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TRACE QUOTIENT MODULES

BY

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This paper is dedicated to the memory of Irving Reiner

1. Definitions and basic properties

This paper studies properties of a class of modules over an artin algebra called trace quotient modules which we define below. We were led to consider these modules during our investigation of the structure of submodules of endomorphism rings [2]. The main impetus for studying trace quotient modules is that whether or not a module is a trace quotient module has very interesting homological consequences. Our results show that although it is rare for all finitely generated modules to be trace quotient modules, the class of trace quotient modules is large. Throughout this paper, Λ will denote an artin algebra. All modules will be finitely generated, left Λ-modules unless otherwise stated. We let $\operatorname{mod}(\Lambda)$ denote the category of finitely generated left Λ-modules and $\operatorname{rad}(\Lambda)$ denote the Jacobson radical of Λ.

If A and X are Λ-modules, the *trace of A in X*, denoted by $\tau_A(X)$, is the submodule of X generated by images of morphisms from A to X. A Λ-module C is called a *trace quotient module* if there exists a nonsplit short exact sequence of Λ-modules

$$(*) \qquad 0 \to Y \xrightarrow{f} X \to C \to 0$$

such that $\tau_A(X) = f(Y)$. In case there is a nonsplit short exact sequence of the form $(*)$, we will sometimes say C is a *trace quotient module by A* or C is a *trace quotient module of X by A*. Finally, if C is a trace quotient module of X by A then we will use the notation

$$0 \to \tau_A(X) \to X \xrightarrow{\pi_x} C \to 0$$

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to mean a nonsplit short exact sequence with the map $\tau_A(X) \to X$ being the inclusion map. We list some basic properties.

PROPOSITION 1.1. (a) *If A and X are Λ-modules, then for some $n \geq 0$, there is a surjection*

$$\coprod_n A \to \tau_A(X),$$

where $\coprod_n A$ denotes the direct sum of n copies of A.

(b) *If A and X are Λ-modules then $\tau_A(X) = \tau_{\tau_A(X)}(X)$.*

(c) *$\tau_A(X_1 \coprod X_2) = \tau_A(X_1) \coprod \tau_A(X_2)$ for all Λ-modules A, X_1 and X_2.*

(d) *If C is a trace quotient of X by A then*

$$\mathrm{Hom}_\Lambda(A, \pi_X)\colon \mathrm{Hom}_\Lambda(A, X) \to \mathrm{Hom}_\Lambda(A, C)$$

is the zero map.

(e) *If C is a trace quotient module of X by A then C is a trace quotient module of X by $\tau_A(X)$.*

(f) *If C is a trace quotient module by A then no nonzero Λ-morphism from A to C factors through a projective Λ-module.*

(g) *If C is an indecomposable trace quotient module by A then C is a trace quotient module of X by A for some indecomposable Λ-module X.*

Proof. We leave (a)–(d) to the reader. Property (e) is an immediate consequence of (b). To prove (f), let C be a trace quotient module of X by A and consider the sequence

$$0 \to \tau_A(X) \to X \xrightarrow{\pi_X} C \to 0.$$

Let $p\colon P \to C$ be a Λ-projective cover of C. Suppose that $f\colon A \to C$ is a nonzero Λ-morphism which factors through a projective Λ-module. Consider the diagram

$$\begin{tikzcd} P \arrow[dr, "p"] & A \arrow[d, "f"] & \\ X \arrow[r, "\pi_X"'] & C \arrow[r] & 0. \end{tikzcd}$$

Since f factors through a projective, there is a map $g\colon A \to P$ so that $pg = f$. Since P is a projective Λ-module, there is a map $h\colon P \to X$ so that $\pi_X h = p$.

Thus we get a commutative diagram:

$$\begin{tikzcd} P & A \arrow[l, "g"'] \arrow[d, "f"] \\ X \arrow[u, leftarrow, "h"] \arrow[r, "\pi_X"'] & C \arrow[r] & 0. \end{tikzcd}$$

But this contradicts $\tau_A(X) = \mathrm{Ker}(\pi_X)$ and part (f) is proved.

Finally, to prove (g), assume that C is an indecomposable trace quotient module of X by A. If X is not indecomposable, decompose X to get $X = \coprod_{i=1}^{n} X_i$ for some indecomposable Λ-modules $X_1, \ldots, X_n$. Now

$$\tau_A\left(\coprod_{i=1}^{n} X_i\right) = \coprod_{i=1}^{n} \tau_A(X_i)$$

by (c). It follows that $C = X/\tau_A(X)$ is isomorphic to $\coprod_{i=1}^{n} X_i/\tau_A(X_i)$. Since C is assumed to be indecomposable, we conclude that $\tau_A(X_i) = X_i$ except for exactly one i, say i_0. Then

$$0 \to \tau_A(X_{i_0}) \to X_{i_0} \to C \to 0$$

is exact and nonsplit. Thus C is a trace quotient module of X_{i_0} by A and X_{i_0} is an indecomposable Λ-module. ■

If A and X are Λ-modules we let $P(A, C)$ be the $\mathrm{End}(A)^{\mathrm{op}}$-submodule of $\mathrm{Hom}_\Lambda(A, C)$ consisting of the maps from A to C which factor through a projective Λ-module. In this notation, (f) of Proposition 1.1 can be restated as follows: If C is a trace quotient module by A then $P(A, C) = (0)$.

We introduce some useful terminology. Given two maps f: $X \to C$ and g: $Y \to C$ we say *f lifts to g* if there exists h: $X \to C$ making the following diagram commute:

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[ur, "h"] \arrow[r, "g"'] & C. \end{tikzcd}$$

The next result is concerned with submodules of trace quotient modules.

PROPOSITION 1.2. *If*

$$0 \to \tau_A(X) \to X \xrightarrow{\pi_X} C \to 0$$

is a nonsplit exact sequence of Λ-modules and C' is a submodule of C then either the inclusion map $C' \to C$ lifts to π_X: $X \to C$ or the pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau_A(X) & \longrightarrow & Y & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \text{inclusion}} & & \\ 0 & \longrightarrow & \tau_A(X) & \longrightarrow & X & \xrightarrow[\pi_X]{} & C & \longrightarrow & 0 \end{array}$$

induces a nonsplit short exact sequence $0 \to \tau_A(X) \to Y \to C' \to 0$ and C' is a trace quotient module of Y by A.

Proof. The sequence $0 \to \tau_A(X) \to Y \to C' \to 0$ induced from the pullback of

$$\begin{array}{ccc} & & C' \\ & & \downarrow{\scriptstyle \text{inclusion}} \\ X & \xrightarrow[\pi_X]{} & C \end{array}$$

is nonsplit if and only if inclusion: $C' \to C$ does not lift to π_X. Suppose that $0 \to \tau_A(X) \to Y \to C' \to 0$ is not split. Then $\tau_A(Y) \subseteq \tau_A(X)$ since $Y \subseteq X$. But we see that $\tau_A(X) \subseteq \tau_A(Y)$ since $\tau_A(X) \subseteq Y$ and we conclude that C' is a trace quotient module of Y by A. ■

As an immediate consequence of this result, we have the following.

COROLLARY 1.3. *If*

$$0 \to \tau_A(X) \to X \xrightarrow{\pi_X} C \to 0$$

is a nonsplit exact sequence of Λ-modules and if $\tau_A(X)$ is an essential submodule of X, then each submodule C' of C is a trace quotient module by A.

