ILLINOIS JOURNAL OF MATHEMATICS
Volume 32, Number 3, Fall 1988

MINIMAL ASYMPTOTIC BASES WITH PRESCRIBED DENSITIES

BY

PAUL ERDŐS AND MELVYN B. NATHANSON[1]

Dedicated to the memory of Irving Reiner

Let $h \geq 2$. The set A of integers is an *asymptotic basis of order h* if every sufficiently large integer can be represented as the sum of h elements of A. If A is an asymptotic basis of order h such that no proper subset of A is an asymptotic basis of order h, then the asymptotic basis A is *minimal*. It follows that if A is minimal, then for every element $a \in A$ there must be infinitely many positive integers n, each of whose representations as a sum of h elements of A includes the number a as a summand. Stöhr [6] introduced the concept of minimal asymptotic basis, and Härtter [2] proved that minimal asymptotic bases of order h exist for all $h \geq 2$. Erdös and Nathanson [1] have reviewed recent progress in the study of minimal asymptotic bases.

For any set A of integers, the *counting function* of A, denoted $A(x)$, is defined by $A(x) = \text{card}(\{a \in A \mid 1 \leq a \leq x\})$. If A is an asymptotic basis of order h, then $A(x) > c_1 x^{1/h}$ for some constant $c_1 > 0$ and all x sufficiently large. For every $h \geq 2$, Nathanson [3], [4] has constructed minimal asymptotic bases that are "thin" in the sense that $A(x) < c_2 x^{1/h}$ for some $c_2 > 0$ and all x sufficiently large.

Let A be a set of integers. The *lower asymptotic density* of A, denoted $d_L(A)$, is defined by $d_L(A) = \liminf_{x \to \infty} A(x)/x$. If $\alpha = \lim_{x \to \infty} A(x)/x$ exists, then α is called the *asymptotic density* of A, and denoted $d(A)$. Nathanson and Sárközy [5] proved that if A is a minimal asymptotic basis of order h, then $d_L(A) \leq 1/h$. In this paper we construct for each $h \geq 2$ a class of minimal asymptotic bases A of order h with $d(A) = 1/h$. This result is best possible in the sense that it gives the "fattest" examples of minimal asymptotic bases. We also prove that for every $\alpha \in (0, 1/(2h-2))$ there exists a minimal asymptotic basis A of order h with $d(A) = \alpha$.

Received January 4, 1988.

[1]The research of the second author was supported in part by a grant from the PSC-CUNY Research Award Program of the City University of New York.

DEFINITIONS. Let $\mathbf{N}$ denote the set of nonnegative integers. Let A be a subset of $\mathbf{N}$. The *h-fold sumset hA* is the set of all integers of the form $a_1 + a_2 + \cdots + a_h$, where $a_i \in A$ for $i = 1, 2, \ldots, h$. Let

$$n = a_1 + \cdots + a_h = a'_1 + \cdots + a'_h$$

be two representations of n as a sum of h elements of A. These representations are *disjoint* if $a_i \neq a'_j$ for all $i, j = 1, \ldots, h$.

The set B of nonnegative integers is a *B_k-sequence* if it satisfies the following property: If $u_i, v_i \in B$ for $i = 1, \ldots, k$ with $u_1 \leq \cdots \leq u_k$ and $v_1 \leq \cdots \leq v_k$, and if $u_1 + \cdots + u_k = v_1 + \cdots + v_k$, then $u_i = v_i$ for $i = 1, \ldots, k$. If B is a B_k-sequence, then B is also a B_j-sequence for every $j < k$.

Let $|S| = \text{card}(S)$ denote the cardinality of the set S. Let $\{x\}$ denote the fractional part of the real number x.

LEMMA. *Let $k \geq 2$, and let $B = \{b_i\}_{i=1}^{\infty}$ satisfy $b_1 > 0$ and $b_{i+1} > k \cdot b_i$ for all $i \geq 1$. Then*:

(0.1) *B is a B_k-sequence.*

(0.2) *$B(x) = O(\log x)$.*

(0.3) *If $\delta \in (0,1)$ and $k^{-t} \leq \delta$, then $B(x) \leq B(\delta x) + t$ for all $x \geq 0$. In particular, $B(x) \leq B(x/k) + 1$.*

Proof. Let $u_i, v_i \in B$ for $i = 1, \ldots, j$, where $j \leq k$, $u_1 \leq \cdots \leq u_j$, and $v_1 \leq \cdots \leq v_j$. Suppose that

$$u_1 + \cdots + u_j = v_1 + \cdots + v_j.$$

Let $v_j = \max\{u_j, v_j\}$. If $u_j < v_j$, then

$$u_1 + \cdots + u_j \leq j \cdot u_j \leq k \cdot u_j < v_j \leq v_1 + \cdots + v_j,$$

which is absurd. Therefore, $u_j = v_j$, and so

$$u_1 + \cdots + u_{j-1} = v_1 + \cdots + v_{j-1}.$$

It follows that $u_i = v_i$ for $i = 1, \ldots, j$. In particular, B is a B_k-sequence. This proves (0.1).

