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WILD CATEGORIES OF PERIODIC MODULES

BY

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Dedicated to the memory of Irving Reiner, friend, colleague and teacher

Let K be a field of characteristic $p > 0$ and $G = \langle x, y \rangle$ be an elementary abelian group of order p^2. It has long been known (Heller-Reiner [2]), that if $p > 2$ the category of left KG-modules is wild. Basically this means that there is no possibility of classifying the indecomposable objects in the category. However, there seems to be a general misconception that suitable subcategories such as the full subcategory of periodic modules, should be better behaved. The purpose of this note is to demonstrate that such is not the case. Indeed we show that, generally, the full subcategory of all KG-modules whose cohomology rings are annihilated by a fixed non-nilpotent element ζ of $H^2(G, K)$ is a wild category. We will not belabor the point by proving it in every possible case. For simplicity, only the case in which $p \geq 7$ and ζ is the Bockstein of a nonzero element of $H^1(G, F_p)$ will be considered. We hope that the reader will regard this example as sufficient.

Considering the terminology and notation of varieties and cohomology rings we refer the reader to [1].

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For convenience let $X = x - 1$ and $Y = y - 1$. Then a KG-module may be considered as a $K[X, Y]$-module in which $X^pM = 0$ and $Y^pM = 0$. Let $\zeta \in H^2(G, k)$ be the Bockstein of the element $\eta \in H^1(G, K)$ represented by $\eta\colon \Omega(K) \to K$, where $\eta(X) = 1$ and $\eta(Y) = 0$, and $\Omega(K)$ is the kernel of the augmentation $KG \to K$.

LEMMA 1. *Let M be a KG-module such that $M_{\langle x \rangle}$ is a free $K\langle x \rangle$-module and*

$$Y^{(p-1)/2}M = 0.$$

Then ζ annihilates $\operatorname{Ext}^*_{KG}(M, M)$.

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Proof. The basic trick is to note that ζ is represented by the sequence

$$E\colon 0 \to K \to K_{\langle y\rangle}^{\uparrow G} \to K_{\langle y\rangle}^{\uparrow G} \to K \to 0,$$

where $K_{\langle y\rangle}^{\uparrow G}$ is the module induced from the trivial $K\langle y\rangle$-module, and the middle map is multiplication by X. If I is the identity of $\operatorname{Ext}_{KG}^{*}(M, M)$, then ζI is represented by $E \otimes M$. For $(\alpha, \beta) \in K^2$ and $u = 1 + \alpha X + \beta Y$, it is easy to see that $M_{\langle u\rangle}$ is free as a $K\langle u\rangle$-module provided $\alpha \neq 0$, hence

$$V_G(M) = \{(0, \beta);\ \beta \in K\}.$$

So M is periodic. Moreover, M must be periodic of period 2. Therefore we have the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & M & \xrightarrow{j} & P_2 & \longrightarrow & P_1 & \longrightarrow & M & \longrightarrow 0 \\
& \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_1} & & \| & \\
E \otimes M\colon 0 \longrightarrow & M & \longrightarrow & K_{\langle y\rangle}^{\uparrow G} \otimes M & \longrightarrow & K_{\langle y\rangle}^{\uparrow G} \otimes M & \longrightarrow & M & \longrightarrow 0
\end{array}$$

where the upper sequence is part of a projective resolution of M, and μ is the chain map coming from the projectivity of P_1 and P_2.

Now $j(M) \subset Y^{(p+1)/2}P_2$, so

$$\mu_2 j(M) \subset Y^{(p+1)/2}\left(K_{\langle y\rangle}^{\uparrow G} \otimes M\right) = 0,$$

since $Y^{(p+1)/2}M = 0$ and $Y \cdot K_{\langle y\rangle}^{\uparrow G} = 0$. Therefore $\mu' = 0$.

Let $\underline{C}_{\zeta}$ be the full subcategory of KG-modules M such that $\zeta \operatorname{Ext}_{KG}^{*}(M, M) = 0$. For any $Z \in K[X, Y]$ and any submodule $L \subseteq M$, let $Z^{-n}L = \{m \in M;\ Z^n m \in L\}$, and let $\psi\colon M \to M/\operatorname{Rad} M$ be the natural quotient. Let $\underline{M}$ be the full subcategory of $K[X, Y]$-modules consisting of all M that satisfy conditions (1) to (7), given below.

(1) $X^p M = 0$.
(2) $Y^3 M = 0$.
(3) $\operatorname{Dim}_K X^{p-1}M = \operatorname{Dim}_K M/XM$.
(4) $YM \subseteq XM$.
(5) $Y^2 M \subseteq X^{p-1}M$.
(6) $\psi(X^{1-p}YX^{-1}YX^{-1}YY^{-2}(0)) \subset \psi(X^{-1}YY^{-2}(0))$.
(7) $\psi(X^{1-p}Y^2 M) \subset \psi(X^{-1}YY^{-2}(0))$.

Note that condition (3) guarantees that M is free as a $K\langle x\rangle$-module. Thus, by the lemma, for $p \geq 7$, conditions (1), (2) and (3) say that any module in $\underline{M}$ is an object in $\underline{C}_{\zeta}$.

