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A THEOREM ON MODULAR ENDOMORPHISM RINGS

BY

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In memory of Irving Reiner

1. Introduction

The object of this note is to present a new version[1] (Theorem (4.2)) of R. Brauer's well-known "reciprocity theorem" for modular decomposition numbers [2, p. 257], [4, p. 434], and to show its application to a theorem of G.D. James (see Section 5).

Let R be a complete discrete valuation ring with quotient field K, maximal ideal πR, and residue class field $F = R/\pi R$. Both K and F can be regarded as R-modules. If k is one of K, F, and if M is any object which (like Λ and X, see below) is a free, finitely-generated R-module, we shall write kM for the k-space $k \otimes_R M$, and θ_k: $M \to kM$ for the R-map which takes $m \to 1_k \otimes m$ $(m \in M)$. The map θ_K is injective, and may be used to identify M with a sub-R-module of kM. The map θ_F is surjective and has kernel πM; hence $FM \cong M/\pi M$. It is clear that

$$\dim_K KM = \dim_F FM, \tag{1.1}$$

both sides of (1.1) being equal to the R-rank of M.

Now let Λ be an R-order, i.e., Λ is an R-algebra with 1, which is free and finitely-generated as R-module. Then $k\Lambda$ is naturally a k-algebra ($k \in \{K, F\}$); Λ is usually regarded as a subring of $K\Lambda$ via θ_K: $\Lambda \to K\Lambda$. A (left) Λ-lattice is, by definition, a (left) Λ-module X which is free and finitely-generated as R-module. Then kX is naturally a finitely-generated (left) $k\Lambda$-module.

We shall need the following notation and terminology.

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[1]In the original version of this paper, Theorem (4.2) was described as a "generalization" of Brauer's theorem. However, E.C. Dade has kindly pointed out that Theorem (4.2) is deducible from Brauer's formula (1.3). I am indebted to Professor Dade for permission to use his proof of Theorem (4.2), which is shorter than mine.

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Notation. If X, Y are Λ-lattices, then $(Y, X)_R$, $(Y, X)_\Lambda$, $E(Y)$ denote $\mathrm{Hom}_R(Y, X)$, $\mathrm{Hom}_\Lambda(Y, X)$, $\mathrm{End}_\Lambda(Y)$, respectively. If $k \in \{K, F\}$ and if X', Y' are $k\Lambda$-modules, then $(Y', X')_k$, $(Y', X')_{k\Lambda}$, $E(Y')$ denote $\mathrm{Hom}_k(Y', X')$, $\mathrm{Hom}_{k\Lambda}(Y', X')$, $\mathrm{End}_{k\Lambda}(Y')$, respectively; also $e(Y') := \dim_k E(Y')$.

Components. A Λ-lattice Y_1 is said to be a *component* of Y, if it is isomorphic to a direct Λ-summand of Y. A similar definition holds for components of $k\Lambda$-modules.

R-forms. If $\mathbf{X}$ is any finitely-generated $K\Lambda$-module, it is always possible to find a Λ-lattice X such that $KX \cong \mathbf{X}$ as $K\Lambda$-modules; such a Λ-lattice X is called an *R-form* of $\mathbf{X}$. (See [4], pp. 409, 410, or [12], p. 55. If X is contained in $\mathbf{X}$, Curtis and Reiner call it a *full* Λ*-lattice* in $\mathbf{X}$.)

From now on we assume that $K\Lambda$ is a semisimple K-algebra. Let $\mathbf{X}_1, \dots, \mathbf{X}_t$ be a full set of simple (= irreducible) left $K\Lambda$-modules, and let $E_1, \dots, E_a$ be a full set of simple left $F\Lambda$-modules. Take fixed suffices $i \in \{1, \dots, t\}$, $\alpha \in \{1, \dots, a\}$. We choose an R-form X_i of $\mathbf{X}_i$, and an indecomposable component Λ_α of the left Λ-lattice ${}_\Lambda\Lambda$ which covers E_α—this means that $F\Lambda_\alpha$, which is an indecomposable component of the left $F\Lambda$-module ${}_{F\Lambda}F\Lambda$, satisfies $F\Lambda_\alpha/\mathrm{rad}\, F\Lambda_\alpha \cong E_\alpha$ (see [4], pp. 130–132 or [12], p. 11).

