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ON GENERALIZED ZETA-FUNCTIONS AT NEGATIVE INTEGERS

BY

HELMUT MÜLLER

1. In analytic number theory the Riemann zeta function $\zeta(s)$, which is defined for complex s with $R(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{1}$$

plays an important role. It is well known that $\zeta(s)$ represents an analytic function of s in the whole complex plane except for a simple pole at $s = 1$ with residue 1. By Euler's identity, for $R(s) > 1$, $\zeta(s)$ can also be defined by an absolutely convergent infinite product, namely

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1} \tag{2}$$

where the product is taken over all primes p. Formula (2) is the basis for many analytical methods to get informations about the distribution of primes.

Zeta functions were also considered in a more general context, in particular, the zeta functions in the theory of Beurling numbers [1], [3], [9]. Here the primes p in (2) are replaced by certain real numbers. The corresponding "zeta function" was studied especially as an analogue to the Riemann zeta function [6], [10], [11]. Another kind of generalization was considered in [2].

Without using any information about the multiplicative structure of the integers, in his thesis [7] J.H. Hawkins studied zeta-functions $z(\nu, s)$ where the integers n in (1) are replaced by real numbers $\lambda_n = \lambda_n(\nu)$; hence

$$z(\nu, s) = \sum_{n=1}^{\infty} \lambda_n^{-s}, \quad s \in \mathbf{C},\ R(s) > 1. \tag{3}$$

Here λ_n is the n-th positive zero of the Bessel function J_ν of the first kind of order ν, $\nu > -1$. Since $\lambda_n(\frac{1}{2}) = \pi \cdot n$, $n = 1, 2, 3 \ldots$, the case of the Riemann

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zeta function is included. It is easy to see that (3) converges uniformly on compact subsets in $R(s) > 1$, so $z(\nu, s)$ is analytic in that half plane. It has an analytical continuation into the whole complex plane except for simple poles at $s = 1, -1, -3, -5, \ldots$. The corresponding residues can be computed recursively. More recently F.T. Howard [8] has studied some arithmetical properties of $z(\nu, 2n)$ for several ν, $n \in \mathbf{N}$. In [5], E. Elizalde gave asymptotic expansions for similar generalized zeta functions which turn out to be of interest in theoretical physics.

In this note we take up the considerations of Hawkins again. But the only condition which the numbers λ_n in (3) now have to satisfy is an asymptotic behavior for large n instead of the condition being zeros of a solution of a certain differential equation (like the Bessel function). Therefore, the conclusions become a bit more complicated, but the results are more general, of course. The following theorem will be proved:

THEOREM. *Let $P(x)$ be a real function defined for $x \geq 1$ with an asymptotic expansion*

$$P(x) \sim \sum_{k=-1}^{\infty} A_k x^{-k}, \qquad A_k \in \mathbf{R}, \tag{4}$$

with $A := A_{-1} > 0$. Let $\lambda_n := P(n)$ for $n = 1, 2, 3, \ldots$. If

$$\zeta_P(s) := \sum_{n=1}^{\infty} \lambda_n^{-s}, \quad s \in \mathbf{C},\ R(s) > 1, \tag{5}$$

then $\zeta_P(s)$ represents an analytic function in the whole complex plane except for a simple pole at $s = 1$ (with residue A^{-1}) and perhaps simple poles at $s = -1, -2, -3, \ldots$. The corresponding residues can be computed recursively in terms of the coefficients A_k.

Note: (4) means that for every $N = 0, 1, 2, \ldots$,

$$P(x) - \sum_{k=-1}^{N} A_k x^{-k} = O(x^{-N-1}), \quad x \to \infty.$$

Examples. (1) The Riemann zeta function $\zeta(s)$ is included by setting $A = 1$, $A_k = 0$ for $k = 0, 1, 2, \ldots$ and equality in (4).

(2) With $A = 1$, $A_0 = a$, $-1 < a \leq 0$, $A_k = 0$ for $k = 1, 2, 3, \ldots$ the Hurwitz zeta function $\zeta(s, 1 - a)$ is also included.

