x_2}\frac{\partial\phi}{\partial x_1}(\sigma_\tau x) - 2\tau O(|x|)\Bigg(i\tau g(x)$$
$$+\tau^{1/2}e^{i\tau x_2}\sum\frac{\partial\phi}{\partial x_i} + \cdots\Bigg)\Bigg\|_q$$

where dots denotes harmless terms. Hence this is bounded above by $c\tau^{3/2-(1+3(n-2)/4)1/q}$ since $|x| < c\tau^{-1/2}$ in support of g. By comparison we prove the claim.

III. Proof of Theorem 1

A change of variable, $u = e^{-\tau\phi(x)}v$, reduces inequalities (1) and (2) to

(7a)
$$\|(D + i(1+y)\tau^N)v\|_{L^2(U)} \le c(p,n)\tau^{\alpha}\| |D + i\tau(1+y)N|^2(v)\|_{L^{p_1}(U)}$$

and

(7b)
$$\|v\|_{L^q(U)} \le c\| |D + i\tau(1+y)N|^2(v)\|_{L^{p_0}(U)}$$

where $N = (0, \ldots, 0, 1) \in \mathbf{R}^n$, and we have the same ranges of p's and q's.

1. We are going to take a left inverse of

$$|D + i\tau(1+y)N|^2 = \sum_{i=1}^{n-1} D_i^2 - \left(\frac{\partial}{\partial y} - (1+y)\tau\right)^2.$$

Observe this operator has constant coefficients with respect to x'-variables and variable coefficients with respect to the last one y. Then it is natural to take the Fourier transform ($\hat{\ }$) with respect to $\mathbf{R}^{n-1}$ variables. We get

$$(|D + i\tau(1+y)N|^2(v))^{\wedge}(y,\xi') = \left[|\xi'|^2 - \left(\frac{\partial}{\partial y} - (1+y)\tau\right)^2\right]\hat{v}(y,\xi')$$

which is a Fourier multiplier in the x'-variable.

Then our aim is to invert the ordinary differential operator with parameters ξ' and τ given by

$$\left(|\xi'| - \frac{\partial}{\partial y} + (1+y)\tau\right)\circ\left(|\xi'| + \frac{\partial}{\partial y} - (1+y)\tau\right).$$

We will take the composition of the left inverses of

$$\Omega_{j,\xi'} = |\xi'| + (-1)^j\left(\frac{\partial}{\partial y} - (1+y)\tau\right), \quad j = 1,2.$$

A left inverse of $\Omega = (d/dz) - z$ (see [3] and [6] for all the claimed properties) is given by the one-variable pseudodifferential operator with symbol

$$b(z,\eta) = \sqrt{2}\left(\int_0^{\infty}\exp(-t^2 - 2t)\,dt\right)\exp\left(-iz\eta - \frac{z^2-\eta^2}{2}\right) - \int_0^{\infty}\exp\left(-\frac{t^2}{2} - t(z - i\eta)\right)dt, \quad z,\eta \in \mathbf{R},$$

which satisfies

$$b(-z,\eta) = -\overline{b(z,\eta)}, \tag{8}$$

$$\left|\frac{\partial^k}{\partial z}\frac{\partial^j}{\partial \eta}b(z,\eta)\right| \le \frac{C_{jk}}{(1+|z+i\eta|)^{i+j+k}}. \tag{9}$$

Our inverses are obtained by the change of variable $z = s(1+y) + (-1)^{j-1}s^{-1}|\xi'|$ with $s^2 = \tau^2$, since

$$s\left(\frac{d}{dz} - z\right) = \frac{d}{dy} - s^2(1+y) + (-1)^{j-1}|\xi'| = (-1)^j\Omega_{j,\xi'}.$$

Thus $\Omega_{j,\xi'}$ has symbol

$$P_{j,s}(y,\eta,|\xi'|) = (-1)^j s^{-1} b\big(s(1+y) + (-1)^j s^{-1}|\xi'|, s^{-1}\eta\big). \tag{10}$$

