

ON THE GLOBAL DIMENSION OF ALGEBRAS OVER REGULAR LOCAL RINGS

BY

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To the memory of Professor Irving Reiner

Throughout this work, R denotes a commutative noetherian complete local ring and \mathfrak{M} its maximal ideal. We will assume that the residue field $k := R/\mathfrak{M}$ is algebraically closed. Let Λ be a basic and indecomposable R -algebra, we will always assume that Λ is finitely generated as R -module. We will be concerned with the estimation of the global dimension of Λ ($\text{gl dim } \Lambda$). Our methods will be of diagrammatical nature: associated with the k -algebra $\bar{\Lambda} = \Lambda/\mathfrak{M}\Lambda$ we have a quiver $Q_{\bar{\Lambda}}$ (i.e., an oriented graph) such that Λ is a quotient of the path algebra $R[Q_{\bar{\Lambda}}]$. We will derive properties of $\text{gl dim } \Lambda$ from the geometric structure of $Q_{\bar{\Lambda}}$.

In Section 1 we show that for any R -algebra Λ , with $Q_{\bar{\Lambda}}$ without oriented cycles, a necessary and sufficient condition for $\text{gl dim } \Lambda < \infty$ is that $\text{gl dim } e_i \Lambda e_i < \infty$, for a complete set e_1, \dots, e_n of pairwise orthogonal primitive idempotents of Λ (observe that the rings $e_i \Lambda e_i$ are quotients of R since $Q_{\bar{\Lambda}}$ has no oriented cycles).

In Section 2 we consider the case where R is a discrete valuation domain and Λ is a Schurian order (tilted order in the notation of [8]). The problem of determining $\text{gl dim } \Lambda$ in this case has been considered before. In [8], Jategaonkar shows that for a triangular Schurian order, $\text{gl dim } \Lambda < \infty$ iff $\text{gl dim } \Lambda \leq n - 1$, with n as above. In [9], this result is generalized to the case where R is a commutative noetherian domain. In [10], among other things, a necessary condition for $\text{gl dim } \Lambda < \infty$ is given. This condition plays an important role in our approach, see (2.1). In [21], Roggenkamp and Wiedemann give a more or less complete description of all Schurian orders of global dimension two. They use quiver methods.

For a Schurian order Λ , we relate the problem of evaluating $\text{gl dim } \Lambda$ to a problem of posets and their representations. We use some arguments of [22] and covering techniques ([3] and [14]). We give a sample of our results: Let

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$R = k[[t]]$ (the power series ring in one variable). Assume that at each vertex of $Q_{\bar{\Lambda}}$ at least two arrows start and at least two end, then $\text{gl dim } \Lambda \geq 3$. This result is related with recent research in [27] and old conjectures in [10] and [26]; see (2.9).

By $\text{Mod } \Lambda$ (resp. $\text{mod } \Lambda$) we denote the category of left Λ -modules (resp. finitely generated Λ -modules). Modules will be sometimes treated as representations (for example, see [6]). In particular, Λ will be considered also as an R -category (this is necessary when quivers are infinite). Some special modules (or representations) are S_x and P_x , the simple and projective module associated with the vertex x of $Q_{\bar{\Lambda}}$, respectively.

We denote by $\text{p dim}_{\Lambda} M$ the projective dimension of a Λ -module M and we let

$$\text{gl dim } \Lambda = \sup\{\text{p dim } M : M \in \text{Mod } \Lambda\}.$$

When R is a local noetherian ring and Λ is an f.g. R -algebra (our case),

$$\text{gl dim } \Lambda = \sup\{\text{p dim}_{\Lambda} S : S \text{ is a simple } \Lambda\text{-module}\}$$

(see [25], Cor 4.6).

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1. Algebras whose quiver has no oriented cycle

1.1. Let Λ be a basic R -algebra finitely generated as R -module. Consider $\bar{\Lambda} = \Lambda/\mathfrak{M}\Lambda$. Then $\bar{\Lambda}$ is a basic finite-dimensional k -algebra. Associated with $\bar{\Lambda}$ we have the quiver $Q_{\bar{\Lambda}}$ (for instance, see [6]). Consider the path algebra $R[Q_{\bar{\Lambda}}]$ and define a morphism $\varphi: R[Q_{\bar{\Lambda}}] \rightarrow \Lambda$ as follows: let $J = \text{rad } \Lambda$ and $1 = e_1 + \dots + e_n$ be a decomposition of 1 as a sum of primitive pairwise orthogonal idempotents. Choose a basis $\{y_{\alpha}\}_{\alpha}$ of $e_j J / (J^2 + \mathfrak{M}\Lambda)e_i$, where α runs over the arrows of $Q_{\bar{\Lambda}}$ with starting point i and target point j . Let $x_{\alpha} \in e_j J e_i$ be a representative of y_{α} and set $\varphi(\alpha) = x_{\alpha}$. Furthermore, φ sends the trivial path at the vertex i to the idempotent e_i for all i . By [19, (6,iii)], we know that φ is surjective. We are interested in the ideal $\ker \varphi$.

DEFINITION. Let Q be any quiver. Let M be the ideal of $R[Q]$ generated by the arrows. A two sided ideal I of $R[Q]$ is said to be *admissible* iff:

(A1) $I \subset \mathfrak{M}R[Q] + M^2$.