Note that $b_j > k \cdot b_{j-1} > k^2 \cdot b_{j-2} > \cdots > k^{j-1} \cdot b_1 = c \cdot k^j$, where $c = b_1/k$. Let $x \geq c \cdot k$. Choose j such that $c \cdot k^j \leq x < c \cdot k^{j+1}$. Then

$$B(x) \leq j \leq \log(x/c)/\log k \leq c' \log x$$

for some $c' > 0$ and x sufficiently large. Thus, $B(x) = O(\log x)$. This proves (0.2).

If $x/k < b_1$, then $x < k \cdot b_1 < b_2$, and $B(x) \leq 1 = B(x/k) + 1$. If $x/k \geq$

b_1, choose $i \geq 2$ such that $b_{i-1} \leq x/k < b_i$. Then $x < k \cdot b_i < b_{i+1}$ and so

$$B(x) \leq i = B(x/k) + 1.$$

Let $1/k^t \leq \delta$. Then

$$B(x) \leq B(x/k) + 1 \leq B(x/k^2) + 2 \leq \cdots \leq B(x/k^t) + t \leq B(\delta x) + t.$$

This proves (0.3).

THEOREM 1. *Let $h \geq 2$. Let A be an asymptotic basis of order h of the form $A = B \cup C$, where B and C are disjoint sets of nonnegative integers. Let $r(n)$ denote the cardinality of the largest set of pairwise disjoint representations of n in the form*

$$n = b_1' + b_2' + \cdots + b_{h-1}' + c, \tag{1}$$

where $c \in C$, $b_1', \ldots, b_{h-1}' \in B$, and $b_1' < b_2' < \cdots < b_{h-1}'$. Let W be the set of all integers $w \in hA$ such that if $w = a_1 + \cdots + a_h$ with $a_i \in A$ for $i = 1, \ldots, h$, then $a_j = c \in C$ for at most one j. Let

$$\Omega(n) = \{c \in C \mid n - c \in (h-1)B\}.$$

Suppose that for some $\delta \in (0,1)$ the following conditions are satisfied:

(1.1) *$B = \{b_i\}_{i=1}^{\infty}$, where $b_{i+1} > (2h-2)b_i$ for $i \geq 1$.*

(1.2) *$r(n) \to \infty$ as $n \to \infty$.*

(1.3) *For every $c \in C$ there exist infinitely many choices of $b_1', \ldots, b_{h-1}' \in B$ such that $w = b_1' + b_2' + \cdots + b_{h-1}' + c \in W \setminus B$ and $c' > \delta w$ for all $c' \in \Omega(w) \setminus \{c\}$.*

(1.4) *For every $b_1' \in B$, at least one of the following holds: (1.4a) there exist infinitely many choices of $b_2', \ldots, b_{h-1}' \in B$ and $c \in C$ such that $w = b_1' + b_2' + \cdots + b_{h-1}' + c \in W \setminus hB$ and $c' > \delta w$ for all $c' \in \Omega(w) \setminus \{c\}$; (1.4b) there exist infinitely many choices of $b_2', \ldots, b_h' \in B$ such that $w = b_1' + b_2' + \cdots + b_h' \in W$ and $c' > \delta w$ for all $c' \in \Omega(w)$.*

Then there exists $C' \subseteq C$ such that $A' = B \cup C'$ is a minimal asymptotic basis of order h and $(C \setminus C')(x) \leq 2B(x)^{h-1}$ for $x \geq w_1$. In particular, $d(C \setminus C') = 0$ and $d_L(A') = d_L(A)$.

Proof. We shall construct the minimal asymptotic basis A' by induction. Choose t such that $(2h-2)^{-t} \leq \delta$. Choose N_1 such that

$$(B(n) + t)^{h-1} < (3/2)B(n)^{h-1} \tag{2}$$

and $r(n) \geq 2$ for all $n \geq N_1$. Let $A_0 = A$ and $C_0 = C$. Choose $c \in C_0$. Let