2. Construction and existence results

We now show that, except for semi-simple artin algebras, there always exist trace quotient modules. Recall that a proper nonzero submodule X' of X is a *waist in X* if every submodule Y of X is either contained in X' or contains X'. Note that if M is a Λ-module of length at least 2 and $M/\text{rad}(\Lambda)M$ is a simple Λ-module, then $\text{rad}(\Lambda)M$ is a waist in M. If M is a Λ-module of length at least 2 and if the socle of M is a simple Λ-module, then $\text{soc}(M)$, the socle of M, is a waist in M. Finally, if U is a uniserial Λ-module of length at least 2 and V is a proper nonzero submodule of U, then V is a waist in U. For a more detailed discussion of modules with waists, see [3], [6].

PROPOSITION 2.1. *Let X be a Λ-module with waist X'. Then $C = X/X'$ is a trace quotient module of X by X'.*

Proof. First note that $\tau_{X'}(X)$ contains X'. Suppose $\tau_{X'}(X) \neq X'$. Then there is a Λ-morphism f: $X' \to X$ such that $f(X') \not\subset X'$. But then it follows that $X' \subsetneqq f(X')$ since X' is a waist in X. This is impossible since the length of $f(X')$ is not greater than the length of X'. It follows that $X' = \tau_{X'}(X)$. Finally, $0 \to X' \to X \to X/X' \to 0$ is not split since X, containing a waist, is indecomposable by [3]. ■

As an immediate consequence, we can prove the claim made earlier in this section.

COROLLARY 2.2. (a) *Every simple nonprojective Λ-module is a trace quotient module by a simple module.*

(b) *If Λ is a Nakayama algebra then every indecomposable nonprojective Λ-module is a trace quotient module by a simple Λ-module.*

Proof. Let S be a simple Λ-module which is not a projective Λ-module. Then there exists a simple Λ-module T such that $\mathrm{Ext}^1_\Lambda(S, T) \neq 0$. Thus there is a nonsplit short exact sequence

$$0 \to T \xrightarrow{a} X \xrightarrow{b} S \to 0.$$

Since $a(T) = \mathrm{soc}(X)$, $a(T)$ is a waist in X and (a) follows.

If Λ is a Nakayama algebra and C is a nonprojective Λ-module, let π: $P \to C$ be a Λ-projective cover of C. Since Λ is a Nakayama algebra, P is a uniserial Λ-module and hence $0 \neq \mathrm{Ker}\,\pi$ is a waist is P. Thus C is a trace quotient module of P by $\mathrm{Ker}\,\pi$. It is immediate that C is a trace quotient module of $P/\mathrm{rad}(\Lambda)\mathrm{Ker}\,\pi$ by $\mathrm{Ker}\,\pi/\mathrm{rad}(\Lambda)\mathrm{Ker}\,\pi$ and that $\mathrm{Ker}\,\pi/\mathrm{rad}(\Lambda)\mathrm{Ker}\,\pi$ is a simple Λ-module.

Our next result has a consequence, the fact that if Λ is a group algebra of a finite group over an algebraically closed field, then there are many indecomposable trace quotient Λ-modules.

PROPOSITION 2.3. *Let Λ be an artin algebra. Let C be a Λ-module such that $\mathrm{Ext}^1_\Lambda(C, C) \neq 0$ and $\mathrm{End}_\Lambda(C)$ is a division ring. Then C is a trace quotient module by C.*

Proof. Let $0 \to C \to X \to C \to 0$ be a nonsplit exact sequence. Then $\tau_C(X) = C$ since any map from C to C is an isomorphism or zero. ■

The next result provides another method of constructing trace quotient modules in some instances. We denote the first syzygy of a Λ-module C by

$\Omega(C)$. That is, $\Omega(C) = \ker(P \to C)$ where $P \to C$ is a Λ-projective cover of C.

PROPOSITION 2.4. *Let C be a Λ-module. Suppose that there is a simple summand S of*

$$\Omega(C)/\mathrm{rad}(\Lambda)\Omega(C)$$

such that $\mathrm{Hom}_\Lambda(S, C) = 0$. *Then C is a trace quotient module of S.*

Proof. Consider the exact sequence

$$0 \to \Omega(C)/\mathrm{rad}(\Lambda)\Omega(C) \xrightarrow{f} P/\mathrm{rad}(\Lambda)\Omega(C) \xrightarrow{g} C \to 0.$$

This sequence is not split since $\mathrm{Im}\, f \subseteq \mathbf{r}(P/\mathbf{r}\Omega(C))$ where $\mathbf{r} = \mathrm{rad}(\Lambda)$. Let u: $\Omega(C)/\mathbf{r}\Omega(C) \to S$ be an epimorphism. Consider the pushout

$$\begin{tikzcd}
0 \arrow[r] & \Omega(C)/\mathbf{r}\Omega(C) \arrow[r, "f"] \arrow[d] & P/\mathbf{r}\Omega(C) \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & S \arrow[r] & X \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

The bottom row is not split since $X/\mathbf{r}X$ is isomorphic to $C/\mathbf{r}C$. Since $\mathrm{Hom}_\Lambda(S, C) = 0$, $\tau_S(X) = S$ and we are done. ∎

We end this section with two more elementary methods of constructing trace quotient modules.

PROPOSITION 2.5. (a) *Suppose that the Loewy length of X is greater than i and X' is the unique maximal submodule of X of Loewy length less than i, then X/X' is a trace quotient module of X by X'.*

(b) *If X' is the socle of X then X/X' is a trace quotient module of X by X'.*

Proof. To prove (a), note that if f: $X' \to X$ then $f(X')$ has Loewy length $\leq i$ and hence $f(X') \subseteq X'$. Thus $\tau_{X'}(X) = X'$ and the result follows. To prove (b), note that if f: $X' \to X$ then $f(X')$ is semisimple and hence $f(X') \subseteq$ socle of $X = X'$. We conclude that $\tau_{X'}(X) = X'$ and again the result follows. ∎

3. Further construction and existence results

As we have seen in §1, if C is a trace quotient module by A then $P(A,C) = 0$ by Proposition 1.1 (f). Assuming $P(A, C) = 0$ we find necessary and sufficient conditions on Λ-modules A and C so that C is a trace quotient module by A. Our first result deals with the case when $\operatorname{Hom}_\Lambda(A, C) = 0$. We assume the reader is familiar with the dual of the transpose, Dtr, where D: $\operatorname{mod}(\Lambda^{\mathrm{op}}) \to \operatorname{mod}(\Lambda)$ is the usual artin algebra duality and tr: $\operatorname{mod}(\Lambda) \to \operatorname{mod}(\Lambda^{\mathrm{op}})$ is the transpose. We refer the reader to [8] for definitions and basic results about these functors and about almost split sequences.

THEOREM 3.1. *Given a Λ-module A and a nonprojective indecomposable Λ-module C such that* $\operatorname{Hom}_\Lambda(A, C) = (0)$ *then the following statements are equivalent.*

(a) *C is a trace quotient module by A.*
(b) $\operatorname{Hom}_\Lambda(A, \operatorname{Dtr}(C)) \neq (0)$.
(c) $\tau_A(\operatorname{Dtr}(C)) \neq 0$.