(A2) There are natural numbers l, m satisfying

$$M^l \subset I + \mathfrak{M}R[Q] \text{ and } \mathfrak{M}R[Q] \subset M_m + I,$$

where M_m denotes the R -submodule of $R[Q]$ spanned by the paths of length at most m .

The following result is straightforward.

PROPOSITION. *Let $\varphi: R[Q_{\bar{\Lambda}}] \rightarrow \Lambda$ be as above. Then $\ker \varphi$ is admissible. Conversely, given an admissible ideal I of $R[Q]$, the R -algebra $\Lambda = R[Q]/I$ is finitely generated as R -module and $Q_{\bar{\Lambda}} = Q$. \square*

1.2. Let Q be the quiver of $\bar{\Lambda} = \Lambda/\mathfrak{M}\Lambda$ as above, denote by Q_0 the set of vertices of Q .

LEMMA. *Assume $\text{gl dim } \Lambda < \infty$.*

- (i) *If b is a source of Q and $e = \sum_{b \neq x \in Q_0} e_x$ then $\text{gl dim}(e\Lambda e) < \infty$*
- (ii) *If Q has no oriented cycle then $\text{gl dim}(e_x\Lambda e_x) < \infty$ for each $x \in Q_0$.*

Proof. (i) Let $x \neq b$. Since b is a source, the minimal Λ -projective resolution of S_x is also an $e\Lambda e$ -projective resolution of S_x .

(ii) Induction on the number of vertices n . The case $n = 1$ is trivial. Thus assume $n > 1$ and take b a source of Q . Let $e = \sum_{x \neq b} e_x$ as in i. Hence $\text{gl dim}(e\Lambda e) < \infty$ and by induction hypothesis $\text{gl dim}(e_x\Lambda e_x) < \infty$ for all $x \neq b$. Now, consider the simple Λ -module $S_b = ke_b$. Consider a minimal Λ -projective resolution for S_b , $0 \rightarrow Q_l \rightarrow \dots \rightarrow Q_0 \rightarrow S_b \rightarrow 0$. As Q has no oriented cycle and b is a source, $e_b Q_i = e_b Q_i e_b$. Therefore we get the $e_b\Lambda e_b$ -projective resolution

$$0 \rightarrow e_b Q_l \rightarrow \dots \rightarrow e_b Q_0 \rightarrow S_b \rightarrow 0.$$

Since S_b is the unique simple $e_b\Lambda e_b$ -module, we have

$$\text{gl dim}(e_b\Lambda e_b) \leq \text{p dim}_{\Lambda} S_b < \infty. \quad \square$$

1.3. **THEOREM.** *Let Λ be a basic finitely generated R -algebra. Assume that the quiver Q of $\bar{\Lambda} = \Lambda/\mathfrak{M}\Lambda$ has no oriented cycle. Assume that*

$$n = \max\{\text{gl dim}(e_x\Lambda e_x) : x \in Q_0\}$$

is finite. Let m be the maximal possible length of paths in Q . Then

$$\text{gl dim } \Lambda \leq n + nm + m.$$

Proof. We proceed by induction on m . When $m = 0$ the result is trivial. For each $x \in Q_0$, let

$$R_x = \Lambda e_x / (M + I) \Lambda e_x$$

(notice that $R_x = e_x \Lambda e_x$). We claim that $\text{p dim}_\Lambda(R_x) \leq nm + m$. Consider the canonical exact Λ -sequence

$$0 \rightarrow K \rightarrow P_x \xrightarrow{\mu} R_x \rightarrow 0.$$

Then, for any source b of Q , $e_b K = 0$: if $b \neq x$, then $e_b P_x = 0$ and $e_b K = 0$; if $b = x$, then

$$e_b P_x \xrightarrow{\mu_b} R_b$$

is an isomorphism and $e_b K = 0$. Hence, K is a $\Lambda/\Lambda e$ -module where $e = \sum_{b \text{ source}} e_b$. As the maximal possible lengths of paths in $Q_{\overline{\Lambda/\Lambda e}}$ is $m - 1$, by induction hypothesis we get $\text{p dim}_{\Lambda/\Lambda e} K \leq n + n(m - 1) + m - 1 = nm + m - 1$. But any projective $\Lambda/\Lambda e$ -module is also Λ -projective, therefore $\text{p dim}_\Lambda K \leq \text{p dim}_{\Lambda/\Lambda e} K$. Hence,

$$\text{p dim}_\Lambda R_x \leq \text{p dim}_\Lambda K + 1 \leq nm + m$$

as desired. On the other hand, $\text{gl dim } R_x = \text{p dim}_{R_x} S_x$. Thus

$$\text{p dim}_\Lambda S_x \leq \text{p dim}_\Lambda R_x + n \leq nm + m + n.$$

Since $\text{gl dim } \Lambda = \sup\{\text{p dim}_\Lambda S_x : x \in Q_0\}$ we are done. \square

1.4. In some cases our bound (1.3) may be improved. We need the following:

LEMMA. *Let N be a finitely generated R -module with depth $N > 0$. Then there exists an element $r \in \mathfrak{M} - \mathfrak{M}^2$ which is not a zero divisor in N .*