$a_1 = c$. By condition (1.3), we can choose $b'_1, \ldots, b'_{h-1} \in B$ such that

$$w_1 = b'_1 + b'_2 + \cdots + b'_{h-1} + c \in W \setminus hB, \tag{3}$$

and $w_1 \geq N_1$ and $c' > \delta w_1$ for all $c' \in \Omega(w_1) \setminus \{c\}$. Let $F_1 = \Omega(w_1) \setminus \{c\}$. Let $C_1 = C \setminus F_1$ and let $A_1 = B \cup C_1$. Then

$$C \setminus C_1 = F_1 \subseteq (\delta w_1, w_1]. \tag{4}$$

If $c' \in F_1$, then there exist integers $v'_i \in B$ for $i = 1, \ldots, h-1$ such that $w_1 = v'_1 + \cdots + v'_{h-1} + c'$. Since $v'_i \leq w_1$, it follows that there are at most $B(w_1)$ choices for each v'_i, and so

$$(C \setminus C_1)(x) = |F_1| \leq B(w_1)^{h-1} \tag{5}$$

for $x \geq w_1$. Since $w_1 \in W \setminus hB$, it follows that, except for permutations of the summands, (3) is the unique representation of w_1 as a sum of h elements of A_1.

Let $n \geq N_1$ and $n \neq w_1$. Since $r(n) \geq 2$ for $n \geq N_1$, it follows that n has at least two disjoint representations of the form (1) of hA. That is, there exist integers u'_i and $u''_i \in B$ for $i = 1, \ldots, h-1$, and $c', c'' \in C$ such that

$$n = u'_1 + \cdots + u'_{h-1} + c' \tag{6}$$

and

$$n = u''_1 + \cdots + u''_{h-1} + c'', \tag{7}$$

where $c' \neq c''$ and $u'_i \neq u''_j$ for all $i, j = 1, \ldots, h-1$.

Either $c' \in C_1$ or $c'' \in C_1$. If not, then

$$c' \in \Omega(w_1) \setminus \{c\} \quad \text{and} \quad c'' \in \Omega(w_1) \setminus \{c\},$$

and so there exist integers v'_i and $v''_i \in B$ for $i = 1, \ldots, h-1$ such that

$$w_1 = v'_1 + \cdots + v'_{h-1} + c' \tag{8}$$

and

$$w_1 = v''_1 + \cdots + v''_{h-1} + c''. \tag{9}$$

Subtracting (8) from (6) and (9) from (7), we get two representations of $n - w_1$, and these yield the relation

$$u''_1 + \cdots + u'_{h-1} + v''_1 + \cdots + v''_{h-1} = u''_1 + \cdots + u''_{h-1} + v'_1 + \cdots + v'_{h-1}.$$

By Lemma 1, the growth condition (1.1) on the elements of B implies that B is a B_{2h-2}-sequence; hence

$$\{u_1',\dots,u_{h-1}',v_1'',\dots,v_{h-1}''\} = \{u_1'',\dots,u_{h-1}'',v_1',\dots,v_{h-1}'\}.$$

Since the representations (6) and (7) are disjoint, it follows that $u_i' \neq u_j''$ for all $i, j = 1,\dots, h-1$, and so

$$\{u_1',\dots,u_{h-1}'\} \subseteq \{v_1',\dots,v_{h-1}'\}.$$

Since $u_1' < \cdots < u_{h-1}'$, it follows that

$$\{u_1',\dots,u_{h-1}'\} = \{v_1',\dots,v_{h-1}'\}.$$

Equations (6) and (8) imply that $n = w_1$, which is false. It follows that either $c' \notin F_1 = \Omega(w_1)\setminus\{c\}$ or $c'' \notin F_1 = \Omega(w_1)\setminus\{c\}$, and so

$$n \in h(B \cup C_1) = hA_1 \quad \text{for all } n \geq N_1.$$

Let $k \geq 2$. Suppose that for each $j < k$ we have constructed

(1.5) an integer $w_j \in W$ with $w_{j-1} < \delta w_j$ for $2 \leq j < k$,

(1.6) a finite set $F_j \subseteq C \cap (\delta w_j, w_j]$ with $|F_j| \leq B(w_j)^{h-1}$,

(1.7) a set $C_j = C\setminus(F_1 \cup \cdots \cup F_j)$ and an integer $a_j \in A_j = B \cup C_j$ such that w_j has a unique representation as a sum of h elements of A_j, and a_j is a summand that is used in this representation, and $n \in hA_j$ for all $n \geq N_1$.

To perform the induction, we choose N_k so large that

(1.8) $N_k > w_{k-1}$,

(1.9) $B(N_k)^{h-1} > 4B(w_{k-1})^{h-1}$, and

(1.10) $r(n) \geq 2 + \sum_{j=1}^{k-1}|F_j| = 2 + |A\setminus A_{k-1}|$ for $n \geq N_k$.

Let $a_k \in A_{k-1} = B \cup C_{k-1}$. There are two cases.