(3) Let χ be a Dirichlet character modulo m and $L(s, \chi) := \sum_{k=1}^{\infty} \chi(n) n^{-s}$ the Dirichlet series. As $L(s, \chi) = m^{-s} \sum_{k=1}^{m} \chi(k) \zeta(s, k/m)$ this case is covered by Example 2.

(4) Let $\nu > -1$ and $P(x) = P_\nu(x)$ defined by $P(0) = 0$ and

$$J_\nu(P(x))\cos(\pi x) + Y_\nu(P(x))\sin(\pi x) = 0,$$

where Y_ν is the Bessel function of second kind. Then the zeros of J_ν are given by $j_{\nu,n} = P(n)$, $n = 1, 2, 3, \ldots$. By McMahon's expansion we have (4) with $A = \pi$, $A_0 = 0$, $A_1 = (1 - 4\nu^2)/8\pi$, $A_2 = 0$,

$$A_3 = \frac{(1 - 4\nu^2)(28\nu^2 - 31)}{384\pi^3}, \quad A_4 = 0,$$
$$A_5 = \frac{(1 - 4\nu^2)(1328\nu^4 - 3928\nu^2 + 3779)}{15360\pi^5}, \ldots$$

(see [1, p. 115] or [12, p. 506]). As the zeta function $z(\nu, s)$, which was studied by Hawkins, is built like this $\zeta_P(s)$ we obtain the same analytical behavior, of course.

2. The proof of the theorem is divided into two steps: in step I we fix an integer $N \geq 1$ and consider the case

$$\lambda_n = An + A_0 + A_1 n^{-1} + \cdots + A_N n^{-N}.$$

In step II we return to the general case

$$\lambda_n = An + A_0 + A_1 n^{-1} + \cdots + A_N n^{-N} + R_N \quad \text{with} \quad R_N = O(n^{-N-1}).$$

Before we start the proof we introduce some simplifications. Because we are interested only in the analytic behavior of $\zeta_P(s)$ and because λ_n^{-s} is an entire function of s we can omit finitely many terms in (5). As λ_n diverges with n we can assume that

$$\lambda_n > \lambda_1 > 0 \tag{6}$$

holds for $n = 2, 3, 4, \ldots$. For $t > 0$ let

$$f(t) := \sum_{n=1}^{\infty} \exp(-\lambda_n t). \tag{7}$$

With (6) we get

$$\begin{aligned} f(t) &= \exp(-\lambda_1 t)(1 + \exp(-(\lambda_2 - \lambda_1)t) + \exp(-(\lambda_3 - \lambda_1)t) + \cdots) \\ &= O(\exp(-\lambda_1 t) \quad \text{for} \quad t \to \infty. \end{aligned}$$

This means that the series in (7) is absolutely convergent for $t > 0$ and even uniformly convergent in $0 < t_0 \leq t \leq t_1$.

Step I. Following the classical methods of Riemann and D.B. Zagier [12] we have to study the asymptotic behavior of $f(t)$ for $t \to 0+$. With

$$\lambda_n = An + A_0 + A_1 n^{-1} + \cdots + A_N n^{-N}$$

we obtain

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \exp\left(-t\left(An + A_0 + A_1 n^{-1} + \cdots + A_N n^{-N}\right)\right) \\ &= \exp(-tA_0) g(t). \end{aligned}$$

As $\exp(-tA_0)$ is analytic in $t = 0$ it is sufficient to look at

$$g(t) = \sum_{n=1}^{\infty} \exp\left(-t\left(An + A_1 n^{-1} + \cdots + A_N n^{-N}\right)\right).$$

Hence we are led to sums like

$$F_k(t) := \sum_{n=1}^{\infty} n^{-k} \exp(-tAn), \quad k \in \mathbf{N},\ t > 0. \tag{8}$$

Here we need the following:

LEMMA. *Let $t > 0$, $A > 0$, $k \in \mathbf{N}$ and $F_k(t)$ defined by* (8). *Then for $t \to 0+$*

$$F_k(t) = (-1)^k \frac{(At)^{k-1}}{(k-1)!} \log t + P_{(N-1)}(k; t) + O(t^N)$$

where $P_{(N-1)}(k; t)$ is a polynomial in t with a degree less than N.