Proof. Let $\text{Ass}(N) = \{p_1, \dots, p_r\}$ be the set of associated prime ideals of N . Then $\cup_{i=1}^r p_i$ is the set of all non zero divisors in N . As depth $N > 0$ then $\cup_{i=1}^r p_i \subsetneq \mathfrak{M}$. By Nakayama's lemma, $\bar{p}_i = (p_i + \mathfrak{M}^2)/\mathfrak{M}^2 \subsetneq \mathfrak{M}/\mathfrak{M}^2$. As $\mathfrak{M}/\mathfrak{M}^2$ is a k -vector space and k is infinite (since $k = \bar{k}$), we obtain $\cup_{i=1}^r \bar{p}_i \subsetneq \mathfrak{M}/\mathfrak{M}^2$. Hence, we may choose an element $r \in \mathfrak{M} - \mathfrak{M}^2$ such that $\bar{r} \notin \cup \bar{p}_i$, which implies $r \notin \cup p_i$. \square

1.5. **PROPOSITION.** *Let R be a regular ring with $n = \text{gl dim } R$. Assume that Λ is an R -algebra such that the quiver Q of Λ has no oriented cycle and $e_x \Lambda e_x = R$ for each $x \in Q_0$ (i.e., $I \subset \mathfrak{M}M + M^2$ in the notation of 1.1). Let m be the maximal possible length of paths in Q and $d = \text{depth } \Lambda$. Then,*

$$dm + \text{gl dim } \Lambda \leq n + nm + m.$$

Proof. By induction on d . The case $d = 0$ follows from (1.3). Let $d > 0$ and take $r \in \mathfrak{M} - \mathfrak{M}^2$, a non-zero divisor in Λ (1.4). Then, $\Lambda' = \Lambda/r\Lambda$ is an $R' = R/rR$ -algebra with the same quiver Q and $e_x \Lambda' e_x = R'$ for all $x \in Q_0$. Therefore R' is regular and $\text{gl dim } R' = n - 1$. As $\text{depth } \Lambda' = d - 1$, we obtain

$$\text{gl dim } \Lambda' \leq (n - 1) + (n - 1)m + m - (d - 1)m = n + nm + m - dm - 1.$$

By [11, III Theorem 10], $\text{gl dim } \Lambda = \text{gl dim } \Lambda' + 1$ and the result follows. \square

COROLLARY. *Let R and Λ be as above. Assume that Λ is a free R -algebra. Then $\text{gl dim } \Lambda \leq n + m$.*

Proof. In this case, $\text{depth } \Lambda = n$. \square

Remark. The argument in the proposition above may be used to show the following.

Let R be regular with $\text{gl dim } R = n$. Assume that Λ is a free R -algebra and that $\bar{\Lambda} = \Lambda/\mathfrak{M}\Lambda$ is a k -algebra with $d = \text{gl dim } \bar{\Lambda} < \infty$. Then $\text{gl dim } \Lambda = n + d$. Observe that the corollary also follows from this remark.

1.6. The bound given in Theorem 1.3 is the best possible. In fact, let R be a regular local ring with $n = \text{gl dim } R$. Consider the quiver

$$Q: 1 \xrightarrow{\alpha} 2$$

and the admissible ideal $I = \mathfrak{M}\alpha$ of $R[Q]$ and let $\Lambda = R[Q]/I$. Notice that

$$\Lambda = \begin{pmatrix} R & 0 \\ k & R \end{pmatrix}.$$

We shall prove that $\text{gl dim } \Lambda = 2n + 1$.

Let (x_1, \dots, x_n) be a regular system of parameters of R generating \mathfrak{M} . Consider the R -modules $M_i = R/\langle m_1, \dots, m_i \rangle$ for $i = 0, \dots, n$. Then $M_i e_1$ is a Λ -module if we set $\alpha M_i e_1 = 0$. For $i = 0$, $0 \rightarrow k\alpha \rightarrow P_1 \rightarrow M_0 e_1 \rightarrow 0$ is exact and $k\alpha \xrightarrow{\sim} ke_2$. Thus $\text{p dim}_\Lambda M_0 e_1 = n + 1$. For $i > 0$ let

$$0 \rightarrow R^{(m_n)} \xrightarrow{p_n} \dots \rightarrow R^{(m_1)} \xrightarrow{p_1} R^{(m_0)} \xrightarrow{p_0} k \rightarrow 0$$

and

$$0 \rightarrow R^{(l_i)} \xrightarrow{q_i} \dots \rightarrow R^{(l_1)} \xrightarrow{q_1} R^{(l_0)} \xrightarrow{q_0} M_i \rightarrow 0$$

be minimal projective resolutions in mod R . Let $N_s = \ker p_s$ and $K_s = \ker q_s$, in particular $N_{-1} = k$. We shall construct a minimal projective resolution for

$M_i e_1$ in mod Λ . As $K_0 \subset \text{rad } R^{(l_0)} = (\text{rad } R)^{(l_0)}$, then $K_0 e_1$ as submodule of $P_1^{(l_0)}$ satisfies $\alpha K_0 e_1 = 0$. Hence, we have an exact Λ -sequence

$$0 \rightarrow K_0 e_1 \oplus (N_{-1} e_2)^{(l_0)} \rightarrow P_1^{(l_0)} \rightarrow M_i e_1 \rightarrow 0,$$

where \oplus is only an R -direct sum. Assume that the j^{th} -syzygy of the minimal projective resolution of $M_i e_1$ is

$$L_j = K_j e_1 \oplus \left(\bigoplus_{s+t=j-1} N_s^{(l_j)} e_2 \right).$$

Let $m = \sum_{s+t=j-1} m_s l_t$. Then we have the exact sequence

$$0 \rightarrow (K_{j+1} e_1 \oplus (k\alpha)^{(l_j)}) \oplus \left(\bigoplus_{s+t=j-1} N_{s+1}^{(l_j)} e_2 \right) \rightarrow P_1^{(l_{j+1})} \oplus P_2^m \rightarrow L_j \rightarrow 0.$$