From (9),

$$\left|\frac{\partial^k}{\partial y^k}\frac{\partial^\alpha}{\partial \xi^\alpha}P_{j,s}\right| \le c_{k,\alpha}\big(s + |s^2(1+y) + (-1)^j|\xi'| + \eta|\big)^{-1-|\alpha|-k} \tag{11}$$

holds for any non-negative integer k, and multiindex $\alpha \in \mathbf{N}^n$. Taking the inverse Fourier transform $\mathbf{R}^{n-1}$ we have

$$v(x',y) = c\int e^{ix'\xi'}\Omega_{2,\xi'}^{-1}\Omega_{1,\xi'}^{-1}\big(|D + i\tau(1+y)N|^2 v\big)^{\wedge}(\xi',y)\,d\xi'$$

and a left inverse of $|D + i\tau(1+y)N|^2$ is given by

$$B_2(y,D)B_1(y,D), \text{ also } B_1(y,D)B_2(y,D),$$

where $B_j(y,D)$ is the pseudodifferential operator with symbol

$$P_{j,s}(y,\eta,|\xi'|) \quad \text{in } \mathbf{R}^{n-1}\times[-3/4,3/4]$$

given by (10).

From (11) we see that $P_{2,s}$ is a classical symbol in the Kohn-Nirenberg class $\mathcal{S}^{-1}(\mathbf{R}_{x'}^{n-1}\times[-3/4,3/4])$.

2. *Proof of (6a).* We want the estimates

$$\|T_j B_2(y,D)B_1(y,D)v\|_{L^2(U)} \le c\tau^\alpha\|v\|_{L^p(U)} \tag{12}$$

where $T_j = D_j$ $j = 1,\dots,n$ and $T_{n+1}v = \tau(1+y)v$.

Take $h \in C_0^\infty(U)$ with $\|h\|_{L^2} = 1$, and $\chi \in C_0^\infty([-3/4,3/4])$ such that $\chi^2(t) \equiv 1$ in $[-1/2,1/2]$.

By duality, (12) is equivalent to

$$I_j \equiv \int_{\mathbf{R}^n} \chi^2(y) T_j B_2(y, D) B_1(y, D) v(x', y) \cdot h(x', y)\, dx'\, dy$$
$$\leq c\tau^\alpha \|v\|_p.$$

Since $P_{2,s} \in \mathcal{S}^{-1}(\mathbf{R}^{n-1} \times [-3/4, 3/4])$, by the classical calculus of pseudo-differential operators $T_j B_2(y, D)$ is in $\mathcal{S}^0(\mathbf{R}^n \times [-3/4, 3/4])$, bounded in $L^p(U)$ for any p, $1 < p < \infty$ (see Taylor [9]), with operator norms independent of $\tau = s^2$. The same is true of their adjoints $(T_j B_2(y, D))^*$.

By the Schwartz inequality

$$I_j \leq c\|\chi(y) B_1(y, D)\|_{L^2}.$$

So we are reduced to proving

$$\|Bv\|_{L^2(U)} \leq c\tau^\alpha \|v\|_{L^p(U)}$$

where $Bv(x)$ is given by

(13)
$$\int_{\mathbf{R}} \chi(y) s^{-1} \int_{\mathbf{R}^{n-1}} b(s(1+y) - s^{-1}|\xi'|, s^{-1}\eta) \hat{u}(\xi', \eta) e^{i\xi' \cdot x'}\, d\xi'\, e^{i\eta y}\, dy$$

and ^ denotes, the $\mathbf{R}^n$-Fourier transform. Now, roughly, for y, η fixed, $|y| \leq 3/4$, we will decompose the above $\mathbf{R}^{n-1}$-Fourier multiplier, and will bound each piece by means of Fourier transform restriction theorems. To do so, we take $\phi \in C_0^\infty([3/4, 2])$ such that $\phi(t) \equiv 1$ in $[1, 3/2]$ and $\sum_{k=1}^\infty \phi(t/2^k) \equiv 1$ for $t \geq 1$. Let

$$\phi_0 = 1 - \sum_{k=0}^\infty \phi\left(\frac{t}{2^k}\right).$$

Then

$$(14) \qquad s^{-1} b_{s,y,\eta} \equiv s^{-1} b(s(1+y) - s^{-1}|\xi'|, s^{-1}\eta)$$
$$= \sum_{k=0}^{L-1} s^{-1} \phi\left(\frac{s^{-1}|s^2(1+y) - |\xi'| + i\eta|}{2^k}\right) b_{s,y,\eta}(\xi')$$
$$+ s^{-1}\phi_L(s^{-2}|s^2(1+y) - |\xi'| + i\eta|) b_{s,y,\eta}(\xi')$$

where $L = \log s - \log 20$ and

$$(15) \qquad \phi_L(t) \equiv 1 \text{ for } t > \frac{1}{10}, \quad \phi_L(t) \equiv 0 \text{ for } t < \frac{1}{20}.$$