Proof. Let

$$(*) \qquad 0 \to \operatorname{Dtr} C \to E \to C \to 0$$

be the almost split sequence for C. If $\operatorname{Hom}_\Lambda(A, \operatorname{Dtr}(C)) \neq 0$ then $\tau_A(E) \neq 0$. Since $\operatorname{Hom}_\Lambda(A, C) = 0$, we conclude that $\tau_A(E)$ is contained in $\operatorname{Dtr}(C)$. Since $(*)$ is an almost split sequence, we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \tau_A(E) \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \operatorname{Dtr}(C) \arrow[r] & E \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

It is easy to check that $\tau_A(E) = \tau_A(E')$ and since $E' \to C$ is not a split epimorphism, we conclude that C is a trace quotient module by A. We have shown that (b) $\Rightarrow$ (a).

Next we assume that C is a trace quotient module by A. We have a nonsplit sequence

$$0 \to \tau_A(X) \to X \xrightarrow{\pi_X} C \longrightarrow 0.$$

From the properties of an almost split sequence, we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \tau_A(X) \arrow[r] \arrow[d, "u"'] & X \arrow[r, "\pi_X"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \operatorname{Dtr} C \arrow[r] & E \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

with $u \neq 0$. Thus $\tau_A(\operatorname{Dtr} C) \neq 0$ since $u(\tau_A(X)) \subseteq \tau_A(\operatorname{Dtr} C) \neq 0$. We conclude that (a) $\Rightarrow$ (c).

Finally, if $\tau_A(\operatorname{Dtr} C) \neq 0$ we immediately conclude that $\operatorname{Hom}_\Lambda(A, \operatorname{Dtr}(C)) \neq 0$ and this completes the proof. ■

We now turn our attention to the case where A and C are Λ-modules with the property that $P(A, C) = 0$ but $\operatorname{Hom}_\Lambda(A, C) \neq 0$ and determine necessary and sufficient conditions for C to be a trace quotient module by A. The answer is a bit more complicated than the case $\operatorname{Hom}_\Lambda(A, C) = 0$, which was handled in the above theorem. We now need to use results from [1], [2] and we briefly recall some definitions from [2] that are used in this paper.

Let A and C be Λ-modules and let Γ denote the ring $\operatorname{End}_\Lambda(C)^{\text{op}}$. There is an exact sequence

$$0 \to K_0 \to B_0 \xrightarrow{f_0} C,$$

called the *sequence associated to the submodule* (0) *of* $\operatorname{Hom}_\Lambda(A, C)$, having the following properties:

(i) The map f_0 is right minimal; i.e., no direct summand of B_0 is contained in K_0.
(ii) The Γ-image of $\operatorname{Hom}_\Lambda(A, f_0)$: $\operatorname{Hom}_\Lambda(A, B_0) \to \operatorname{Hom}_\Lambda(A, C)$ is (0).
(iii) The map f_0 is A-determined; i.e., if f: $X \to C$ has the property that the image of $\operatorname{Hom}_\Lambda(A, g)$: $\operatorname{Hom}_\Lambda(A, X) \to \operatorname{Hom}_\Lambda(A, C)$ is zero, then f can be lifted to f_0.

The basic properties of the sequences associated to (0) in $\operatorname{Hom}_\Lambda(A, C)$ are developed in [2] and we will use some of them here.

Let

$$0 \to K_0 \to B_0 \xrightarrow{f_0} C$$

be the sequence associated to (0) in $\operatorname{Hom}_\Lambda(A, C)$. Since $\operatorname{Hom}_\Lambda(A, f_0)$ is the zero map, it follows that $\tau_A(B_0)$ is contained in K_0. Since we are assuming that $P(A, C) = 0$, we have that f_0 is a surjection [1]. In particular, we get a short exact sequence

$$0 \to K_0 \to B_0 \xrightarrow{f_0} C \to 0.$$

We denote this sequence by ϵ_0 and let $[\epsilon_0]$ denote its class in $\operatorname{Ext}^1_\Lambda(C, K_0)$.

THEOREM 3.2. *Let A and C be Λ-modules such that* $\operatorname{Hom}_\Lambda(A, C) \neq 0$ *and* $P(A, C) = 0$. *Then C is a trace quotient module by A if and only if the inclusion map* μ: $\tau_A(B_0) \to K_0$ *has the property that* $[\epsilon_0]$ *is in the image of* $\operatorname{Ext}^1_\Lambda(C, \mu)$: $\operatorname{Ext}^1_\Lambda(C, \tau_A(B_0)) \to \operatorname{Ext}^1_\Lambda(C, K_0)$.

Proof. First suppose that C is a trace quotient module by A. Let

$$0 \to \tau_A(X) \to X \xrightarrow{\pi_X} C \to 0$$

be a nonsplit exact sequence. Since $\operatorname{Hom}_\Lambda(A, \pi_X)$: $\operatorname{Hom}_\Lambda(A, X) \to \operatorname{Hom}_\Lambda(A, C)$ is the zero map, by Proposition 1.1 (d), we get a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \tau_A(X) \arrow[r] \arrow[d, "u"'] & X \arrow[r] \arrow[d, "v"'] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_0 \arrow[r] & B_0 \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Now u: $\tau_A(X) \to K_0$ lifts to μ: $\tau_A(B_0) \to K_0$ since $\operatorname{Im}(\mu)$ is contained in $\tau_A(B_0)$. Thus $[\epsilon_0]$ is in the image of $\operatorname{Ext}_\Lambda(C, \mu)$ since $[\epsilon_0]$ is in the image of $\operatorname{Ext}_\Lambda(C, u)$ and $u = \mu u'$ for some u': $\tau_A(X) \to \tau_A(B_0)$.

Next assume that $[\epsilon_0]$ in in the image of $\operatorname{Ext}_\Lambda(C, \mu)$. Then we get a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \tau_A(B_0) \arrow[r] \arrow[d, "\mu"'] & B' \arrow[r] \arrow[d, "v"'] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_0 \arrow[r] & B_0 \arrow[r, "f_0"] & C \arrow[r] & 0. \end{tikzcd}$$

Since ϵ_0 is not split, it follows that $0 \to \tau_A(B_0) \to B' \to C \to 0$ is not split. Finally, it is easy to check that $\tau_A(B') = \tau_A(B_0)$ and hence C is a trace quotient module of B' by A. ∎

We end this section with an easy consequence of the above result. We recall some notation from [2]. Let A be a Λ-module. There is a one-to-one correspondence between the simple $\operatorname{End}_\Lambda(A)^{\mathrm{op}}$-modules and indecomposable summands of A, given as follows: corresponding to indecomposable summand, A' of A is the simple $\operatorname{End}_\Lambda(A)^{\mathrm{op}}$-module S where S is isomorphic to

$$\operatorname{Hom}_\Lambda(A, A')/\operatorname{rad}\left(\operatorname{End}_\Lambda(A)^{\mathrm{op}}\right)\operatorname{Hom}_\Lambda(A, A').$$

Given a simple $\operatorname{End}_\Lambda(A)^{\mathrm{op}}$-module S we denote a fixed summand corresponding to S by A_S. For further details see [2].