As $\alpha K_{j+1} e_1 = 0$ in $P_1^{(l_{j+1})}$, then L_{j+1} has the desired form. Thus, $L_{n+i+1} \xrightarrow{\sim} P_2^{(l_i)}$ is the first L_j which is Λ -projective. Hence, $\text{p dim }_{\Lambda} M_i e_1 = n + i + 1$. In particular $\text{p dim }_{\Lambda} k e_1 = 2n + 1$ and $\text{gl dim } \Lambda = 2n + 1$.

1.7. In general there is no strong relation between $\text{gl dim } \Lambda$ and $\text{gl dim } \bar{\Lambda}$, where $\bar{\Lambda} = \Lambda / \mathfrak{M} \Lambda$. For instance, if Λ is any non-trivial hereditary algebra over a discrete valuation domain R , then $\text{gl dim } \bar{\Lambda} = \infty$.

Now consider a discrete valuation domain R with maximal ideal $\mathfrak{M} = \pi R$. Let Q be the quiver

$$\begin{array}{ccc} 1 & \xrightarrow{\alpha} & 2 \\ \delta \uparrow & & \downarrow \beta \\ 4 & \xleftarrow{\gamma} & 3 \end{array}$$

and let $I = \langle \beta\alpha, \delta\gamma\beta \rangle + \mathfrak{M} M^2$, an admissible ideal of $R[Q]$. Let $\Lambda = R[Q]/I$. Then

$$\bar{\Lambda} \xrightarrow{\sim} k[Q] / \langle \beta\sigma, \delta\gamma\beta \rangle,$$

with $\text{gl dim } \bar{\Lambda} = 3$. On the other hand $\text{p dim }_{\Lambda} R e_1 = \infty$.

2. Schurian orders

Let R be a complete valuation domain with maximal ideal $\mathfrak{M} = \pi R$, and K the quotient field of R . Let Λ be an R -order in the full $n \times n$ -matrix algebra $(K)_n$ contained in $(R)_n$, containing a complete set of pairwise orthogonal primitive idempotents $\{e_i; i = 1, \dots, n\}$. We may assume that $e_i = e_{ii}$ are the

usual matrix idempotents. Then, $\Lambda = (\mathfrak{M}^{m_{ij}})$, $m_{ij} \in \mathbb{N} \cup \{0\}$ such that $m_{ii} = 0$ and $m_{ik} + m_{kj} \geq m_{ij}$, for all $i, j, k = 1, \dots, n$. Following [22], we call Λ a *Schurian order*. We also assume that Λ is basic, by [9] Lemma 1.6, this means that $m_{ij} + m_{ji} \geq 1$ whenever $i \neq j$. In this situation the radical of Λ is $J = (\mathfrak{M}^{m'_{ij}})$ with $m'_{ii} = 1$ and $m'_{ij} = m_{ij}$ for $i \neq j$ (see [9], Lemma 1.3). Finally, Λ is indecomposable as a ring, i.e., $Q_{\bar{\Lambda}}$ is connected.

2.1. DEFINITION. Λ is said to satisfy the *Jategaonkar condition* iff for each $1 \leq i \leq n$, there exists $1 \leq i \neq \mu(i) \leq n$ such that $m_{i, \mu(i)} + m_{\mu(i), i} = 1$.

PROPOSITION. (i) *If $\text{gl dim } \Lambda < \infty$, then Λ satisfies the Jategaonkar condition.*

(ii) *Assume Λ satisfies the Jategaonkar condition. Then:*

(a) *P_i is not isomorphic to a direct summand of $P(\text{rad } P_i)$, $i = 1, \dots, n$. (Here $P(M)$ denotes the projective cover of M).*

(b) *$Q_{\bar{\Lambda}}$ has no loops and no double arrows.*

Proof. (i) This is Lemma 2.7 of [10].

(ii) (a) This is Lemma 2.3 of [21].

(ii) (b) Since $\mathfrak{M}\Lambda \subset J^2$, (a) implies that $e_i J / J^2 e_i = 0$. Therefore, $Q_{\bar{\Lambda}}$ has no loops. The second assertion is Lemma 1.3 of [21]. \square

2.2. Let Λ be as before and satisfy Jategaonkar condition. By (2.1), $(Q_{\bar{\Lambda}})^{\text{op}}$ coincides with the quiver $Q(\Lambda)$ constructed in [21], Section 2. As there, we can provide $Q_{\bar{\Lambda}}$ with a valuation ν , such that $\nu(\alpha) = m_{ji}$ for an arrow $\alpha: i \rightarrow j$. We extend ν to a valuation in all the walks (i.e., non oriented paths) of $Q_{\bar{\Lambda}}$. (Thus, $\nu(\alpha\beta^{-1}\gamma) = \nu(\alpha) - \nu(\beta) + \nu(\gamma)$). We denote by L the ideal of $k[Q_{\bar{\Lambda}}]$ generated by the differences $\gamma - \delta$ of paths with the same extremal points and such that $\nu(\gamma) = \nu(\delta)$. Clearly, $L \subset \text{rad}^2 k[Q_{\bar{\Lambda}}]$.