THEOREM 2. *The category $\underline{M}$ is a wild category, and hence $\underline{C}_\xi$ is wild if $p \geq 7$.*

Proof. The proof follows Ringel [3]. Let Q be the quiver

$$\begin{tikzcd} \overset{1}{\cdot} \arrow[r] & \overset{2}{\cdot} \arrow[loop right] \end{tikzcd}$$

A K-representation of Q consists of a quadruple $R = (V, W, \gamma, \phi)$, where W is the K-space corresponding to the vertex 2, V is the space corresponding to vertex 1, the map $\gamma\colon V \to W$ corresponds to the arrow from 1 to 2, and the linear transformation $\phi\colon W \to W$ corresponds to the loop on 2. The category of K representations of Q, $\operatorname{Rep}(Q)$, is known to be a wild category.

To any $R = (V, W, \gamma, \phi)$ in $\operatorname{Rep}(Q)$ we associate a KG-module M_R in $\underline{M}$ represented by the following diagram:

$$\begin{tikzcd}
W_1 \arrow[dd, "X"'] & & \\
& W_1' \arrow[dl, "Y"'] \arrow[dd, "X"] & \\
W_2 \arrow[dd, "X"'] & & \\
& W_2' \arrow[dl, "Y"'] \arrow[dd, "X"] & \\
W_3 \arrow[dd, "X"'] & & \\
& \vdots \arrow[dd, "X"] & \\
\vdots \arrow[dd, "X"'] & & V_1 \arrow[dl, "Y"'] \arrow[dd, "X"] \\
& W_{p-1}' \arrow[dl, "Y"'] \arrow[dd, "X"] & \\
W_p & & V_2 \arrow[dl, "Y"'] \arrow[dd, "X"] \\
& W_p' & \\
& & \vdots \arrow[dd, "X"] \\
& & \\
& & V_p
\end{tikzcd}$$

That is, as a K-space,

$$M_R = W_1 \oplus \cdots \oplus W_p \oplus W_1' \oplus \cdots \oplus W_p' \oplus V_1 \oplus \cdots \oplus V_p,$$

where

$$W_1 \cong \cdots \cong W_p \cong W_1' \cong \cdots \cong W_p' \cong W,$$

and

$$V_1 \cong \cdots \cong V_p \cong V.$$

The action of X on M is given by the identity maps

$$I_W\colon W_i \to W_{i+1}, \quad I_W\colon W_i' \to W_{i+1}', \quad I_V\colon V_i \to V_{i+1},$$

for $i = 1, \ldots, p-1$, and X is 0 on W_p, W_p' and V_p.

The action of Y on M is given by

$$\phi\colon W_1 \to W_p', \quad I_W\colon W_i' \to W_{i+1}$$

for $i = 1, \ldots, p-1$, and by the maps $\gamma\colon V_1 \to W_{p-1}'$, $\gamma\colon V_2 \to W_p'$. Finally Y is 0 on $W_2, \ldots, W_p, V_3, \ldots, V_p$.

It is clear that $X^p M_R = Y^3 M_R = 0$, and $XY = YX$ on M_R. Also

$$M_R/\operatorname{Rad} M_R \equiv W_1 \oplus W_1' \oplus V_1 \cong W_p \oplus W_p' \oplus V_p' = X^{p-1}M_R.$$

It is easy to verify conditions (4) and (5), and the proofs of (6) and (7) are also straightforward.

Then define a functor $f\colon \underline{\operatorname{Rep}(Q)} \to \underline{M}$ by $f(R) = M_R$. A morphism

$$\theta\colon R = (V, W, \gamma, \phi) \to \hat{R} = (\hat{V}, \hat{W}, \hat{\gamma}, \hat{\phi})$$

is such that $\hat{\gamma}\theta_V = \theta_W\gamma$ and $\theta_W\phi = \hat{\phi}\theta_W$. So define a KG-module homomorphism $\mu\colon M_R \to M_{\hat{R}}$ by $\mu = \theta_W$ on W_i or W_i', and $\mu = \theta_V$ on V_i, for $i = 1, \ldots, p$.

We now define a functor $g\colon \underline{M} \to \underline{\operatorname{Rep}(Q)}$. Given $M \in \underline{M}$, let

$$W = \psi(X^{-1}YY^{-2}(0)), V \cong \psi(X^{1-p}Y^2M) \oplus V',$$

where

$$V' \cong \psi(Y^{-1}(0))/\psi(Y^{-1}(0) \cap X^{-1}YY^{-2}(0))$$

Define $\gamma\colon V \to W$ to be the inclusion $\psi(X^{1-p}Y^2M) \subseteq W$ on the first summand, while $\gamma(V') = 0$.

In order to define $\phi\colon W \to W$, given $w \in W$ choose a representative $\overline{w} \in X^{-1}YY^{-2}(0)$. Then $XY\overline{w} = 0$. Choose w_1 such that $Xw_1 = Y\overline{w}$. Then

$$Yw_1 \in YX^{-1}XX^{-1}YY^{-2}(0),$$

and it is easily seen that Yw_1 is uniquely determined by w. Since $Yw_1 \in X^{p-1}M$, we have $Yw_1 = X^{p-1}w_2$, where $w_2 \in X^{1-p}YX^{-1}YX^{-1}YY^{-2}(0)$. Again, modulo $\operatorname{Rad} M$, w_2 is uniquely determined by w. Let then $\phi(w) = \psi(w_2)$. Define now $g\colon \underline{M} \to \underline{\operatorname{Rep}(Q)}$ by $g(M) = (V, W, \gamma, \phi)$, with V, W, γ and ϕ as above.

Any KG-homomorphism, θ: $M \to M'$, must preserve all of the subspaces used in the definition. So it must induce a morphism $\hat{\theta}$: $g(M) \to g(M')$.

Finally note that for all R in $\underline{\text{Rep}}(Q)$, $gf(R) \cong R$.

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