Brauer's proof. It will be useful to review Brauer's proof of his theorem. This rests on the equation

$$\dim_K(K\Lambda_\alpha, \mathbf{X}_i)_{K\Lambda} = \dim_F(F\Lambda_\alpha, FX_i)_{F\Lambda} \tag{1.2}$$

(see [2], (8), p. 257). The left side of (1.2) is easily calculated using Schur's lemma, since the $K\Lambda$-module $K\Lambda_\alpha$ is semisimple: it is equal to $\delta^*_{i\alpha}.e(\mathbf{X}_i)$, where $\delta^*_{i\alpha}$ denotes the multiplicity of $\mathbf{X}_i$ as component of $K\Lambda_\alpha$, and $e(\mathbf{X}_i) := \dim_K E(\mathbf{X}_i)$ ($E(\mathbf{X}_i) := \mathrm{End}_{K\Lambda}(\mathbf{X}_i)$). Since $F\Lambda_\alpha$ is a projective cover of E_α, we may calculate also the right side of (1.2) [3, Thm. (54.19), p. 376]: it is equal to $\delta_{i\alpha}.e(E_\alpha)$, where $\delta_{i\alpha}$ denotes the multiplicity of E_α as composition factor of the $F\Lambda$-module FX_i, and $e(E_\alpha) := \dim_F E(E_\alpha)$ ($E(E_\alpha) := \mathrm{End}_{F\Lambda}(E_\alpha)$). Therefore (1.2) gives Brauer's "reciprocity theorem"

$$\delta^*_{i\alpha}.e(\mathbf{X}_i) = \delta_{i\alpha}.e(E_\alpha) \quad \textit{for } i \in \{1, \dots, t\},\ \alpha \in \{1, \dots, a\}. \tag{1.3}$$

This shows incidentally that the decomposition number $\delta_{i\alpha}$ is independent of the R-form X_i of $\mathbf{X}_i$ which has been used to define it, because the left side of (1.3) depends only on the $K\Lambda$-isomorphism class of $\mathbf{X}_i$.

2. F-endostable Λ-lattices

Our "new version" of Brauer's theorem comes by replacing the Λ-lattice ${}_\Lambda\Lambda$ by an arbitrary (non-zero) Λ-lattice Y which is *F-endostable*, in the sense now to be defined.

If Y, X are Λ-lattices then $(Y, X)_\Lambda$ is an R-pure sublattice of the R-lattice $(Y, X)_R$, and it follows easily that, for $k \in \{K, F\}$, the k-isomorphism

$$k(Y, X)_R \to (kY, kX)_k$$

which takes $c \otimes f \to c(\mathrm{Id}_k \otimes f)$ ($c \in k$, $f \in (Y, X)_R$; Id_k denotes the identity map on k) induces a k-map

$$(2.1) \qquad \psi_k\colon k(Y, X)_\Lambda \to (kY, kX)_{k\Lambda}$$

which is injective. If $k = K$, then (2.1) is always an isomorphism, so that

$$(2.2) \qquad K(Y, X)_\Lambda \cong (KY, KX)_{K\Lambda} \quad \textit{as } K\textit{-spaces}$$

(see [12] Lemma 14.5, p. 57, or [4] (2.39), p. 36).

In general, ψ_F is not surjective. If it is, then

$$F(X, Y)_\Lambda \cong (FX, FY)_{F\Lambda} \quad \textit{as } F\textit{-spaces},$$

and we say that the pair Y, X is *F-stable*. This is clearly equivalent to the condition that the map

$$(2.3) \qquad \phi_F\colon (Y, X)_\Lambda \to (FY, FX)_{F\Lambda}$$

which takes $f \to \mathrm{Id}_F \otimes f$ ($f \in (Y, X)_\Lambda$) should be surjective. Notice that in any case ϕ_F has kernel $\pi(Y, X)_\Lambda$, for it is the composite of ψ_F with the natural map $\theta_F\colon (Y, X)_\Lambda \to F(Y, X)$.

The proof of the next lemma is an easy exercise.