Now, let $\mathbf{P}(\Lambda)$ denote the poset associated to Λ in the following form (see [13] or [22]): the vertices of $\mathbf{P}(\Lambda)$ are $\{1, \dots, n\} \times \mathbb{Z}$ and the order in $\mathbf{P}(\Lambda)$ is generated by $(i, \alpha) \leq (j, \beta)$ iff $(i = j \text{ and } \alpha \leq \beta \leq \alpha + 1)$ or $(\beta = \alpha + m_{ji})$.

Associated with $\mathbf{P}(\Lambda)$ we have a quiver Δ ($\Delta_0 = \mathbf{P}(\Lambda)$) and there is an arrow $p \rightarrow q$ in Δ whenever $p < q$ and there is no $v \in \mathbf{P}(\Lambda)$ with $p < v < q$. Let I be the ideal generated by all commutativity relations in $k[\Delta]$. Let $k[\mathbf{P}(\Lambda)] = k[\Delta]/I$.

2.3. For the notions used in the next proposition the reader is referred to [3] and [14].

PROPOSITION. *Assume Λ satisfies the Jategaonkar condition. Then there is a Galois covering $\pi: k[\mathbf{P}(\Lambda)] \rightarrow k[Q_{\bar{\Lambda}}]/L$ with group \mathbb{Z} .*

Proof. Fix a vertex $x_0 \in Q_{\bar{\Lambda}}$ and let W be the set of all walks starting at x_0 . Let \sim be the equivalence relation in W defined by $\gamma \sim \delta$ iff γ, δ have the same extremal vertices and $\nu(\gamma) = \nu(\delta)$. Consider the quiver Q with vertices $Q_0 = W/\sim$ and arrows $\alpha: [\gamma] \rightarrow [\alpha\gamma]$ for α an arrow in $Q_{\bar{\Lambda}}$. Let $\tilde{\Lambda}$ be the quotient $k[Q]/\tilde{L}$, where \tilde{L} is the ideal generated by all possible commutativity relations. Then

$$\pi: \tilde{\Lambda} \rightarrow k[Q_{\bar{\Lambda}}]/L, [\gamma] \mapsto e(\gamma) = \text{ending point of } \gamma$$

yields a covering in the sense of [14]. As Q has no oriented cycles (indeed, if

$$\delta: x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_n = x_1$$

is an oriented cycle in $Q_{\bar{\Lambda}}$ with $\nu(\delta) = 0$, then

$$m_{x_1x_2} = 0 \quad \text{and} \quad m_{x_2x_1} \leq \sum_{i=2}^{n-1} m_{x_i x_{i+1}} = 0,$$

contradicting the fact that Λ is basic) there is an associated poset $\tilde{\Delta}$ with $k[\tilde{\Delta}] \xrightarrow{\sim} \tilde{\Lambda}$. We shall prove that $\mathbf{P}(\Lambda) \xrightarrow{\sim} \tilde{\Delta}$.

Define $\varphi: \tilde{\Delta} \rightarrow \mathbf{P}(\Lambda), \gamma \mapsto (e(\gamma), \nu(\gamma))$. Clearly, φ is an injective order-preserving map.

It is not hard to see that for any two vertices $i, j = 1, \dots, n$, there is a path δ from i to j with $\nu(\delta) = m_{ji}$. For the surjectivity of φ , let (i, α) be a vertex of $\mathbf{P}(\Lambda)$. As $Q_{\bar{\Lambda}}$ is connected, there is a vertex $[\gamma] \in Q_0$ with $e(\gamma) = i$. By Jategaonkar condition there is some $j \in Q_0$ with $m_{ij} + m_{ji} = 1$. Therefore, from our claim above we get a closed path δ at i with $\nu(\delta) = 1$. Let $\beta = \alpha - \nu(\gamma) \in \mathbf{Z}$ and consider the point $[\delta^\beta \gamma] \in Q_0$. Then we have $\varphi[\delta^\beta \gamma] = (i, \nu(\delta^\beta \gamma))$, where $\nu(\delta^\beta \gamma) = \beta \nu(\delta) + \nu(\gamma) = \alpha$.

Let $\theta: \mathbf{P}(\Lambda) \rightarrow \tilde{\Delta}$ be the inverse map of φ . We just have to show that θ is an order preserving map. Let us consider the case $(i, \alpha) \leq (j, \beta)$ with $\beta = \alpha + m_{ji}$, the other case being similar. There is a path δ from i to j with $\nu(\delta) = m_{ji}$. Let $\theta(i, \alpha) = [\gamma]$; then $\theta(j, \beta) = [\delta\gamma]$ and $\theta(i, \alpha) \leq \theta(j, \beta)$. Finally the automorphisms of $k[\mathbf{P}(\Lambda)]$ preserving π are the T_α given by

$$(i, \beta) \rightarrow (i, \alpha + \beta) \quad \text{for } \alpha \in \mathbf{Z}.$$

Hence, $k[\mathbf{P}(\Lambda)]$ is a Galois covering of $k[Q_{\bar{\Lambda}}]/L$ with group \mathbf{Z} . \square