Case 1. Suppose $a_k = c \in C_{k-1}$. By condition (1.3) of the theorem, there exist integers $b_i' \in B$ for $i = 1, 2, \dots, h-1$ such that

$$b_1' + b_2' + \cdots + b_{h-1}' + c = w_k \in W\setminus hB,$$

where $\delta w_k > N_k$ and $c' > \delta w_k$ for all $c' \in F_k = \Omega(w_k)\setminus\{c\}$. Let

$$C_k = C_{k-1}\setminus F_k \quad \text{and} \quad A_k = B \cup C_k.$$

Then the element w_k has a unique representation (up to permutations of the summands) as a sum of h elements of A_k, and the integer $a_k = c$ is one of the summands in this representation.

Case 2. Suppose $a_k = b_1' \in B$. If condition (1.4a) is satisfied, there exist integers $b_i' \in B$ for $i = 2, 3, \ldots, h-1$ and $c \in C$ such that

$$b_1' + b_2' + \cdots + b_{h-1}' + c = w_k \in W \setminus hB,$$

where $\delta w_k > N_k$ and $c' > \delta w_k$ for all $c' \in F_k = \Omega(w_k) \setminus \{c\}$. If condition (1.4b) is satisfied, there exist integers $b_i' \in B$ for $i = 2, 3, \ldots, h$ such that

$$b_1' + b_2' + \cdots + b_h' = w_k \in W,$$

where $\delta w_k > N_k$ and $c' > \delta w_k$ for all $c' \in F_k = \Omega(w_k)$. With either condition (1.4a) or (1.4b), let $C_k = C_{k-1} \setminus F_k$ and $A_k = B \cup C_k$. Then the element w_k has a unique representation (up to permutations of the summands) as a sum of h elements of A_k, and this representation includes the integer $a_k = b_1'$.

In both cases, $F_k \subseteq C_{k-1} \cap (\delta w_k, w_k]$ and $|F_k| \leq B(w_k)^{h-1}$. Let $n \geq N_1$. We shall show that $n \in hA_k$. Since $n \in hA_{k-1}$ and $c' > \delta w_k > N_k > w_{k-1}$ for all $c' \in F_k = A_{k-1} \setminus A_k$, it follows that $n \in hA_k$ for $N_1 \leq n \leq \delta w_k$. Let $n > \delta w_k$ and $n \neq w_k$. Since $r(n) \geq 2 + |A \setminus A_{k-1}|$ for $n \geq N_k$ by condition (1.10), it follows that n has at least two disjoint representations of the form (1) in hA_{k-1}. That is, there exist integers u_i' and $u_i'' \in B$ for $i = 1, \ldots, h-1$, and $c', c'' \in C_{k-1}$ such that

$$n = u_1' + \cdots + u_{h-1}' + c' \tag{10}$$

and

$$n = u_1'' + \cdots + u_{h-1}'' + c'', \tag{11}$$

where $c' \neq c''$ and $u_i' \neq u_j''$ for all $i, j = 1, \ldots, h-1$. If $c' \in F_k$ and $c'' \in F_k$, then there exist integers v_i' and $v_i'' \in B$ for $i = 1, \ldots, h-1$ such that

$$w_k = v_i' + \cdots + v_{h-1}' + c' \tag{12}$$

and

$$w_k = v_1'' + \cdots + v_{h-1}'' + c''. \tag{13}$$

Subtracting (12) from (10) and (13) from (11), we get two representations of $n - w_k$, and these yield the relation

$$u_1' + \cdots + u_{h-1}' + v_1'' + \cdots + v_{h-1}'' = u_1'' + \cdots + u_{h-1}'' + v_1' + \cdots + v_{h-1}'.$$

Since B is a B_{2h-2}-sequence, the argument used at the beginning of this proof shows that $n \in h(B \cup C_k) = hA_k$. Thus, $n \in hA_k$ for all $n \geq N_1$. This completes the induction.

We now define

$$C' = \bigcap_{k=1}^{\infty} C_k = C \setminus \bigcup_{k=1}^{\infty} F_k \quad \text{and} \quad A' = B \cup C'.$$

Let $n \geq N_1$. Choose $w_k > n$. Then $n \in hA_k$. Since

$$a' > w_k > n \quad \text{for all } a' \in A_k \setminus A' = \bigcup_{j=k+1}^{\infty} F_j,$$

it follows that $n \in hA'$. Thus, A' is an asymptotic basis of order h.

Here is the critical idea in the proof: At the k-th step of the induction, we could choose *any* element $a_k \in A_k = B \cup C_k$. We must make these choices in such a way that if $a' \in A'$, then $a' = a_k$ for *infinitely many* k. This implies that for every $a' \in A'$ there are infinitely many integers w_k such that $w_k \in hA'$, but $w_k \notin h(A' \setminus \{a'\})$, and so A' is a minimal asymptotic basis of order h.