Define $P_k(y, \xi', \eta)$ and $P_L(y, \xi', \eta)$ as the terms in (14) and let $B_k(y, \eta, D')$ be the corresponding Fourier multiplier operators in $\mathbf{R}^{n-1}$ with symbols P_k, $k = 0, \ldots, L-1$. Then

$$Bu(x) = \sum_{k=0}^{L-1} \int \chi(y) B_k(y, \eta, D') \hat{u}(x', \eta) e^{i\eta y}\, d\eta + B_L u \tag{16}$$

where $\hat{}$ denotes the y-variable Fourier transform in $\mathbf{R}$.

Observe that for η and y fixed, $P_k(y, \eta, \xi')$ is in $C_0^\infty(\mathbf{R}^{n-1})$ and supported on

$$\{\xi' \in \mathbf{R}^{n-1}, s^{-1}|s^2(1+y) - |\xi'| + i\eta| < 2^k\},$$

i.e., the strip around the sphere $S^{n-1}(s^2(1+y))$ and width $s2^k$. By (11), this multiplier has L^∞-norm bounded by $2^{-k}s^{-1}$. It is natural to use the following Stein-Tomas result (see [10]).

LEMMA 2. *If f is $L^p(\mathbf{R}^{n-1})$ for some p, $1 \le p \le 2n/(n+2)$, then*

$$\int_{S^{n-2}} |\hat{f}(\theta)|^2\, d\theta \le c_p \|f\|_p^2.$$

Then for $v(x') \equiv \hat{u}(x', \eta) = c\int e^{-i\eta y} u(x', y)\, dy$ we have

$$\begin{aligned}
&\|B_k(y, \eta, D')\hat{u})(x', \eta)\|_{L^2(dx')} \\
&\quad = \left(\int_{\mathbf{R}^{n-1}} |P_k(y, \eta, \xi')\hat{v}(\xi')|^2\, d\xi\right)^{1/2} \\
&\quad = \left(\int_{s^2(1+y)-s2^k}^{s^2(1+y)+s2^k} \left(\int_{S_1^{n-2}} |P_k(y, \eta, r\zeta)\hat{v}(r\zeta)|^2 r^{n-2}\, d\theta(\zeta)\right) dr\right)^{1/2} \\
&\quad = \left(\int_{(1+y)-s^{-1}2^k}^{(1+y)+s^{-1}s^k} \left(\int_{S_1^{n-2}} |P_k(y, \eta, s^2t\zeta)\hat{v}(s^2t\zeta)|^2 s^{2(n-1)} t^{n-2}\, d\theta(\zeta)\right) dt\right)^{1/2} \\
&\quad \le c(s^{-3}2^{-k})^{1/2} \left(\int_{S_{1+y}} |\hat{v}(s^2\zeta)|^2 s^{2(n-1)}\, d\theta(\zeta)\right)^{1/2}
\end{aligned}$$

which is bounded by Lemma 2, for $y \in [-1/2, 1/2]$, by

$$c(s^{-3}2^{-k})^{1/2} \left(\int_{\mathbf{R}^{n-1}} |s^{-2(n-1)} v(s^{-2}x')|^P\, dx'\right)^{1/P} s^{n-1}$$

for $p = 2n/(n+2)$. Finally, by dilating we have

$$\|B_k(y, \eta, D')\hat{u}(x', \eta)\|_{L^2(dx')} \le C(s^{-3}2^{-k})^{1/2} s^{(n-1)(2/p-1)} \|v\|_{L^P(dx')}. \tag{17}$$

Using the bounds for derivatives in (11), a similar argument, proves

(18)
$$\left\| \frac{\partial^j}{\partial \eta^j} B_k(y, \eta, D')v \right\|_{L^2(dx')} \le Cs^{-1/2} 2^{k/2} (s2^k)^{-1-j} s^{(n-1)(2/p-1)} \|v\|_{L^p(dx')}.$$