2.4. We make now a strong restriction, namely, we assume that $R = k[[t]]$ is the power series ring in one variable. (Therefore, $\mathfrak{M} = tR$). In this section

we show that

$$\text{gl dim } k[\mathbf{P}(\Lambda)] = \text{gl dim } \Lambda.$$

For this purpose we need some notation and results from [22]. Let $R^0(\mathbf{P}(\Lambda))$ be the category of bounded representations of the poset $\mathbf{P}(\Lambda)$ (i.e., $V \in R^0(\mathbf{P}(\Lambda))$ iff there is a finite dimensional k -vector space $V(\omega)$ such that $V(i, \alpha) = 0$ and $V(i, \beta) = V(\omega)$ for α sufficiently small and β sufficiently large, moreover $V(i, \alpha) \subset V(j, \beta)$ provided $(i, \alpha) \leq (j, \beta)$). Let ${}_{\Lambda}\mathfrak{M}^0$ denote the category of left Λ -lattices. Consider the functor

$$\mathbf{F}: R^0(\mathbf{P}(\Lambda)) \rightarrow {}_{\Lambda}\mathfrak{M}^0$$

defined on objects as follows: for $V = (V(\omega), (V(i, \alpha))_{(i, \alpha)}) \in R^0(\mathbf{P}(\Lambda))$ we have

$$\mathbf{F}(V)(j) = \bigoplus_{\alpha \in \mathbf{Z}} t^\alpha V(j, \alpha), \quad j \in Q_0.$$

We concentrate some information about \mathbf{P} in the following:

PROPOSITION [22, (1.10) and (3.1)]. (i) \mathbf{F} is an exact functor.

(ii) If $X \in R^0(\mathbf{P}(\Lambda))$ is indecomposable, then $\mathbf{F}X$ is indecomposable.

(iii) $X \in R^0(\mathbf{P}(\Lambda))$ is indecomposable projective iff $\mathbf{F}X$ is indecomposable projective (furthermore, $\mathbf{F}P_{(i, \alpha)} = P_i$ and $\mathbf{F} \text{ rad } P_{(i, \alpha)} = \text{rad } P_i$ for every $(i, \alpha) \in \mathbf{P}(\Lambda)$).

(iv) For each $M \in {}_{\Lambda}\mathfrak{M}^0$, a representation $\mathcal{V}(M) \in R^0(\mathbf{P}(\Lambda))$ is given such that:

(a) $\mathcal{V}(\mathbf{F}V) \xrightarrow{\sim} V$ for each $V \in R^0(\mathbf{P}(\Lambda))$.

(b) If $M' \subset M$ in ${}_{\Lambda}\mathfrak{M}^0$, then there is an induced monomorphism $\mathcal{V}(M') \rightarrow \mathcal{V}(M)$ in $R^0(\mathbf{P}(\Lambda))$.

LEMMA. $\text{gl dim } k[\mathbf{P}(\Lambda)] = \text{gl dim } R^0(\mathbf{P}(\Lambda)) + 1.$

Proof. For $q, q' \in \mathbf{P}(\Lambda)$, let $\mathbf{P}_q = \{\nu \in \mathbf{P}(\Lambda): q \leq \nu\}$, $\mathbf{P}^{q'} = \{\nu \in \mathbf{P}(\Lambda): \nu \leq q'\}$ and $\mathbf{P}_q^{q'} = \mathbf{P}_q \cap \mathbf{P}^{q'}$. As a $\mathbf{P}(\Lambda)$ -module M is projective iff $M|_{\mathbf{P}_q}$ is \mathbf{P}_q -projective for each q (see [4], 1.4), then

$$\text{gl dim } \mathbf{P}(\Lambda) = \sup \{ \text{gl dim } \mathbf{P}_q : q \in \mathbf{P}(\Lambda) \}.$$

By duality,

$$\text{gl dim } \mathbf{P}_q = \sup \{ \text{gl dim } \mathbf{P}_q^{q'} : q' \in \mathbf{P}_q \}$$

and

$$\text{gl dim } \mathbf{P}(\Lambda) = \sup\{\text{gl dim } \mathbf{P}_q^{q'} : q, q' \in \mathbf{P}(\Lambda)\}.$$

Therefore,

$$\text{gl dim } \mathbf{P}(\Lambda) = \sup\{p \dim_{\mathbf{P}(\Lambda)} S_x : x \in \mathbf{P}(\Lambda)\}.$$

For $x \in \mathbf{P}(\Lambda)$, consider the exact $\mathbf{P}(\Lambda)$ -sequence $0 \rightarrow K \rightarrow P_x \rightarrow S_x \rightarrow 0$. Clearly, $K \in R^0(\mathbf{P}(\Lambda))$ and therefore

$$\text{gl dim } \mathbf{P}(\Lambda) \leq \text{gl dim } R^0(\mathbf{P}(\Lambda)) + 1.$$

Now, let $V \in R^0(\mathbf{P}(\Lambda))$. We shall prove that

$$p \dim_{R^0(\mathbf{P}(\Lambda))} V + 1 \leq \text{gl dim } \mathbf{P}(\Lambda).$$

As $\text{gl dim } \mathbf{P}(\Lambda) \geq 1$, we may assume that V is not projective. By [20, Theorem 10.6], there is a projective Λ -lattice $P \in {}_{\Lambda} \mathfrak{M}^0$ such that $\mathbf{F}V \subset P$. By (iii) in the proposition, there is a projective $\mathbf{P}(\Lambda)$ -representation $Q \in R^0(\mathbf{P}(\Lambda))$ with $\mathbf{F}Q = P$. Therefore, $\mathbf{F}V \subset \mathbf{F}Q$ and $V \xrightarrow{\sim} \mathcal{V}\mathbf{F}V \subset \mathcal{V}\mathbf{F}Q \xrightarrow{\sim} Q$ which implies the desired inequality. \square

THEOREM. $\text{gl dim } k[\mathbf{P}(\Lambda)] = \text{gl dim } \Lambda$.