COROLLARY 3.3. *Let A and C be Λ-modules satisfying $P(A, C) = 0$ and $\operatorname{Hom}_\Lambda(A, C) \neq 0$. Suppose that the $\operatorname{End}_\Lambda(A)^{\mathrm{op}}$-socle of $\operatorname{Hom}_\Lambda(A, C)$ is $\coprod_{i=1}^n m_i S_i$ with S_i nonisomorphic simple $\operatorname{End}_\Lambda(A)^{\mathrm{op}}$-modules.*

(a) *If C is a trace quotient module by A then $\operatorname{Hom}_\Lambda(A, \operatorname{Dtr}(\coprod_{i=1}^n A_{S_i})) \neq 0$.*

(b) *If $\tau_A(\operatorname{Dtr}(A)) = \operatorname{Dtr} A$ then C is a trace quotient module by A.*

Proof. Suppose that C is a trace quotient module by A. Let $\mu\colon \tau_A(B_0) \to K_0$ be the inclusion map. Then $\mu \neq 0$ since $[\epsilon_0]$ is in the image of $\operatorname{Ext}^1_\Lambda(C, \mu)$. Thus $\operatorname{Hom}_\Lambda(\tau_A(B_0), K_0) \neq 0$. By [1], $K_0 = \coprod_{i=1}^n m_i \operatorname{Dtr}(A_{S_i})$. Since there is an epimorphism $\coprod A \to \tau_A(B_0)$, we conclude that

$$Hom_\Lambda\left(A, \textstyle\coprod_{i=1}^n Dtr\left(A_{S_i}\right)\right) \neq 0.$$

Thus part (a) follows. Next, suppose that $\tau_A(\operatorname{Dtr}(A)) = \operatorname{Dtr} A$. Since K_0 is a direct sum of Λ-modules of the form $\operatorname{Dtr}(A_i)$ where the A_i's are summands of A, we conclude that $\tau_A(K_0) = K_0$. But $\tau_A(B_0) = \tau_A(K_0) = K_0$ and hence μ is an isomorphism. Thus $\operatorname{Ext}^1_\Lambda(C, \mu)$ is an isomorphism and we get the desired result. ■

4. Relationship to Dtr C

In the last section, given Λ-modules A and C, we found necessary and sufficient conditions for C to be a trace quotient module by A. In this section, fix a Λ-module C and try to determine if it is a trace quotient module by some Λ-module. We begin this investigation by constructing a unique submodule of $\operatorname{Dtr}(C)$ which plays a central role in the following discussion. For the remainder of the section, C will denote an indecomposable nonprojective Λ-module.

Consider the following sequence of submodules of $\operatorname{Dtr}(C)$ defined inductively. Let

$$L_0 = \operatorname{Dtr}(C) \quad \text{and} \quad L_1 = \bigcap_{f\colon \operatorname{Dtr}(C) \to C} \operatorname{Ker} f.$$

Having defined L_{n-1}, define

$$L_n = \bigcap_{f\colon L_{n-1} \to C} \operatorname{Ker} f.$$

In this fashion, we get a chain of submodules of $\operatorname{Dtr}(C)$, namely

$$\operatorname{Dtr}(C) = L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots \supseteq L_n \supseteq \cdots.$$

We define $\sigma(C)$ to be the smallest nonzero L_i. Of course, $\sigma(C)$ exists since $\operatorname{Dtr}(C)$ is of finite length. The next result introduces the two cases we will be interested in.

LEMMA 4.1. *Let C be an indecomposable nonprojective Λ-module. Then either there is a monomorphism from $\sigma(C)$ into a finite direct sum of copies of C or* $\operatorname{Hom}_\Lambda(\sigma(C), C) = 0$.

Proof. Suppose that $\sigma(C) = L_n$. Then either $L_{n+1} = L_n$ or $L_{n+1} = 0$. If $L_{n+1} = 0$ then

$$\bigcap_{f\colon L_n \to C} \operatorname{Ker} f = 0$$

and, since L_n is of finite length, we get a monomorphism $L_n\colon \to \coprod_n C$ for some $n > 0$. On the other hand, if $L_{n+1} = L_n$ then

$$\bigcap_{f\colon L_n \to C} \operatorname{Ker} f = L_n.$$

Thus, $\operatorname{Hom}_\Lambda(L_n, C) = 0$ and we are done. ∎

This result shows that indecomposable nonprojective Λ-modules fall into one of two classes: either $\operatorname{Hom}_\Lambda(\sigma(C), C)$ is zero or not. The next result relates these cases to the question of whether or not C is a trace quotient module.

PROPOSITION 4.2. *Let C be a nonprojective indecomposable Λ-module.*
(a) *If $\operatorname{Hom}_\Lambda(\sigma(C), C) = 0$ then C is a trace quotient module by $\sigma(C)$.*
(b) *If $\operatorname{Hom}_\Lambda(\sigma(C), C) \neq 0$ and C is a trace quotient module by Λ-module A, then*

$$\operatorname{Hom}_\Lambda(A, C) \neq 0, \quad \operatorname{Hom}_\Lambda(A, \operatorname{Dtr}(C)) \neq 0 \quad \textit{and} \quad \operatorname{Hom}_\Lambda(A, \operatorname{Dtr}(A)) \neq 0.$$

Proof. Let

$$0 \to \operatorname{Dtr}(C) \to E \to C \to 0$$

be an almost split sequence. First suppose that $\operatorname{Hom}_\Lambda(\sigma(C), C) = 0$. Since $\sigma(C)$ is a nonzero submodule of $\operatorname{Dtr}(C)$, there is a commutative diagram with row exact, nonsplit sequences:

$$\begin{tikzcd}
0 \arrow[r] & \sigma(C) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Dtr}(C) \arrow[r] & E \arrow[r] & C \arrow[r] & 0 .
\end{tikzcd}$$

Since $\operatorname{Hom}_\Lambda(\sigma(C), C) = 0$, $\tau_{\sigma(C)}(X) = \sigma(C)$ and we conclude that C is a trace quotient module of X by $\sigma(C)$.

For the second part of the result, assume that $\operatorname{Hom}_\Lambda(\sigma(C), C) \neq 0$ and C is a trace quotient module by A. Let

$$0 \to \tau_A(X) \to X \xrightarrow{\pi_X} C \to 0$$

be a nonsplit exact sequence of Λ-modules. Then, by the mapping properties of an almost split sequence, we get a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \tau_A(X) \arrow[r] \arrow[d, "u"'] & X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Dtr}(C) \arrow[r] & E \arrow[r] & C \arrow[r] & 0. \end{tikzcd}$$

Since neither sequence is split, $u \neq 0$ and we see that

$$\mathrm{Hom}_\Lambda(A, \mathrm{Dtr}(C)) \neq 0 \quad \text{and} \quad \tau_A(E) \neq 0.$$

If $\mathrm{Hom}_\Lambda(A, C) = 0$ then $\mathrm{Hom}_\Lambda(\tau_A(\mathrm{Dtr}(C)), C) = 0$ and hence $\tau_A(\mathrm{Dtr}(C)) \subseteq L_i$ for all i. This contradicts $\mathrm{Hom}_\Lambda(\sigma(C), C) \neq 0$ since there is a monomorphism $\sigma(C) \to \coprod C$. We have shown that $\mathrm{Hom}_\Lambda(A, C) \neq 0$. Next, since C is a trace quotient module by A, $P(A, C) = 0$ and so we have a short exact sequence

$$0 \to K_0 \to B_0 \xrightarrow{f_0} C \to 0$$

associated to the $\mathrm{End}_\Lambda(A)^{\mathrm{op}}$-submodule (0) of $\mathrm{Hom}_\Lambda(A, C)$. Since $\mathrm{Hom}_\Lambda(A, \pi_X) = 0$, we get a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \tau_A(X) \arrow[r] \arrow[d, "v"'] & X \arrow[r, "\pi_X"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_0 \arrow[r] & B_0 \arrow[r, "f_0"] & C \arrow[r] & 0. \end{tikzcd}$$

Again we have $v \neq 0$ and, since K_0 is a summand of a direct sum of copies of $\mathrm{Dtr}(A)$, we conclude that $\mathrm{Hom}_\Lambda(A, \mathrm{Dtr}(A)) \neq 0$. This completes the proof. ∎

The above result shows that if $\mathrm{Hom}_\Lambda(\sigma(C), C) = 0$ then C is a trace quotient module by some Λ-module and if $\mathrm{Hom}_\Lambda(\sigma(C), C) \neq 0$ then we get necessary conditions for C to be a trace quotient module by some Λ-module. We point out that in general, there are examples of modules C with $\mathrm{Hom}_\Lambda(\sigma(C), C) \neq 0$, some of which are trace quotient modules and others of which are not. As we see in the next result, if a module C has the property that $\mathrm{Hom}_\Lambda(\sigma(C), C) \neq 0$ then it has an unusual mapping property relative to all Λ-modules.