Finally, we must prove that for $x \geq w_1$,

$$(C \setminus C')(x) \leq 2B(x)^{h-1}. \tag{14}$$

By (5), $(C \setminus C')(w_1) \leq B(w_1)^{h-1}$. Suppose that (14) holds for $w_1 \leq x \leq w_{k-1}$. Since $(C \setminus C') \cap (w_{k-1}, \delta w_k] = \varnothing$, then (14) holds for $x \leq \delta w_k$. Let $\delta w_k < x \leq w_k$. Then by (1.6), (0.3), (1.9), and (2) we have

$$\begin{aligned}
(C \setminus C')(x) &\leq (C \setminus C')(w_k) = (C \setminus C')(w_{k-1}) + |F_k| \\
&\leq 2B(w_{k-1})^{h-1} + B(w_k)^{h-1} \\
&\leq 2B(w_{k-1})^{h-1} + \left(B(\delta w_k) + t\right)^{h-1} \\
&\leq \tfrac{1}{2}B(\delta w_k)^{h-1} + \tfrac{3}{2}B(\delta w_k)^{h-1} \\
&= 2B(\delta w_k)^{h-1} \\
&\leq 2B(x)^{h-1}.
\end{aligned}$$

Thus, (14) holds for all $x \geq w_1$. Since the set B is a $B_{(2h-2)}$-sequence, it follows from the lemma that $B(x) = O(\log x)$, and so $d(C \setminus C') = 0$ and $d_L(A') = d_L(A)$. This completes the proof.

We shall now use Theorem 1 to construct examples of minimal asymptotic bases of order h with prescribed positive densities.

THEOREM 2. *Let* $h \geq 2$. *Let* $B = \{b_i\}_{i=1}^{\infty}$ *be a set of positive integers such that*

(2.1) $b_{i+1} > (2h-1)b_i$ *for* $i \geq 1$,
(2.2) $B_0 = \{b_i \in B | b \equiv 0 \pmod h\}$ *is infinite*,
(2.3) $B_1 = \{b_i \in B | b \equiv 1 \pmod h\}$ *is infinite*,
(2.4) $B = B_0 \cup B_1$.

Let $C = \{c \geq 0 | c \equiv 0 \pmod h\} \setminus B_0$. *Then there exists a set* $C' \subseteq C$ *such that* $A' = B \cup C'$ *is a minimal asymptotic basis of order* h, *and* $d(A') = 1/h$.

Proof. The set $A = B \cup C$ is an asymptotic basis of order h, and $d(A) = 1/h$. We shall show that conditions (1.1)–(1.4) of Theorem 1 are satisfied with $\delta = 1/(h+1)$. Note that condition (1.1) in Theorem 1 follows immediately from condition (2.1) in Theorem 2. The lemma implies that $B(x) = O(\log x)$.

To show condition (1.2), choose a large integer m. Let

$$e \in \{0, 1, \ldots, h-1\}.$$

By (2.2) and (2.3), we can choose $m+1$ pairwise disjoint sets

$$\{b_{j,1}, \ldots, b_{j,h-1}\} \subseteq B$$

such that $b_{j,1} < \cdots < b_{j,h-1}$ and $b_{j,h-1} < b_{j+1,1}$ for $j = 1, \ldots, m$ and

$$e_j = b_{j,1} + \cdots + b_{j,h-1} \equiv e \pmod h$$

for $j = 1, \ldots, m+1$. Then $e_1 < \cdots < e_{m+1}$. Choose

$$b_k > \max\{e_1, \ldots, e_{m+1}\}.$$

Let $n \equiv e \pmod h$ and $n \geq b_{k+1}$. Then $n - e_j > 0$ and $n - e_j \equiv 0 \pmod h$ for $j = 1, \ldots, m+1$. Suppose that $n - e_i = b_u \in B$ and $n - e_j = b_v \in B$ for some $i < j$. Then $b_u > b_v$ and

$$b_v = n - e_j > b_{k+1} - b_k > b_k > e_j > e_j - e_i = b_u - b_v > b_v$$

which is absurd. Therefore, $n - e_j \in C$ for at least m different e_j, and so $r(n) \geq m$ for all sufficiently large $n \equiv e \pmod h$. It follows that $r(n) \to \infty$ as $n \to \infty$, and condition (1.2) is satisfied.