Define

$$K_k(y, z) = \int_{\mathbf{R}} \chi(y) B_k(y, \eta, D') e^{iz\eta}\, d\eta.$$

From (16), Bu is given by a sum of operators:

$$(B_k u)(x', y) = \int_{\mathbf{R}} K_k(y, z - y) v(z, \cdot)(x')\, dz.$$

If we notice that $B_k(\eta, y, D') = 0$ if $|\eta| > Cs2^k$, then integration by parts gives

$$K_k(y, z) = c \int_{\mathbf{R}} \frac{\partial^j}{\partial \eta^j} B_k(\eta, y, D') \frac{1}{(iz)^j} e^{iz\eta}\, d\eta.$$

Then

$$\|K_k(y, z)v\|_{L^2(dx')} \le C_j (2^k sz)^{-j} s^{-1/2} 2^{k/2} s^{(n-1)(2/p-1)} \|v\|_{L^p(dx')}.$$

So for any non-negative integer N,

(19)
$$\|K_k(y, z)v\|_{L^2(dx')} \le C_N (1 + |2^k sz|)^{-N} s^{-1/2} 2^{k/2} s^{(n-1)(2/p-1)} \|v\|_{L^p(dx')}.$$

Interpolation with the obvious estimates

$$\|K_k(y, z)v\|_{L^2(dx')} \le C(2^k sz)^{-j} \|v\|_{L^2(dx')} \tag{20}$$

allows us to claim that

$$\|K_k(y, z)v\|_{L^2(dx')} \le C_N (1 + |2^k sz|)^{-N} (s^{-1/2} 2^{k/2} s^{(n-1)(2/p-1)})^{1-t} \|v\|_{L^{p_1}(dx')}$$

for

$$\frac{1}{p_1} = \frac{t}{2} + (1 - t)\frac{n+2}{2n}, \quad 0 \le t \le 1.$$

The following lemma allows us to obtain bounds in both y and x' variables.

LEMMA 3. *Let $H(y, z-y)$ be a bounded operator from $L^p(\mathbf{R}^{n-1})$ to $L^q(\mathbf{R}^{n-1})$ with operator norm bounded by $h(z-y)$ for each $y, z \in \mathbf{R}$. Suppose $h \in L^r(\mathbf{R})$ for $1/r + 1/p = 1 + 1/q$. Then*

$$Tf(y, x') = \int_{-\infty}^{\infty} H(y, z-y) f(z, \cdot)(x')\, dz$$

satisfies

$$\|Tf\|_{L^q(\mathbf{R}\times\mathbf{R}^{n-1})} \leq \|h\|_{L^r(\mathbf{R})} \|f\|_{L^p(\mathbf{R}^n)}.$$

The proof is an application of Minkowski's and Young's inequalities. Lemma 3 gives our case

$$\|B_k u\|_{L^2(U)} \leq (2^k s)^{-1/r} \big[s^{-1/2+(n-1)(2/p-1)} 2^{k/2}\big]^{1-t} \|u\|_{L^{p_1}(U)},$$
$$\frac{1}{r} + \frac{1}{p_1} - 1 = \frac{1}{2} \quad \text{and} \quad p = \frac{2n}{n+2}.$$

Hence the sum in (16) has L^2-norm bounded by

$$\sum_{k=0}^{L-1} (2^k s)^{1/p_1 - 3/2} \big(s^{-1/2+2(n-1)/n} 2^{k/2}\big)^{1-t} \|u\|_{L^{p_1}(U)},$$
$$\frac{1}{p_1} = \frac{t}{2} + (1-t)\frac{n+2}{2n},$$

which converges for all the range of t, $0 \leq t \leq 1$, $n > 2$, and is bounded by

$$C s^{((3n-2)\gamma-2)/2} \|u\|_{L^{p_1}(U)} \quad \text{for} \quad 0 \leq \gamma = \frac{1}{p_1} - \frac{1}{2} \leq \frac{1}{n}, \; s = \tau.$$

Only the B_L term in (16) remains to be bounded; it has symbol

$$\phi_L\big(s^{-2}|s^2(1+y) - |\xi'| + i\eta|\big) s^{-1} b\big(s(1+y) - s^{-1}|\xi'|, s^{-1}\eta\big) = P_L$$

which, by (11) and (15), satisfies the following estimates, with $C_{\alpha,j}$ independent of s:

$$\left| \frac{\partial^j}{\partial \eta^j} \frac{\partial^\alpha}{\partial \eta^\alpha} P_L \right| \leq \frac{C_{\alpha,j}}{(s + |\xi'| + |\eta|)^{j+|\alpha|+1}}.$$

Hence it behaves like the corresponding fractional integral, and is bounded $L^{p_1} \to L^2$ for $1/p_1 - 1/2 \leq 1/n$.