THEOREM 4.3. *Let C be a nonprojective indecomposable Λ-module.*

(a) *There is a Λ-module A such that C is a trace quotient module by A with $\mathrm{Hom}_\Lambda(A, C) = 0$ if and only if C is a trace quotient by $\sigma(C)$ and $\mathrm{Hom}_\Lambda(\sigma(C), C) = 0$.*

(b) *If* $\operatorname{Hom}_\Lambda(\sigma(C), C) \neq 0$ *then for each* Λ*-module* A, $\operatorname{Hom}_\Lambda(A, C) = 0$ *implies that*

$$\operatorname{Hom}_\Lambda(A, \operatorname{Dtr}(C)) = 0.$$

Proof. Let $0 \to \operatorname{Dtr}(C) \to E \to C \to 0$ be an almost split sequence. To prove part (a), suppose that C is a trace quotient module by A, for some Λ-module A with $\operatorname{Hom}_\Lambda(A, C) = 0$. As usual, we get a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \tau_A(X) \arrow[r] \arrow[d, "u"'] & X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \operatorname{Dtr}(C) \arrow[r] & E \arrow[r] & C \arrow[r] & 0. \end{tikzcd}$$

The morphism $u \neq 0$ and hence, if we let $Z = \tau_A(\operatorname{Dtr}(C))$ we see that $Z \neq 0$. By assumption $\operatorname{Hom}_\Lambda(A, C) = 0$ and hence $\operatorname{Hom}_\Lambda(Z, C) = 0$. We see that $Z \subseteq \operatorname{Ker}(f\colon \operatorname{Dtr}(C) \to C)$ for all f and so $Z \subseteq L_1$. Next we see that $Z \subseteq \operatorname{Ker}(f\colon L_1 \to C)$ for all f. Continuing in this fashion, we conclude, by induction, that $Z \subseteq L_i$ for all i. But then $Z \subseteq \sigma(C) = L_n$ and $Z \subseteq L_{n+1}$. As a result, we have that $L_n = L_{n+1}$ and $\operatorname{Hom}_\Lambda(\sigma(C), C) \neq 0$. Part (a) follows.

To prove part (b), suppose that

$$\operatorname{Hom}_\Lambda(\sigma(C), C) \neq 0 \quad \text{and} \quad \operatorname{Hom}_\Lambda(A, C) = 0.$$

If $\operatorname{Hom}_\Lambda(A, \operatorname{Dtr}(C)) \neq 0$ then $\tau_A(\operatorname{Dtr}(C)) \neq 0$ and yet $\operatorname{Hom}_\Lambda(\tau_A(\operatorname{Dtr}(C)), C) = 0$. By the induction argument given above applied to $\tau_A(\operatorname{Dtr}(C))$, we see that $\tau_A(\operatorname{Dtr}(C)) \subseteq L_n$ for all n. Hence we cannot embed $\sigma(C)$ in a direct sum of copies of C since $\operatorname{Hom}_\Lambda(\tau_A(\operatorname{Dtr}(C)), C) = 0$. Thus

$$\operatorname{Hom}_\Lambda(\sigma(C), C) = 0,$$

which is a contradiction and we are done. ■

We have the following interesting consequences of the above result.

COROLLARY 4.4. *Let C be an indecomposable nonprojective Λ-module. Suppose that*

$$0 \to \operatorname{Dtr}(C) \to \coprod_{i=1}^{n} E_i \to C \to 0$$

is an almost split sequence with the property that each E_i is an indecomposable

Λ-module of length less than the length of C. Then $\sigma(C) = \mathrm{Dtr}(C)$. In particular, for all Λ-modules A, $\mathrm{Hom}_\Lambda(A, C) = 0$ implies $\mathrm{Hom}_\Lambda(A, \mathrm{Dtr}(C)) = 0$.

Proof. Note that irreducible maps must be either monomorphisms or epimorphisms. Since each E_i has length less than the length of C, the induced irreducible maps $E_i \to C$ are all monomorphisms. If we compose these maps, we get a monomorphism $\coprod_{i=1}^n E_i \to \coprod_{i=1}^n C$. It follows that $\mathrm{Dtr}(C)$ embeds in a direct sum of copies of C. The result now follows from the definition of $\sigma(C)$ ∎

The proof of the next result is omitted since it is similar to the proof of the last corollary.

COROLLARY 4.5. *Let C be an indecomposable Λ-module with $\mathrm{Hom}_\Lambda(\sigma(C), C) = 0$. If*

$$0 \to \mathrm{Dtr}(C) \to \coprod_{i=1}^n D_i \to C \to 0$$

is any nonsplit sequence, then at least one of the induced maps $D_i \to C$ is not a monomorphism.

Our next result gives another description of $\sigma(C)$ in the case when $\mathrm{Hom}_\Lambda(\sigma(C), C) = 0$. Let

$$\mathbf{X} = \{A \in \mathrm{mod}(\Lambda) \mid \mathrm{Hom}_\Lambda(A, C) = 0\}.$$

PROPOSITION 4.6. $\mathrm{Hom}_\Lambda(\sigma(C), C) = 0$ *if and only if $\sigma(C)$ is the submodule of $\mathrm{Dtr}(C)$ generated by $\{\tau_A(\mathrm{Dtr}(C))\}_{A \in \mathbf{X}}$*

Proof. Consider the almost split sequence

$$0 \to \mathrm{Dtr}\, C \to E \to C \to 0.$$

Suppose that $\mathrm{Hom}_\Lambda(A, C) = 0$. Then, as we saw earlier in the section, $\tau_A(\mathrm{Dtr}(C)) \subseteq L_n$, for all n. It follows that $\tau_A(\mathrm{Dtr}(C)) \subseteq \sigma(C)$. If $\mathrm{Hom}_\Lambda(\sigma(C), C) \neq 0$ then, for some n, $L_n = 0$. This implies that $\tau_A(\mathrm{Dtr}(C)) = 0$ for all $A \in \mathbf{X}$. From this we see that $\sigma(C)$ is not the submodule generated by $\{\tau_A(\mathrm{Dtr}(C))\}_{A \in \mathbf{X}}$.