(2.4) LEMMA. *Let X, Y be Λ-lattices.*
(i) *If the pair Y, X is F-stable, then so also is the pair Y_1, X_1, where Y_1, X_1 are any components of Y, X, respectively.*
(ii) *If Y is projective, the pair Y, X is F-stable for any X.*

DEFINITION. We say that a Λ-lattice Y is *F-endostable* if the pair Y, Y is F-stable, i.e., if the map $\phi_F\colon E(Y) \to E(FY)$ (see (2.3)) is surjective.

It is clear that $Y =_\Lambda \Lambda$ is F-endostable. And if $\Lambda = RG$, for a finite group G, then any permutation RG-lattice Y is F-endostable [13], [12, p. 174].

From now on we assume that Y is a non-zero F-endostable Λ-lattice. Then we have $E(FY) \cong E(Y)/\mathrm{Ker}\,\phi_F = E(Y)/\pi E(Y)$; and by (2.2) we may regard $E(Y)$ as an R-order in the K-algebra $E(KY)$. Also $E(KY)$ is a semisimple K-algebra, since KY is a $K\Lambda$-module, and $K\Lambda$ is by assumption a semisimple algebra. Let $\mathbf{Z}_1, \dots, \mathbf{Z}_s$ be a full set of simple $E(KY)$-modules, and let $S_1, \dots, S_l$ be a full set of simple $E(FY)$-modules. Then we may define decomposition number $d_{i\lambda}$ as the multiplicity of S_λ as an $E(FY)$-composition factor of FZ_i, where Z_i is an R-form for $\mathbf{Z}_i$ ($i \in \{1, \dots, s\}$, $\lambda \in \{1, \dots, l\}$). By the argument used in the last section, we have

$$(2.5) \qquad d_{i\lambda}.e(S_\lambda) = \dim_F(\bar{e}_\lambda E(FY), FZ_i),$$

where $e(S_\lambda) = \dim_F E(S_\lambda)$, Z_i is any R-form of $\mathbf{Z}_i$, and $\bar{e}_\lambda$ is a primitive idempotent of $E(FY)$ so chosen that

$$(2.6) \qquad \bar{e}_\lambda E(FY)/\mathrm{rad}\, \bar{e}_\lambda E(FY) \cong S_\lambda.$$

Because the discrete valuation ring R is complete, we may "lift" each $\bar{e}_\lambda$ to a primitive idempotent $e_\lambda \in E(Y)$ such that $\phi_F(e_\lambda) = \bar{e}_\lambda$ [4, Thm. (6.7), p. 123]. A standard theorem [4, Prop. (6.17), p. 130] now tells us that

$$(2.7) \qquad e_1 E(Y), \dots, e_l E(Y)$$

is a full set of indecomposable projective right $E(Y)$-lattices.

3. The functor $(Y, \;\;)$

The transition from Λ-lattices to $E(Y)$-lattices is most easily made by means of the familiar functor

$$T = (Y, \;\;): \mathrm{mod}\, \Lambda \to \mathrm{mod}\, E(Y)^{\mathrm{op}}.$$

Here mod Λ and mod $E(Y)^{\mathrm{op}}$ denote the categories of left Λ-lattices and right $E(Y)$-lattices, respectively. T takes any $X \in \mathrm{mod}\, \Lambda$ to $T(X) := (Y, X)_\Lambda$, which has a natural structure of right $E(Y)$-lattice: $h \in E(Y)$ acts on $f \in (Y, X)_\Lambda$ to give $fh \in (Y, X)_\Lambda$. T takes any Λ-map ξ: $M \to X$ to the $E(Y)$-map

$$T(\xi): (Y, M)_\Lambda \to (Y, X)_\Lambda$$

given by $T(\xi)(g) = \xi g (g \in (Y, M)_\Lambda)$. Also T is an R-functor, which means that, for any M, $X \in \mathrm{mod}\, \Lambda$, the map

$$T_{M,X}: (M, X)_\Lambda \to ((Y, M)_\Lambda, (Y, X)_\Lambda)_{E(Y)}$$

which takes $\xi \to T(\xi)$, is R-linear. It follows that T commutes with finite direct sums.