3. *Proof of (7b).* As for (7a) our first aim is to get rid of $B_2(y, D)$ in the inequality

$$\|B_1(y, D)\cdot B_2(y, D)v\|_{L^q(U)} \le C\|v\|_{L^p(U)}. \tag{17}$$

Take $(1-\Delta)^{-1/2}$ the pseudodifferential operator with symbol

$$\psi(\xi) = \left(1 + |\xi'|^2 + |\eta|^2\right)^{-1/2},$$

and also consider its inverse $(1-\Delta)^{1/2}$ whose principal symbol is

$$\left(1 + |\xi'|^2 + |\eta|^2\right)^{1/2};$$

then we write the left hand side of (17) as

$$\begin{aligned}&\left\|B_1(y, D)(1-\Delta)^{-1/2}(1-\Delta)^{1/2}B_2(y, D)v\right\|_{L^q(U)}\\ &\quad = \left\|B_1(y, D)\cdot(1-\Delta)^{-1/2}u\right\|_{L^q(U)} \quad \text{for} \quad u = (1-\Delta)^{1/2}B_2(y, D)v.\end{aligned}$$

Since we expect $(1-\Delta)^{1/2}B_2(y, D)$ to be bounded from $L^p(U)$ to $L^p(\mathbf{R}^n)$, we are going to bound the operator $B_1(1-\Delta)^{-1/2}$ which has the advantage of being a composition of a Fourier multiplier in $\mathbf{R}^n$ and a pseudodifferential operator. Hence following the line of the proof of (7a), we obtain a decomposition of the symbol given by

$$\begin{aligned}(14\text{b})\quad & s^{-1}b_{s,y,\eta}(\xi')\psi(\xi)\\ &= \sum_{k=0}^{L-1} s^{-1}\phi\left(\frac{s^{-1}|s^2(1+y) - |\xi'| + i\eta|}{2^k}\right)b_{s,y,\eta}(\xi')\psi(\xi)\\ &\quad +\phi_L\cdot b_{s,y,\eta}\cdot\psi(\xi)\cdot\\ &= \sum_{k=0}^{L-1} q_k(y,\eta,\xi') + q_L(y,\eta,\xi').\end{aligned}$$

The supports of q_k are the same as the support of q_k in part 3, but the L^∞-norms of q_k are bounded by $2^{-k}S^{-3}$, since $|\psi(\xi)| < 4s^{-2}$ for $(\xi', \eta) = \xi$ in the support of ϕ_k.

Let denote $Q_k(y, \eta, D')$, $k = 0, \ldots, L-1$, the corresponding $\mathbf{R}^{n-1}$-Fourier multipliers. By taking the dilation $u(x') = f(s^2x')$, we obtain a $\mathbf{R}^{n-1}$-Fourier multiplier with symbol supported in a strip of width $s^{-1}2^k$ around the sphere of radius $1 + y$. Now, as above, we are going to use a restriction theorem, in particular, Sogge's version:

LEMMA 4.

$$\left(\int_{\mathbf{R}^{n-1}}\left|\int_{S_1^{n-1}} \hat{f}(\xi')e^{ix'\cdot\xi'}\,d\theta(\xi')\right|^q dx'\right)^{1/q} \le c\|f\|_{L^p(dx')}$$

for $\delta > 0$ *and*

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{n} + \delta$$

and

$$\frac{2n(n-1)}{n^2+2n-4} < \bar{p} < \frac{2(n-1)}{n}, \quad f \in \mathcal{S}.$$

(See [7]; actually it is obtained from Corollary 5.1 in [7] using duality and interpolation.)