Proof. Since $\text{gl dim } \Lambda = \text{gl dim}_{\Lambda} \mathfrak{M}^0 + 1$, then by the lemma, it is enough to show that

$$\text{gl dim } R^0(\mathbf{P}(\Lambda)) = \text{gl dim}_{\Lambda} \mathfrak{M}^0.$$

Assume first that $\text{gl dim } R^0(\mathbf{P}(\Lambda)) = m < \infty$. Take $i \in Q_0$ and $V = \text{rad } P_{(i,0)}$. Consider a $R^0(\mathbf{P}(\Lambda))$ -projective resolution $0 \rightarrow Q_m \rightarrow \dots \rightarrow Q_0 \rightarrow V \rightarrow 0$. Thus,

$$0 \rightarrow \mathbf{F}Q_m \rightarrow \dots \rightarrow \mathbf{F}Q_0 \rightarrow \text{rad } P_i \rightarrow 0$$

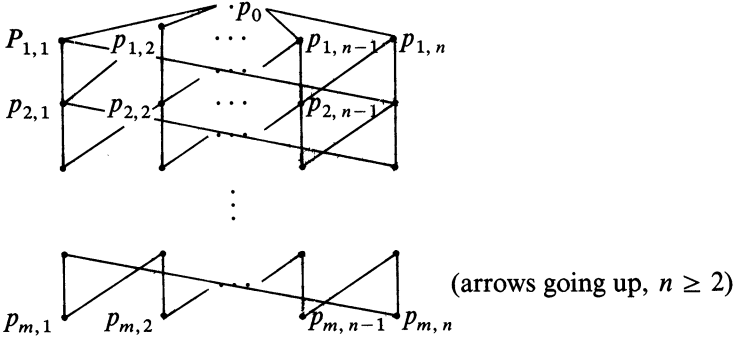
is a ${}_{\Lambda} \mathfrak{M}^0$ -projective resolution and

$$\text{gl dim}_{\Lambda} \mathfrak{M}^0 = \sup\{p \dim_{\mathfrak{M}^0} \text{rad } P_i : i \in Q_0\} \leq \text{gl dim } R^0(\mathbf{P}(\Lambda)).$$

The converse inequality follows similarly from (iii) in the proposition. \square

2.5. To get some applications we need to determine $\text{gl dim } \mathbf{P}$ for some posets \mathbf{P} .

The poset $B_{m,n}$ pictured below is called a (m,n) -braid.



If we add an infimum ω to $B_{m,n}$ we obtain a complete braid $\hat{B}_{m,n}$.

LEMMA. $\text{gl dim } \hat{B}_{m,n} = m + 1$.

Proof. For $n = 2$, a proof may be found in [15, IX, Theorems 10, 12]. We recall here the argument. We denote by $x^{(m)}$ the elements of $B_{m,n}$. By induction on m , it is easy to see that $\text{gl dim } B_{m,n} \leq m$. Clearly, $\text{gl dim } B_{m,n} \leq \text{gl dim } \hat{B}_{m,n}$. Therefore, it is enough to show that $p \dim_{\hat{B}_{m,n}} S_{\omega^{(m)}} = m + 1$. Let $\Delta^{(m)} \in \text{mod } B_{m,n}$ with $\Delta^{(m)}(x) = k$ for all x and $\Delta^{(m)}(\alpha) = \text{id}_k$ for all arrows α . Then, the sequences

$$0 \rightarrow \Delta^{(m)} \rightarrow P_{\omega^{(m)}} \rightarrow S_{\omega^{(m)}} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \Delta^{(m-1)} \rightarrow \bigoplus_{i=1}^m P_{p_{m,i}^{(m)}} \rightarrow \Delta^{(m)} \rightarrow 0$$

are exact. As the $B_{m-1,n}$ -projectives are $\hat{B}_{m,n}$ -projectives we get

$$p \dim_{\hat{B}_{m,n}} \Delta^{(m-1)} = p \dim_{\hat{B}_{m-1,n}} \Delta^{(m-1)} = p \dim_{\hat{B}_{m-1,n}} S_{\omega^{(m-1)}} - 1 = m - 1.$$

Then $p \dim_{\hat{B}_{m,n}} S_{\omega^{(m)}} = m + 1$. \square

2.6 THEOREM. Let $R = k[[t]]$ and Λ a Schurian order in $(R)_n$.

- (i) If $B_{m,n}$ is a retract of $\mathbf{P}(\Lambda)$ then $\text{gl dim } \Lambda \geq m + 1$.
- (ii) $\text{gl dim } \Lambda \geq 2$ iff Fig. 1 is a retract of $\mathbf{P}(\Lambda)$ iff $\mathbf{P}(\Lambda)$ is not linearly ordered.

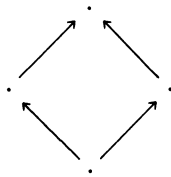


FIG. 1

- (iii) $\text{gl dim } \Lambda \geq 3$ iff $B_{2,n}$ is a retract of $\mathbf{P}(\Lambda)$, for some $n \geq 2$.