Let add Y denote the full subcategory of mod Λ whose objects are all the components of finite direct sums of copies of Y. Since $T(Y) = (Y, Y) = E(Y)_{E(Y)}$, it is clear that $T(M)$ is a projective right $E(Y)$-lattice, for all $M \in \text{add}\, Y$. The next proposition is well known (see M. Auslander [1], Prop. 27(d), p. 193 or [4], Prop. (6.3), p. 120), and follows easily from Lemmas (3.2), (3.3) below.

(3.1) Proposition. *The functor T induces a category equivalence between add Y and the category $\mathfrak{P}(E(Y)^{\text{op}})$ of all finitely generated projective right $E(Y)$-lattices.*

(3.2) Lemma. *If $M \in \text{add}\, Y$, then the R-map $T_{M,X}$ is bijective, for all $X \in \text{mod}\, \Lambda$.*

Proof. First verify that $T_{Y,X}$ is bijective, which is easy. One then shows that $T_{M,X}$ is bijective for any component M of Y [7, Lemma (2.1a), p. 249]; the lemma follows.

(3.3) Lemma. *If e is an idempotent in $E(Y)$, then $T(e(Y)) = (Y, e(Y))_\Lambda$ is isomorphic, as right $E(Y)$-lattice, to $e(Y, Y)_\Lambda = eE(Y)$.*

Proof. Let p: $Y \to e(Y)$ (resp. i: $e(Y) \to Y$) be the projection (resp. inclusion) map. Check that $g \to ig$ ($g \in (Y, e(Y)_\Lambda)$ defines an $E(Y)$-isomorphism $(Y, e(Y))_\Lambda \to e(Y, Y)_\Lambda$, with inverse $f \to pf$ ($f \in e(Y, Y)_\Lambda$).

Now let $e_1, \dots, e_l$ be the primitive idempotents of $E(Y)$ which figure in (2.7). Then for any indecomposable component Y' of Y, there is precisely one $\lambda \in \{1, \dots, l\}$ such that $(Y, Y')_\Lambda \cong e_\lambda E(Y)$ as right $E(Y)$-lattices (the $E(Y)$-lattice $(Y, Y')_\Lambda$ is indecomposable by (3.1)), hence such that $Y' \cong e_\lambda(Y)$ as Λ-lattices (since $(Y, Y')_\Lambda \cong (Y, e_\lambda(Y)_\Lambda)$ by (3.3), and this implies $Y' \cong e_\lambda(Y)$ by (3.1)). Therefore

$$e_1(Y), \dots, e_l(Y)$$

is a full set of indecomposable components of Y. This can be restated as the following proposition.

(3.4) Proposition. *If $Y_1, \dots, Y_l$ is a full set of indecomposable components of Y, then $(Y, Y_1)_\Lambda, \dots, (Y, Y_l)_\Lambda$ is a full set of indecomposable projective right $E(Y)$-modules; in fact the Y_λ can be so numbered that*

$$(3.5) \qquad (Y, Y_\lambda)_\Lambda \cong e_\lambda E(Y) \text{ as right } E(Y)\text{-lattices},$$

for all $\lambda \in \{1, \dots, l\}$.

All the preceding discussion of the functor $(Y, \ )$ holds good for the functor $(kY, \ )$: $\text{mod}\, kY \to \text{mod}\, E(kY)^{\text{op}}$ $(k \in \{K, F\})$; one has only to replace Y by kY, and "lattice" by "finitely-generated module", throughout. For $k = K, F$, an argument analogous to that of Proposition (3.4) gives:

(3.6) *If $U_1, \ldots, U_r$ is a full set of indecomposable components of the $k\Lambda$-module kY, then $(kY, U_1)_{k\Lambda}, \ldots, (kY, U_r)_{k\Lambda}$ is a full set of indecomposable projective right $E(kY)$-modules.*

Returning to the case $k = F$, suppose that $Y_1, \ldots, Y_l$ are as in Proposition (3.4). Then we find

$$(3.7) \qquad (FY, FY_\lambda)_{F\Lambda} \cong \bar{e}_\lambda E(FY), \quad \textit{for all } \lambda \in \{1, \ldots, l\}.$$