Next we show that (1.3) holds. Since $c \equiv 0 \pmod h$ for all $c \in C$, it follows that if $n \equiv h-1 \pmod h$, then $n \in W$. Fix $c \in C$. Choose $b_t \in B$ with $b_t > c$ and $b_t \equiv 1 \pmod h$. Let $w = (h-1)b_t + c$. Then $w \equiv h-1 \pmod h$ and $w \in W$.

We shall prove that $w \in W \setminus hB$. Suppose that there exist $b_1', \dots, b_h' \in B$ such that $w = b_1' + \cdots + b_h'$. Since

$$(h-1)b_t \le w < hb_t \le (2h-2)b_t < b_{t+1},$$

it follows that $b_i' \le b_t$ for all $i = 1, \dots, h$, but $b_i' \ne b_t$ for some $i = 1, \dots, h$. If $b_j' \ne b_t$ for exactly one $j \in \{1, \dots, h\}$, then

$$b_j' = c \in B \cap C = \varnothing,$$

which is absurd. If $b_j' \ne b_t$ and $b_k' \ne b_t$, then

$$w = b_1' + \cdots + b_h' \le (h-2)b_t + 2b_{t-1} < (h-1)b_t \le w,$$

which is also absurd. Therefore, $w \notin hB$.

Let $c' \in \Omega(w) \setminus \{c\}$. Then there exist $b_i' \in B$ for $i = 1, \dots, h-1$ such that $w = b_1' + \cdots + b_{h-1}' + c'$ and $b_j' \ne b_t$ for some j. Then $b_j' \le b_{t-1}$. Since

$$(h-1)b_t \le (h-1)b_t + c = w \le (h-2)b_t + b_{t-1} + c'$$

it follows that

$$c' \ge b_t - b_{t-1} > ((2h-2)/(2h-1))b_t > ((2h-2)/h(2h-1))w \ge \delta w.$$

Thus, condition (1.3) of Theorem 1 holds.

Finally, we consider condition (1.4). Let $b_u \in B = B_0 \cup B_1$. If $b_u \in B_0$, we shall show that (1.4b) holds. Choose $b_t \in B_1$ with $b_t > b_u$. Let

$$w = b_u + (h-1)b_t.$$

Then $w < hb_t < b_{t+1}$. Since $w \equiv h-1 \pmod h$, it follows that $w \in W$. Let $c' \in \Omega(w)$. There exist $b_i' \in B$ such that $w = b_1' + \cdots + b_{h-1}' + c'$, where $b_i' \le b_t$ for all i and $b_j' \le b_{t-1}$ for some j. The same argument as above implies that

$$c' > ((2h-2)/h(2h-1))w \ge \delta w.$$

If $b_u \in B_1$, we shall show that (1.4a) holds. Choose $b_t \in B_1$ with $b_t > b_u$. The interval $(2b_t - b_u, 3b_t - b_u)$ contains $b_t/h + O(1)$ multiples of h, and so $b_t/h + O(\log b_t)$ elements of C. There are at most $B(3b_t)^2 = O(\log^2 b_t)$ integers of the form $b_i + b_j - b_u$ in this interval. It follows that for b_t sufficiently large there exists an integer $c \in C$ such that

$$2b_t < b_u + c < 3b_t \quad \text{and} \quad b_u + c \notin 2B.$$

Let $w = (h-2)b_t + b_u + c$. Then $w \equiv h - 1 \pmod{h}$, hence $w \in W$. If $w \in hB$, there exist $b'_1, \ldots, b'_h \in B$ such that $b'_1 + \cdots + b'_h = w$, but this is impossible, since

$$hb_t < w < (h+1)b_t \leq (2h-1)b_t < b_{t+1}.$$

Therefore, $w \in W \setminus hB$.

Let $c' \in \Omega(w) \setminus \{c\}$. There exist $b'_1, \ldots, b'_{h-1}$ such that

$$w = b'_1 + \cdots + b'_{h-1} + c'.$$

Then $b'_i \leq b_t$ for $i = 1, \ldots, h-1$ and so

$$c' \geq w - (h-1)b_t > b_t > w/(h+1) = \delta w.$$

This completes the proof of Theorem 2.

COROLLARY. *For every $h \geq 2$ there exists a minimal asymptotic basis A' of order h with asymptotic density $d(A') = 1/h$.*

THEOREM 3. *Let $h \geq 2$. For every $\alpha \in (0, 1/(2h-2))$ there exists a minimal asymptotic basis A of order h with asymptotic density $d(A) = \alpha$.*

Proof. Let $\alpha \in (0, 1/(2h-2))$. Let $\Theta > 0$ be irrational. Let $B = \{b_i\}_{i=1}^{\infty}$ be a set of positive integers so that $\{b_i\Theta\}$ is dense in the interval $(0, 1/(h-1))$ and $b_{i+1} > (2h-2)b_i$ for all $i \geq 1$. Let

$$C = \{c \geq 0 \mid \{c\Theta\} < \alpha\} \setminus B.$$

Let $A = B \cup C$. Then $d(B) = 0$ and $d(A) = d(C) = \alpha$. We shall prove that A is an asymptotic basis of order h and satisfies conditions (1.1)–(1.4) of Theorem 1 with $\delta = (2h-3)/h(2h-2) \leq 1/4$.