In the dilated variable we have

$$\|\tilde{Q}_k(y,\eta,D')u(s^{-2}x')\|_{L^{\tilde{q}}(dx')} \le s^{-4}\|u(s^{-2}x')\|_{L^p},$$

and recovering the old variable we have

$$\|Q_k(y,\eta,D')u\|_{L^q(dx')} \le cs^{-4+2(n-1)(1/p-1/q)}\|u\|_{L^p(dx')}.$$

From estimates (11) we can again prove that

(18b)

$$\left\|\frac{\partial^j}{\partial\eta^j}Q_k(y,\eta,D')u\right\|_{L^q(\mathbf{R}^{n-1})} \le (2^k s)^{-1-j}s^{-3}2^k s^{2(n-1)(1/p-1/q)}\|u\|_{L^p}$$

Repetition of above arguments and using Lemma 3 again we have

$$\|Q_k u\|_{L^q(\mathbf{R}^n)} \le C_j(2^k s)^{-1/r}s^{-3+2(n-1)(1/p-1/q)}2^k\|u\|_{L^p(U)}$$

for $1/q = 1/r + 1/p - 1$. Hence

$$\left\|\sum_{k=0}^{L-1} Q_k(y,D)u\right\|_{L^q(\mathbf{R}^n)} \le C\left(\sum_{k=0}^{\log s-\log 20} 2^{1/p-1/q}\right)s^{-4+(1/p-1/q)(2n-1)}\|U\|_{L^p(U)}$$
$$\le Cs^{2n\delta}\|u\|_{L^p}$$

since $1/p - 1/q = 2/n + \delta$. Now take δ small enough and use interpolation with the obvious estimate coming from

$$\|\tilde{K}_k(y, z)v\|_{L^2(dx')} \leq CS^{-2}(2^k sz)^{-j}\|v\|_{L^2(dx')} \tag{20b}$$

which is

$$\left\|\sum_{k=0}^{L-1} Q_k(y, D)u\right\|_{L^2(\mathbf{R}^n)} \leq \sum (2^k s)^{-1} s^{-2}\|u\|_{L^2(\mathbf{R}^n)}.$$

In this way we can gain some power of s which gives the desired estimates for $1/\bar{p} - 1/\bar{q} = 2/n - \varepsilon$. This is the claim of the theorem.

The remainder can be bounded again for the corresponding fractional integral, which is bounded from $L^p \to L^q$, $1/\bar{p} - 1/\bar{q} \leq 2/n$.

Finally we have a comment to convince the reader that $(1 - \Delta)^{1/2} \cdot B_2(y, D)$ is bounded from $L^p(U) \to L^p(\mathbf{R}^n)$, $1 < p < \infty$.

$B_2(y, D)$ is a classical pseudodifferential operator with symbol in

$$\mathscr{S}^{-1}(\mathbf{R}^{n-1} \times [-3/4, 3/4]);$$

it is a multiplier in the $\mathbf{R}^{n-1}$ variable so its composition with $(1 - \Delta)^{1/2}$ has a symbol in $\mathscr{S}^0(\mathbf{R}^{n-1} \times [-3/4, 3/4])$ which also is a multiplier in the non-compact variable x'. Then it must be bounded from $L^p(U)$ to $L^p(\mathbf{R}^n)$ since $U = \mathbf{R}^{n-1} \times [-1/2, 1/2]$. (We refer to Taylor [9].)

IV. Further comments and open questions

(a) We obtain our Sobolev inequalities by taking an exact inverse of the perturbated operators $|D + i\tau(1 + y)N|^2$. This is one of the key ingredients in the proof, and one of the obstacles to generalize the theorem to variable Lipschitz coefficients as in Hormander [1].

(b) Are unique continuation properties also true for worse potentials v and w? As we can see, Carleman inequalities are false outside of $r \geq (3n - 2)/2$, $s \geq n/2$, but we do not know about unique continuation; the counterexamples, as far as we know, are for the stronger unique continuation property, that makes identically zero solutions which are zero at order infinity in a point (see [4]).

(c) Inequality (1) is false for weights $\phi(x) = x_n$. Nevertheless we obtain some range for the convex function $\phi(x) = x_n + x_n^2/2$; this is related to uniform Sobolev inequalities as in [5]. For what lower order perturbations $\Sigma a_j(x)D_j + b(x)$ of the Laplacian does inequality (4′) hold? For this and related topics see Hormander [2] and Strömberg [8].

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