On the other hand, if $\mathrm{Hom}_\Lambda(\sigma(C), C) = 0$ then $\sigma(C) \in \mathbf{X}$. This implies that $\sigma(C)$ is contained in the submodule generated the traces of A in $\mathrm{Dtr}(C)$. But we just saw that if $\mathrm{Hom}_\Lambda(A, C) = 0$ then $\tau_A(\mathrm{Dtr}(C)) \subseteq \sigma(C)$ and we are done. ∎

We end this section by considering when a Λ-module C is a trace quotient module by itself.

PROPOSITION 4.7. *Suppose C is an indecomposable Λ-module which is isomorphic to* $\mathrm{Dtr}(C)$. *Let*

$$0 \to K_0 \to B_0 \xrightarrow{f_0} C$$

be the sequence associated with the $\mathrm{End}_\Lambda(C)^{\mathrm{op}}$*-submodule* (0) *of* $\mathrm{Hom}_\Lambda(C, C)$. *Let C_0 denote the image of f_0. Then $\tau_A(B_0) = K_0$ and hence C_0 is a trace quotient module by C.*

Proof. Since C is an indecomposable module, it follows that K_0 is a finite direct sum of copies of $\mathrm{Dtr}(C)$. By assumption, $\mathrm{Dtr}(C)$ is isomorphic to C. Since $\mathrm{Hom}_\Lambda(C, f_0)$ is the zero map, we see that $\tau_C(K_0) = K_0$ and the result follows. ■

As an immediate consequence we have the following.

COROLLARY 4.8. *If C is isomorphic to* $\mathrm{Dtr}(C)$ *and $P(C, C) = 0$ then C is a trace quotient module by C.*

Proof. If

$$0 \to K_0 \to B_0 \xrightarrow{f_0} C$$

is the sequence associated to the submodule (0) in $\mathrm{Hom}_\Lambda(C, C)$, then $P(C, C) = 0$ implies that f_0 is a surjection. Hence the result follows from the previous proposition. ■

5. Applications to representation theory

In this section, we apply the preceding ideas and results to study the category $\mathrm{mod}(\Lambda)$. Our first result indicates that it is rare for all nonprojective indecomposable Λ-modules to be trace quotient modules. The result uses the fact that if C is a trace quotient module then there is an indecomposable module X such that C is a trace quotient module of X. Recall that Λ is said to be of *finite representation type* if there are only a finite number of nonisomorphic indecomposable Λ-modules. If M is a Λ-module, we say that *M has a simple top* if $M/\mathrm{rad}(\Lambda)M$ is a simple Λ-module and we call *$M/\mathrm{rad}(\Lambda)M$ the top of M.*

PROPOSITION 5.1. *If Λ is of finite representation type and every nonprojective indecomposable Λ-module is a trace quotient module then every indecomposable Λ-module has a simple top.*

Proof. Assume that Λ is of finite representation type and every nonprojective indecomposable Λ-module is a trace quotient module. Let C be an indecomposable nonprojective Λ-module. We show that C has a simple top. Since C is the trace quotient module of some indecomposable Λ-module X_1 (Proposition 1.1 (g)), we have a nonsplit surjection π_{X_1}: $X_1 \to C$. If X_1 is a projective Λ-module then we are done. If X_1 is not a projective module then it is also a trace quotient module of an indecomposable Λ-module X_2. Again we have a nonsplit surjection π_{X_2}: $X_2 \to X_1$. We get a sequence of nonsplit epimorphisms

$$X_2 \xrightarrow{\pi_{X_2}} X_1 \xrightarrow{\pi_{X_1}} C.$$

If X_2 is a projective λ-module then we are done. Since Λ is of finite representation type this process must stop. That is, there is a sequence of nonsplit epimorphisms

$$X_n \to X_{n-1} \to \cdots \to X_1 \to C$$

such that no indecomposable Λ-module maps onto X_n. It follows that X_n is an indecomposable projective Λ-module and the proof is complete. ■

Although the above result is a special case of the next result, it was included because the next section is devoted to investigating the converse. We give necessary and sufficient conditions on an artin algebra Λ so that every indecomposable Λ-module is a trace quotient module. Before starting the next result we recall the definition of the preprojective partition of Λ. Let ind(Λ) be the full subcategory of mod(Λ) whose objects consist of one indecomposable module from each isomorphism class of indecomposable Λ-modules. The *preprojective partition of* Λ consists of a disjoint collection of objects of ind(Λ), $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n, \ldots, \mathbf{P}_\infty$ satisfying the following properties:

(i) For $0 \leq n < \infty$, $\mathbf{P}_n$ is finite.
(ii) $\bigcup_{n=1}^{\infty} \mathbf{P}_n = \text{ind}(\Lambda)$.
(iii) if $i \neq j$, then $\mathbf{P}_i \cap \mathbf{P}_j = \phi$.
(iv) for $0 \leq n < \infty$, if C is an object in ind(Λ) $-$ $\{\mathbf{P}_0 \cup \cdots \cup \mathbf{P}_n\}$ then there exist not necessarily distinct objects $X_1, \ldots, X_m$ of $\mathbf{P}_{n+1}$ such that there is an epimorphism from $\coprod_{i=1}^{m} X_i$ onto C.
(v) Given $\mathbf{P}_0, \ldots, \mathbf{P}_n$, then $\mathbf{P}_{n+1}$ is minimal with respect to property (v).

Preprojective partitions were introduced in [5] by Auslander and Smalø, and we refer the reader there for more information. In particular, they prove the existence and uniqueness of preprojective partitions in that paper. We say an indecomposable Λ-module is a *preprojective module* if it is isomorphic to an object in some $\mathbf{P}_n$ for $0 \leq n < \infty$. Note that $C \in \operatorname{ind}(\Lambda)$ is projective if and only if $C \in \mathbf{P}_0$.

THEOREM 5.2. *Let Λ be an artin algebra. If each nonprojective preprojective Λ-module is a trace quotient module then Λ is of finite representation type. In fact, every indecomposable Λ-module has a simple top.*

Proof. Let $\mathbf{P}_0, \ldots, \mathbf{P}_n, \ldots, \mathbf{P}_\infty$ be the preprojective partition of Λ. Let C be a preprojective nonprojective Λ-module. First we show that C has a simple top. Since C is a trace quotient module there is an indecomposable Λ-module X_1 with length greater than the length of C, which maps epimorphically onto C; say

$$X_1 \xrightarrow{f_1} C \to 0.$$

Suppose that $C \in \mathbf{P}_i$. By the definition of preprojective partition, we see that $X_1 \in \mathbf{P}_{j_1}$ with $j_1 < i$. If X_1 is a projective Λ-module then C has a simple top. If X_1 is not a projective Λ-module, it is a trace quotient module. Thus there is a nonsplit epimorphism $f_2\colon X_2 \to X_1$ for some indecomposable Λ-module X_2. Again we conclude that $X_2 \in \mathbf{P}_{j_2}$, with $j_2 < j_1$. Continuing in this fashion, we get a sequence of nonsplit epimorphisms

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \to X_1 \xrightarrow{f_1} C$$

where each X_l is an indecomposable module and $X_l \in \mathbf{P}_{j_l}$, with $j_1 > j_2 > \cdots > j_n$. Thus we may assume that $X_n \in \mathbf{P}_0$; that is, X_n is an indecomposable projective Λ-module. Thus C has a simple top.

We have shown that each indecomposable preprojective Λ-module has a simple top. It follows that there is a bound on the lengths of the indecomposable preprojective Λ-modules. By the Harada-Sai Theorem [4], [7], if there is a bound on the lengths of a set $\{M_1, \ldots, M_r\}$ then there is a bound on the lengths of sequences of maps

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to \cdots \xrightarrow{f_t} M_t$$

having nonzero composition. Thus the proof would be complete if we could show that if Λ is of infinite representation type then there are arbitrarily large nonzero compositions of maps between indecomposable preprojective modules.