Clearly, B satisfies (1.1). To show that condition (1.2) holds, we first fix an integer $N > 2/\alpha$. Choose m large. For $i = 1, \ldots, h-1$, and $j = 1, \ldots, m+1$, and $k = 1, \ldots, N$, we choose pairwise distinct integers $b(i, j, k) \in B$ such that

(3.1) $b(1, j, k) < b(2, j, k) < \cdots < b(h-1, j, k)$ for all j, k,
(3.2) $b(h-1, j, k) < b(1, j+1, k)$ for $j = 1, 2, \ldots, m$ and all k,
(3.3) $\{b(i, j, k)\Theta\} \in [(k-1)/((h-1)N), k/((h-1)N))$.

Let

$$s(j, k) = \sum_{i=1}^{h-1} b(i, j, k) \in (h-1)B.$$

Conditions (3.1) and (3.2) imply that $s(1,k) < s(2,k) < \cdots < s(m+1,k)$. Also, condition (3.3) implies that

$$\{s(j,k)\Theta\} \in [(k-1)/N, k/N) \quad \text{for } j = 1,\ldots,m+1.$$

Let

$$n > 2 \cdot \max\{s(j,k) | j = 1,\ldots,m+1,\ k = 1,\ldots,N\}.$$

If $\{n\Theta\} \in [1/N, 1)$, then $\{n\Theta\} \in [k/N, (k+1)/N)$ for some $k = 1,\ldots, N-1$, and

$$\{(n - s(j,k))\Theta\} \in [0, 2/N) \subset [0, \alpha)$$

for $j = 1,\ldots,m+1$. If $\{n\Theta\} \in [0, 1/N)$, then

$$\{(n - s(j,N))\Theta\} \in [0, 2/N) \subset [0, \alpha).$$

In all cases, $n - s(j,N) = c_j \in B \cup C$ for $j = 1,\ldots,m+1$, and $c_1 > c_2 > \cdots > c_{m+1}$. Since $s(j,k) \in (h-1)B$ and since B is a B_h-sequence, it follows that $c_j \in B$ for at most one j, and so n has at least m pairwise disjoint representations of the form (1). Thus, A is an asymptotic basis of order h, and $r(n) \to \infty$ as $n \to \infty$. Condition (1.2) is satisfied.

Let W be the set of all integers $w \in hA$ such that if $w = a_1 + \cdots + a_h$ with $a_i \in A$ for $i = 1,\ldots,h$, then $a_j \in C$ for at most one j. Let

$$\beta = (h-2)/(h-1) + 2\alpha.$$

Since $0 < \alpha < 1/(2h-2)$, it follows that $0 < \alpha < \beta < 1$. Let n be a positive integer such that $\{n\Theta\} \geq \beta$. We shall show that $n \in W$. If not, then there exists a representation

$$n = b_1' + \cdots + b_k' + c_{k+1} + \cdots + c_h,$$

where $b_i' \in B$, $c_j \in C$, and $0 \leq k \leq h-2$. Since $\{b_i'\Theta\} < 1/(h-1)$ and $\{c_j\Theta\} < \alpha$, it follows that

$$\begin{aligned}
\{n\Theta\} &< k/(h-1) + (h-k)\alpha \\
&= h\alpha + k(1/(h-1) - \alpha) \\
&\leq h\alpha + (h-2)(1/(h-1) - \alpha) \\
&= (h-2)/(h-1) + 2\alpha \\
&= \beta,
\end{aligned}$$

which contradicts $\{n\Theta\} \geq \beta$. Therefore, $k = h$ or $k = h-1$, and so $n \in W$.