Proof. (i) follows from (2.4) and (2.5). The remaining parts of (ii) and (iii) follow from [4, Theorem 15 and Remark 2], and (2.4). \square

2.7. THEOREM. *Let $R = k[[t]]$ and let Λ be a Schurian order in $(R)_n$. Assume that at each vertex of $Q_{\bar{\Lambda}}$ at least 2 arrows start and at least two arrows end (then we say that $Q_{\bar{\Lambda}}$ is 2 edge connected). Then $\text{gl dim } \Lambda \geq 3$.*

Proof. By (2.1) we can assume that Λ satisfies Jategaonkar condition. By (2.3), there is a bijection $x^+ \xrightarrow{\sim} \pi(x)^+$ (resp. $x^- \xrightarrow{\sim} \pi(x)^-$) where

$$x^+ = \{ y \in \mathbf{P}(\Lambda) : \text{there exists an arrow } x \rightarrow y \}$$

(resp. $x^- = \{ y \in \mathbf{P}(\Lambda) : \text{there exists an arrow } y \rightarrow x \}$). It is easy to see that $\mathbf{P}(\Lambda)$ satisfies the hypothesis of the next lemma and therefore

$$\text{gl dim } \Lambda = \text{gl dim } \mathbf{P}(\Lambda) \geq 3. \quad \square$$

2.8. Let \mathbf{P} be a poset. We recall that q is *cover* of p in \mathbf{P} (dually, a *cocover*) iff $p < q$ and there is no $v \in \mathbf{P}$ with $p < v < q$.

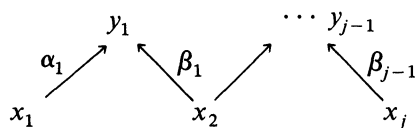
We say that \mathbf{P} is *locally bounded* iff for each finite family $\{x_1, \dots, x_m\} \subset \mathbf{P}$ there are elements $p, q \in \mathbf{P}$ such that $q \leq x_i \leq p$, for every $i = 1, \dots, m$.

We recall that the width $w(\mathbf{P})$ of \mathbf{P} is the supremum of the cardinalities of subsets of pairwise non-comparable elements of \mathbf{P} .

LEMMA. *Let \mathbf{P} be a locally bounded-poset with $w(\mathbf{P}) < \infty$ and such that each $x \in \mathbf{P}$ has at least two covers and at least two cocovers in \mathbf{P} . Then:*

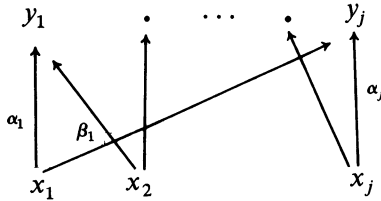
- (i) *There is some $n \geq 2$ such that $\hat{B}_{2,n}$ is a retract of \mathbf{P} .*
- (ii) $\text{gl dim } \mathbf{P} \geq 3$.

Proof. Let $x_1 \in \mathbf{P}$ and $\alpha_1: x_1 \rightarrow y_1$ be a cover of x_1 in \mathbf{P} . Take $\beta_1: y_1 \leftarrow x_2$ a cover with $x_1 \neq x_2$. (Therefore x_1 and x_2 are not comparable.) Assume we have already constructed a sequence



such that each arrow represents a cover in \mathbf{P} , y_i is only comparable with x_i, x_{i+1} and x_i is only comparable with y_{i-1}, y_i for each i . (These are the so called fences; see [12].) As $w(\mathbf{P}) < \infty$ we may assume that this sequence is maximal with these properties. Therefore, taking a cover $\alpha_j: x_j \rightarrow y_j$ with $y_{j-1} \neq y_j$, we have y_j comparable with some x_i or $y_i, i < j$. Without loss of

generality, y_j is comparable with x_1 or y_1 and not with x_i or y_i for $1 < i < j$. Therefore, $x_1 \leq y_j$ and not $y_1 \leq y_j$. The full subposet of \mathbf{P} with vertices $\{x_1, y_1, x_2, \dots, x_j, y_j\}$ is



As \mathbf{P} is locally bounded, there are points $x, y \in \mathbf{P}$ with $x \leq x_i \leq y_i \leq y$ for each i . Clearly, $\hat{B}_{2,j}$ is a full subposet of \mathbf{P} . If $j > 2$, $\hat{B}_{2,j}$ is a complete lattice and then \mathbf{P} contains $\hat{B}_{2,j}$ as a retract [17, Proposition 1.2]. If $j = 2$, one must add to the fullness condition that there is no element in \mathbf{P} preceding both y_1 and y_2 and following both x_1 and x_2 [17, Lemma 35.6]. As $\alpha_1: x_1 \rightarrow y_1$ is a cover, this condition is trivially satisfied. Therefore, by [18, Introduction] (compare with [7] and [11, III Theorem 5]), $\text{gl dim } \mathbf{P} \geq \text{gl dim } \hat{B}_{2,j}$ and the result follows from (2.6). \square

2.9. It would be interesting to get a better bound in Theorem 2.7. Recently, Kirkman and Kuzmanovich [27] have given an example to show that $\text{gl dim } \Lambda$ may be finite under the hypothesis of Theorem 2.7. We thank the referee for pointing out this paper to us.

These results seem to be related with earlier conjectures by Jategaonkar [10, 5] and Tarsey [26].

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