Assume that Λ is of infinite representation type. Then for $i \geq 0, \mathbf{P}_i \neq \phi$ by [5]. Let N be a positive integer and choose $X_n \in \mathbf{P}_N$. Then there exist $Y_i^{N-1} \in \mathbf{P}_{N-1}$ such that there is an epimorphism

$$\coprod_i Y_i^{N-1} \to X_N.$$

Choose one Y_i^{N-1} such that the induced map $Y_i^{N-1} \to X_N$ is not zero. Set $X_{N-1} = Y_i^{N-1}$. Finally suppose we have found $X_{i+1} \to X_{i+2} \to \cdots \to X_N$ with nonzero composition and $X_{i+1} \in \mathbf{P}_{i+1}$. Then there exist $Y_j^i \in \mathbf{P}_i$ so that there is an epimorphism $\coprod_j Y_j^i \to X_{i+1}$ since $X_{i+1} \in \mathbf{P}_{i+1}$. It follows that there is at least one j such that the composition $Y_j^i \to X_{i+1} \to \cdots \to X_N$ is nonzero. Set $X_i = Y_j^i$. In this fashion, we get a nonzero composition $X_1 \to \cdots \to X_N$. Since N was arbitrary the result follows. ■

The next result indicates that for most artin algebras there are indecomposable preprojective Λ-modules which are not trace quotient modules. It uses the fact (see [5]) that if $C \in \mathbf{P}_n$ and X is a Λ-module which is isomorphic to a direct sum is modules in $\bigcup_{i \geq n} \mathbf{P}_i$ then every epimorphism $X \to C$ is a split epimorphism.

COROLLARY 5.3. *Let C be an indecomposable preprojective Λ-module with a nonsimple top. Then there exists an epimorphism $X \to C$ for some indecomposable preprojective Λ-module X which is not a trace quotient module.*

Proof. Let $\mathbf{r}$ denote $\operatorname{rad}(\Lambda)$. Suppose that $C \in \mathbf{P}_n$. We proceed by induction on n. If $n = 0$ then the result is vacuously true since $C/\mathbf{r}C$ is not a simple module. If $n = 1$ then, in fact, C is not a trace quotient module. For, suppose that C were a trace quotient module. There exists a nonsplit epimorphism $P \to C$ for some indecomposable Λ-module P. But then, by the remarks preceeding this corollary, $P \in \mathbf{P}_0$; i.e., P is an indecomposable projective Λ-module. This contradicts the assumption that the top of C is not simple. Thus, if $C \in \mathbf{P}_1$ then C is not a trace quotient module and we may take $X = C$. Now suppose that $C \in \mathbf{P}_n$. Then if C is not a trace quotient module we may take $X = C$. If C is a trace quotient module, then there is a nonsplit epimorphism $C' \to C$ where C' is an indecomposable Λ-module. It follows that $C' \in \mathbf{P}_i$ for some $0 \leq i < n$ and $C'/\mathbf{r}C'$ is not a simple Λ-module since $C/\mathbf{r}C$ is not a simple Λ-module. Thus, by induction, there is an epimorphism $X \to C'$ where X is an indecomposable Λ-module which is not a trace quotient module. The result follows using the composition $X \to C' \to C$. ■

Note that we have shown that an indecomposable Λ-module $C \in \mathbf{P}_1$ either has a simple top or is not a trace quotient module. This remark has an interesting consequence.

THEOREM 5.4. *Suppose that* $\mathrm{Dtr}(C)$ *is a summand of* $\mathbf{r}$, *the radical of* Λ. *Then C is not a trace quotient module if $C/\mathbf{r}C$ is not a simple Λ-module. In this case we have that* $\mathrm{Hom}_\Lambda(\sigma(C), C) \neq 0$ *and if* $\mathrm{Hom}_\Lambda(A, C) = 0$ *then* $\mathrm{Hom}_\Lambda(A, \mathrm{Dtr}(C)) = 0$

Proof. It suffices to show that C is not a trace quotient module. But $\mathrm{tr}\, D(C)$ is a summand of $\mathrm{tr}\, D(\mathbf{r})$. By [5], the objects in $\mathbf{P}_1$ are summands of direct sums of copies of $\mathrm{tr}\, D(\mathbf{r})$. Thus we conclude $C \in \mathbf{P}_1$ and the result follows. ■

6. A classification result

We have seen that if every nonprojective indecomposable module is a trace quotient module then every indecomposable module has a simple top. In this section, we find necessary and sufficient conditions on Λ so that every nonprojective indecomposable Λ-module is a trace quotient module. Note that if Λ is an artin algebra with the property that every indecomposable Λ-module has a simple top then (a) every indecomposable injective Λ-module is uniserial and (b) if $\mathbf{r} = \mathrm{rad}(\Lambda)$ and P is an indecomposable projective Λ-module, then $\mathbf{r}P$ is a sum of at most two uniserial module [4]. This type of ring was first studied by Tachikawa [9]. Before stating the main classification result, we note that at the end of this section we provide an example of an artin algebra, each of whose indecomposable modules has a simple top and yet has a module which is not a trace quotient module.

THEOREM 6.1. *Let Λ be an artin algebra such that each indecomposable Λ-module has a simple top. Then the following statements are equivalent*:

(a) *Every nonprojective indecomposable Λ-module is a trace quotient module by a simple Λ-module.*
(b) *Every nonprojective indecomposable Λ-module is a trace quotient module.*
(c) *If P is an indecomposable projective Λ-module with $\mathbf{r}P = U_1 \amalg U_2$, with each U_i a uniserial module, then for $i \neq j$ the socle of U_i is not a composition factor of U_j.*
(d) *If P is an indecomposable projective Λ-module with $\mathbf{r}P = U_1 \amalg U_2$, with each U_i a uniserial module, then*

$$\{S \mid S \textit{ is a composition factor of } U_1\} \cap \{T \mid T \textit{ is a composition factor of } U_2\} = \phi.$$

Proof. Clearly (a) $\Rightarrow$ (b). We show that (b) $\Rightarrow$ (c). Suppose that every nonprojective indecomposable Λ-module is a trace quotient module. Let P be an indecomposable projective Λ-module such that $\mathbf{r}P = U_1 \amalg U_2$ with each U_i a uniserial Λ-module. Let V be a nonzero submodule of U_1. It suffices to show that the top of V is not isomorphic to $\mathrm{soc}(U_2)$. Let $C = P/V$. Since C is a trace quotient module, there is a nonsplit exact sequence

$$0 \to \tau_A(X) \to X \xrightarrow{\pi_X} C \to 0$$

with X an indecomposable Λ-module. Then the top of X is isomorphic to the top of C and we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \tau_A(X) \arrow[r] & X \arrow[r, "\pi"] & C \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, "u"] & P \arrow[r] \arrow[u, "v"] & C \arrow[r] \arrow[u, equal] & 0 .
\end{tikzcd}$$

Note that V is a uniserial Λ-module and v is an epimorphism. Since the top of V is a simple Λ-module and V maps onto $\tau_A(X)$ we see that the top of A maps onto the top of V. Next, note that $X = P/\mathrm{Ker}(v) = P/\mathrm{Ker}(u)$ and hence we have that $\mathrm{Ker}(v) \subseteq V \subseteq U_1$. From this we see that there is a nonzero composition $\mathrm{soc}(U_2) \to X \to C$. If $\mathrm{top}(V)$ is isomorphic to $\mathrm{soc}(U_2)$ then we get a nonzero composition

$$A \xrightarrow{\text{onto}} \mathrm{top}(V) \xrightarrow{\sim} \mathrm{soc}(U_2) \hookrightarrow X \xrightarrow{f_X} C.$$

This contradicts $\tau_A(X) = \mathrm{Ker}\, f_X$. We conclude that the top of V is not isomorphic to the socle of U_2 and part (c) follows.