We now prove that condition (1.3) holds. Let $c \in C$. Then $\{c\Theta\} < \alpha < \beta$. The set $\{\{b_i\Theta\} | b_i \in B\}$ is dense in $(0, 1/(h-1))$, and so there exist infinitely many $b_t \in B$ such that $b_t > c$ and

$$(\beta - \{c\Theta\})/(h-1) < \{b_t\Theta\} < (1 - \{c\Theta\})/(h-1).$$

Let $w = (h-1)b_t + c$. Then

$$\beta < \{w\Theta\} = (h-1)\{b_t\Theta\} + \{c\Theta\} < 1$$

and so $w \in W$. Since $(h-1)b_t \le w < hb_t < b_{t+1}$, it follows that $w \notin hB$, hence $w \in W \setminus hB$. Let $c' \in \Omega(w) \setminus \{c\}$. Then there exist $b_i' \in B$ such that

$$w = b_1' + \cdots + b_{h-1}' + c',$$

where $b_i' \le b_t$ for all i and $b_j' \le b_{t-1}$ for at least one j. Then

$$(h-1)b_t \le w \le (h-2)b_t + b_{t-1} + c',$$

and so

$$\begin{aligned} c' &\ge b_t - b_{t-1} \\ &> ((2h-3)/(2h-2))b_t \\ &> ((2h-3)/h(2h-2))w \\ &= \delta w. \end{aligned}$$

Thus, A satisfies condition (1.3).

We show next that (1.4b) holds. Let $b_u \in B$. Suppose that $\{b_u\Theta\} < \beta$. Note that this is always true for $h \ge 3$, since

$$\{b_u\Theta\} < 1/(h-1) < (h-2)/(h-1) + 2\alpha = \beta.$$

Then there exist infinitely many $b_t \in B$ such that $b_t > b_u$ and

$$(\beta - \{b_u\Theta\})/(h-1) < \{b_t\Theta\} < (1 - \{b_u\Theta\})/(h-1).$$

Let $w = (h-1)b_t + b_u$. It follows as in the case above that $w \in W$ and $c' > \delta w$ for all $c' \in \Omega(w)$.

Finally, we consider the case $h = 2$ and

$$0 < 2\alpha = \beta \le \{b_u\Theta\} < 1.$$

There exist infinitely many $b_t \in B$ such that $b_t > b_u$ and

$$0 < \{b_t\Theta\} < 1 - \{b_u\Theta\}.$$

Let $w = b_t + b_u$. Then $b_t < w < 2b_t < b_{t+1}$, and

$$\beta \le \{b_u\Theta\} < \{w\Theta\} = \{b_t\Theta\} + \{b_u\Theta\} < 1,$$

hence $w \in W$. Let $c' \in \Omega(w)$. Then there exists $b_1' \in B$ such that $w = b_1' + c'$, where $b_1' \le b_{t-1}$. Then

$$b_t < w \le b_{t-1} + c',$$

and so

$$c' > b_t - b_{t-1} > b_t/2 > w/4 = \delta w.$$

Thus, condition (1.4) is satisfied. This completes the proof of the theorem.

COROLLARY. *If A is a minimal asymptotic basis of order 2, then $d_L(A) \le 1/2$. For every $\alpha \in (0, 1/2]$, there exists a minimal asymptotic basis A with* $d(A) = 1/2$.

Proof. This follows immediately from Theorems 2 and 3 and the result of Nathanson and Sárközy [5].

Open problems. It should be possible to generalize the corollary to Theorem 3 to bases of order $h \ge 3$. If $\alpha \in (0, 1/h)$, prove that there exists a minimal asymptotic basis A of order h with asymptotic density α.

The minimal asymptotic basis $A = \{a_i\}_{i=1}^{\infty}$ of order 2 and density 1/2 constructed in Theorem 2 has the property that $a_{i+1} - a_i \le 4$ for all i and $a_{i+1} - a_i = 4$ for infinitely many i. It is easy to show that there does not exist a minimal asymptotic basis A of order 2 with $\limsup(a_{i+1} - a_i) = 2$. Does there exist a minimal asymptotic basis A of order 2 with $\limsup(a_{i+1} - a_i) = 3$?

REFERENCES

1. P. ERDÖS and M.B. NATHANSON, "Problems and results on minimal bases in additive number theory" in *Number theory*, New York, 1985–86, Lecture Notes in Mathematics, vol. 1240, Springer-Verlag, New York, 1987, pp. 87–96.
2. E. HÄRTTER, *Ein Beitrag zur Theorie der Minimalbasen*, J. Reine Angew. Math., vol. 196 (1956), pp. 170–204.
3. M.B. NATHANSON, *Minimal bases and maximal nonbases in additive number theory*, J. Number Theory, vol. 6 (1974), pp. 324–333.
4. ______, *Minimal bases and powers of 2*, Acta Arith., vol. 51 (1988), pp. 95–102.
5. M.B. NATHANSON and A. SÁRKÖZY, *On the maximum density of minimal asymptotic bases*, Proc. Amer. Math. Soc., to appear.
6. A. STÖHR, *Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe*, II, J. Reine Angew. Math, vol. 194 (1955), pp. 111–140.

MATHEMATICAL INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, HUNGARY
LEHMAN COLLEGE OF THE CITY UNIVERSITY OF NEW YORK
NEW YORK