Proof. We use induction on k. For $k = 1$ we have

$$F_1(t) = \sum_{n=1}^{\infty} n^{-1} \exp(-tAn) = -\log(1 - \exp(-At) \quad \text{(p.v.)}$$

and therefore

$$F_1(t) + \log t = -\log \frac{1 - \exp(-At)}{t}.$$

As $(1 - \exp(-At))/t$ is analytic in $t = 0$ and doesn't vanish for $|t| < 2\pi/A$, $F_1(t) + \log t$ is analytic in this domain, too; i.e., the assertion is true for

$k = 1$. If $k > 1$ and if the lemma is true for $k-1$ then $F_k'(t) = -AF_{k-1}(t)$ implies that for fixed $t_0 > 0$,

$$F_k(t) = A\int_t^{t_0} F_{k-1}(x)\,dx + F_k(t_0)$$
$$= A\int_t^{t_0}\left\{(-1)^{k-1}\frac{(Ax)^{k-2}}{(k-2)!}\log x + P_{(N-1)}(k-1; x) + O(x^N)\right\}dx$$
$$+ F_k(t_0).$$

After computing the integrals and using the estimation $t^m = O(t^N)$ for $m \geq N$, $t \to 0+$ we get the assertion for k.

Then we obtain

$$g(t) = \sum_{n=1}^{\infty} \exp(-tAn)\exp\left(-t\left(A_1 n^{-1} + \cdots + A_N n^{-N}\right)\right)$$
$$= \sum_{n=1}^{\infty} \exp(-tAn) \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \sum_{v_1 + \cdots v_N = k}$$
$$\times \frac{k!}{v_1! \cdots v_N!}\left(A_1 n^{-1}\right)^{v_1} \cdots \left(A_N n^{-N}\right)^{v_N}$$
$$= \sum_{n=1}^{\infty} \exp(-tAn) + \sum_{k=1}^{\infty} (-1)^k \sum_{v_1 + \cdots v_N = k} \frac{A_1^{v_1} \cdots A_N^{v_N}}{v_1! \cdots v_N!} F_{v_1 + 2v_2 + \cdots + Nv_N}$$

Since $\sum_{n=1}^{\infty}\exp(-tAn) - (At)^{-1}$ is analytic at $t = 0$ there exist polynomials $p_{(n-1)}(k; t)$ of degree $\leq N-1$ with

$$g(t) = (At)^{-1} + \sum_{k=1}^{\infty} (-t)^k \sum_{v_1 + \cdots + v_N = k} \frac{A_1^{v_1} \cdots A_N^{v_N}}{v_1! \cdots v_N!} \tag{9}$$
$$\times\left\{p_{(n-1)}(k; t) + (-1)^m \frac{(At)^{m-1}}{(m-1)!}\log t + O(t^N)\right\},$$
$$t \to 0+,$$

where $m = v_1 + 2v_2 + \cdots + Nv_N$. Finally we expand $\exp(-A_0 t)$ into its series. Multiplying term by term we get

$$f(t) = \sum_{n=-1}^{N-2} B_n t^n + \log t \sum_{n=1}^{N-1} C_n t^n + O(t^N), \quad t \to 0+ \tag{10}$$

with certain well defined real numbers B_n, C_n. For $s \in \mathbf{C}$, $R(s) > 1$, using the

Mellin-transform we obtain

$$\Gamma(s)\zeta_P(s) = \int_0^\infty f(t)t^{s-1}\,dt = \left(\int_0^1 + \int_1^\infty\right) f(t)t^{s-1}\,dt$$
$$= I_1 + I_2.$$

By (6) we know that I_2 converges for all s, even absolutely and uniformly on compact sets; i.e., I_2 represents an entire function on s. With

$$\int_0^1 \left(\sum_{n=-1}^{N-2} B_n t^n\right) t^{s-1}\,dt = \sum_{n=-1}^{N-2} \frac{B_n}{n+s}$$

and

$$\int_0^1 \left(\log t \sum_{n=1}^{N-1} C_n t^n\right) t^{s-1}\,dt = -\sum_{n=1}^{N-1} \frac{C_n}{(n+s)^2}$$