Next, we show that part (c) implies part (d). Suppose that given an indecomposable projective Λ-module P with $\mathbf{r}P = U_1 \amalg U_2$ then $\mathrm{soc}(U_j)$ is not a composition factor of U_i for $i \neq j$. Suppose that U_1 and U_2 have a composition factor, say S, in common. For notational convenience, we list the composition factors of U_1 as $V_1, \ldots, V_n$ and those of U_2 as $W_1, \ldots, W_m$, where $\mathrm{soc}(U_1) = V_n$, $\mathrm{soc}(U_1/V_n) = V_{n-1}, \ldots,$ and $\mathrm{soc}(U_2) = W_m, \mathrm{soc}(U_2/W_m) = W_{m-1}, \ldots$. By assumption, we may choose $S = V_i = W_j$, with $i < n$, $j < m$ and $V_{i+1} \neq W_{j+1}$. Set $V = V_{i+1}$ and $W = W_{j+1}$. Note that the Λ-injective envelope, E, of S is a uniserial module. Hence, if we let T denote the socle of E/S, there are two uniserial Λ-modules X and Y such that the socle of X is V, the socle of X/V is S, the top of X is T, the socle of Y is W, the socle of Y/W is S and the top of Y is T. By the uniseriality of E we see that X/V is isomorphic to Y/W. Let C be the pullback of

$$\begin{tikzcd}
X \arrow[r] & X/V \\
 & Y . \arrow[u]
\end{tikzcd}$$

It follows that C is an indecomposable Λ-module of length 4, Loewy length 3 and socle of length 2. The top of C is isomorphic to T. Let Q be the Λ-projective cover of T. Then there is an epimorphism, $Q \to C$ which induces an epimorphism $Q/\mathbf{r}^3Q \to C$. Since $\mathbf{r}^2C = V \amalg W$, we see that Q is not a uniserial Λ-module. Thus, $\mathbf{r}Q$ is a direct sum of two uniserial Λ-modules. From this it follows that $Q/\mathbf{r}^3Q$ has length 5 since $\mathbf{r}^2Q/\mathbf{r}^3Q$ has length 2. Next we note that the map $Q/\mathbf{r}^3Q \to C$ is a monomorphism when restricted to the socle of $Q/\mathbf{r}^3Q$. This contradicts the fact that the length of $Q/\mathbf{r}^3Q$ is greater than the length of C. From this, part (d) follows.

Finally, assume that if P is an indecomposable projective Λ-module and if $\mathbf{r}P = U_1 \amalg U_2$ then there are no composition factors of U_1 and U_2 in common. Let C be a nonprojective indecomposable Λ-module. Let $f\colon P \to C$ be a Λ-projective cover of C. If $\mathbf{r}P$ is a uniserial module then C is uniserial module which is not projective. Applying the results of section 2, we see that C is a trace quotient module by a simple module since, in this case, the socle of $P/\mathbf{r}\operatorname{Ker} f$ is a waist is $P/\mathbf{r}\operatorname{Ker} f$. Thus we may assume that $\mathbf{r}P$ is a direct sum of two uniserial modules U_1 and U_2. Let $K = \operatorname{Ker} f$. Since C is not a projective Λ-module, $\operatorname{Ker} f \neq 0$ and thus $K \cap \operatorname{soc}(P) \neq 0$. By assumption, $\operatorname{soc}(U_1) \neq \operatorname{soc}(U_2)$ and so we may assume that $\operatorname{soc}(U_1) \subseteq K$. Since U_1 and U_2 have distinct composition factors from one another, it follows that

$$K = (K \cap U_1) \amalg (K \cap U_2).$$

First assume that $K \cap U_1 = U_1$. Then C is a uniserial Λ-module. If C is simple module, then C is a trace quotient module by any simple summand of $\mathbf{r}P/\mathbf{r}^2P$. If C is not a simple module, then it follows by the disjointness of the composition factors of U_1 and U_2 that the top of U_1 is not isomorphic to the socle of C. We get a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "u"'] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{top}(U_1) \arrow[r] & X \arrow[r] & C \arrow[r] & 0\,.
\end{tikzcd}$$

The bottom row is a nonsplit exact sequence and $\operatorname{Hom}_\Lambda(\operatorname{top}(U_1), C) = 0$. We conclude that C is a trace quotient module by $\operatorname{top}(U_1)$.

The final case to consider is when $K \cap U_1 \subsetneqq U_1$ and $K \cap U_2 \subsetneqq U_2$. We get the following pushout diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{top}(K \cap U_1) \arrow[r] & P/(\mathbf{r}(K \cap U_1)) \amalg (K \cap U_2) \arrow[r] & C \arrow[r] & 0\,.
\end{tikzcd}$$

But

$$\mathrm{top}(K \cap U_1) \cong \mathrm{soc}(U_1/\mathbf{r}(K \cap U_1) \not\cong \mathrm{soc}(U_2/K \cap U_2).$$

Thus

$$\tau_{\mathrm{top}(K \cap U_1)}(P/(\mathbf{r}(K \cap U_1))\amalg(K \cap U_2)) = \mathrm{top}(K \cap U_1).$$

Thus C is a trace quotient module by $\mathrm{top}(K \cap U_1)$ and the result follows. ■

We end this section with an example of an artin algebra each of whose indecomposable modules has a simple top and yet does not satisfy the equivalent conditions of the above theorem.

Example 6.2. Let Λ be the ring

$$\begin{pmatrix} k[x]/x^2 & 0 \\ k[x]/x^2 & k \end{pmatrix},$$

where k is a field and $k[x]/x^2$ in the lower left hand corner is a $k-k[x]/x^2$-bimodule in the obvious way. It is not hard to check that every indecomposable Λ-module has a simple top. Let

$$e_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$P_1 = \Lambda e_1$, $P_2 = \Lambda e_2$, $S_1 = P_1/\mathbf{r}P_1$ and $S_2 = P_2/\mathbf{r}P_2$, where $r = \mathrm{rad}(\Lambda)$. The socle of P_1 is $S_2 \amalg S_2$. Choosing a fixed submodule S_2 of P_1 we set $C = P_1/S_2$. Suppose that C is a trace quotient module by some Λ-module A. The only indecomposable module of larger length than C is P_1. Consider the short exact sequence $0 \to S_2 \to P_1 \to C \to 0$. By supposition, we must have $\tau_A(P_1) = S_2$. Since the socle of P_1 is $S_2 \amalg S_2$ we see that $\tau_{S_2}(P_1) \neq S_2$. On the other hand, if $\tau_A(P_1) = S_2$, then there is a surjection $A \to S_2$. In this case, $\tau_A(P_1) \supseteq (S_2 \amalg S_2)$ which is a contradiction. We conclude that C is not a trace quotient module.

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