it follows that

$$I_1 = I_1(s) = \sum_{n=-1}^{N-2} \frac{B_n}{n+s} - \sum_{n=1}^{N-1} \frac{C_n}{(n+s)^2}$$
$$+ \int_0^1 \left\{ f(t) - \sum_{n=-1}^{N-2} B_n t^n - \log t \sum_{n=1}^{N-1} C_n t^n \right\} t^{s-1}\,dt,$$

where the integral converges for all complex s with $R(s) > -N$ (see (10)) absolutely and uniformly on compact sets. Hence the function

$$\Gamma(s)\zeta_P(s) - \sum_{n=-1}^{N-2} \frac{B_n}{n+s} + \sum_{n=1}^{N-1} \frac{C_n}{(n+s)^2}$$

has an analytical continuation into $R(s) > -N$. We see that $\Gamma(s)\zeta_P(s)$ is analytic in this complex half plane except for simple poles at $s = 1$ and $s = 0$ with residue B_{-1} (resp. B_0) and double poles at $s = -1, -2, \ldots, -N+1$. It is well known that $\Gamma(s)$ has simple poles at $s = -n$, $n = 0, 1, 2, \ldots$ (with residue $(-1)^n/n!$). Therefore $\zeta_P(s)$ has at most simple poles at $s = 1, -1, -2, \ldots, -N+1$ where a pole at a negative integer only occurs when the coefficient C_n (which can be computed recursively by comparing (9) and (10)) doesn't vanish.

This proves the theorem in the case where $N \geq 1$ is fixed and

$$R_N := P(n) - \sum_{k=-1}^{N} A_k n^{-k} = 0.$$

Step II. Let $\mu_n := P(n) = \Sigma_{k=-1}^{N} A_k n^{-k} + R_N = \lambda_n + R_N$ where $R_N = R_N(n) = O(n^{-N-1})$, $n \to \infty$, and the involved constant may depend only on N. In step I we proved that $\zeta_P(s) = \Sigma_{n=1}^{\infty} \lambda_n^{-s}$ is analytic in $R(s) > -N$ except for at most simple poles at $s = 1, -1, -2, \ldots, -N+1$. Considering

$$d(s) := \zeta_P(s) - \sum_{n=1}^{\infty} \mu_n^{-s} = -\sum_{n=1}^{\infty} (\lambda_n^s - \mu_n^s)(\lambda_n \mu_n)^{-s}$$

with $\lambda_n, \mu_n \sim An$ for $n \to \infty$ and $R_N = O(n^{-N-1})$, we get

$$(\lambda_n^s - \mu_n^s)(\lambda_n \mu_n)^{-s} = O(n^{-N-1-R(s)}), \quad n \to \infty.$$

Hence the series of $d(s)$ converges uniformly on compact subsets contained in the half plane $R(s) > -N$; i.e., $d(s)$ represents in this region an analytic function and therefore $\zeta_P(s)$ has the same singularities as $\zeta_P(s) - d(s)$.

As N can be chosen arbitrarily large the theorem is proved.

3. Finally we give the values of the first residues of $\zeta_P(s)$ where we assume for simplicity $A_0 = 0$.

n	C_n	$\mathrm{Res}_{s=-n}\zeta_P(s)$
1	A_1	A_1
2	$-A_2A$	$2A_2A$
3	$A_3\frac{A^2}{2} + A_1^2\frac{A}{2}$	$3A_3A^2 + 3A_1^2A$
4	$-A_4\frac{A^3}{6} - A_1A_2\frac{A^2}{2}$	$4A_4A^3 + 12A_1A_2A^2$
5	$A_5\frac{A^4}{24} + A_2^2\frac{A^3}{12} + A_1A_3\frac{A^3}{6}$ $+A_1^3\frac{A^2}{12}$	$5A_5A^4 + 10A_2^2A^3 + 20A_1A_3A^3$ $+10A_1^3A^2$

If one replaces A_k by the coefficients of McMahon's expansion (see Example 4)) the same values appear as those determined by Hawkins.

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MATHEMATISCHES SEMINAR DER UNIVERSITÄT HAMBURG,
BUNDESSTRAßE 55, D-2000 HAMBURG 13, BRD.

UNIVERSITÄT HAMBURG
HAMBURG, WEST GERMANY