For our assumption that Y is F-endostable, together with (2.4)(i), shows that the maps ϕ_F: $e_\lambda E(Y) \to \bar{e}_\lambda E(FY)$ and ϕ_F: $(Y, Y_\lambda)_\Lambda \to (FY, FY)_{F\Lambda}$ are both surjective (remember that $\phi_F(e_\lambda) = \bar{e}_\lambda$). Thus

$$(FY, FY_\lambda)_{F\Lambda} \cong (Y, Y_\lambda)_\Lambda / \pi (Y, Y_\lambda)_\Lambda \cong e_\lambda E(Y)/\pi e_\lambda E(Y) \cong \bar{e}_\lambda E(FY).$$

Finally, combining (3.7) with (2.6) we have

$$(3.8) \quad (FY, FY_\lambda)_{F\Lambda}/\text{rad}(FY, FY_\lambda)_{F\Lambda} \cong S_\lambda, \quad \textit{for all } \lambda \in \{1, \ldots, l\}.$$

Any one of the (equivalent) conditions (3.5), (3.7), (3.8) serves to show how the numbering of the components Y_λ, is 'compatible' with that of the simple $E(FY)$-modules S_λ.

4. The theorem

From now on we arrange the simple $K\Lambda$-modules $\mathbf{X}_1, \ldots, \mathbf{X}_t$ (see Section 1) so that $\mathbf{X}_1, \ldots, \mathbf{X}_s$ are components of KY, while for $i > s$, $\mathbf{X}_i$ is not a component of KY. Then (remembering that both $K\Lambda$ and $E(KY)$ are semisimple K-algebras) $\mathbf{X}_1, \ldots, \mathbf{X}_s$ is a full set of indecomposable $K\Lambda$-components of KY, so by (3.6),

$$(KY, \mathbf{X}_1)_{K\Lambda}, \ldots, (KY, \mathbf{X}_s)_{K\Lambda}$$

is a full set of simple right $E(KY)$-modules. Write $\mathbf{Z}_i = (KY, \mathbf{X}_i)_{k\Lambda}$ $(i \in \{1, \ldots, s\})$, and use this numbering to define the decomposition numbers $d_{i\lambda}$ of Section 2 $(\lambda \in \{1, \ldots, l\})$.

Suppose now that Y_λ is an indecomposable Λ-component of Y such that $Y_\lambda \cong e_\lambda(Y)$ (see (3.3)). Then KY_λ is a $K\Lambda$-component, in general not indecomposable, of KY.

(4.1) DEFINITION. For any $i \in \{1,\ldots,s\}$, $\lambda \in \{1,\ldots,l\}$, $d^*_{i\lambda}$ is the multiplicity of $\mathbf{X}_i$ as component of KY_λ.

We are now at last in a position to state our theorem.

(4.2) THEOREM. *Let Λ be an R-order in a semisimple K-algebra $K\Lambda$, and let Y be a non-zero F-endostable Λ-lattice. Let $\mathbf{X}_1,\ldots,\mathbf{X}_s$ be a full set of simple $K\Lambda$-modules which are components of KY. Let $S_1,\ldots,S_l$ be a full set of simple $E(FY)$-modules. Then the numbers $d_{i\lambda}$, $d^*_{i\lambda}$ defined above are connected by the equation*

$$(4.3) \qquad d^*_{i\lambda}.e(\mathbf{X}_i) = d_{i\lambda}.e(S_\lambda),$$

for all $i \in \{1,\ldots,s\}$, $\lambda \in \{1,\ldots,l\}$. Here

$$e(S_\lambda) := \dim_F E(S_\lambda) \quad \textit{and} \quad e(\mathbf{X}_i) := \dim_K E(\mathbf{X}_i).$$

Proof (E.C. Dade). Since $Y \cong e_\lambda(Y)$, it follows from (3.5) and (2.6) that $(Y, Y_\lambda)_\Lambda$ is a projective $E(Y)$-lattice which covers the simple $E(FY)$-module S_λ. So if we replace Λ, $\mathbf{X}_i$, Λ_α, E_α in Brauer's formula (1.3) by $E(Y)$, $\mathbf{Z}_i$, $(Y, Y_\lambda)_\Lambda$, S_λ, respectively, we get

$$(4.4) \qquad \delta^*_{i\lambda}.e(\mathbf{Z}_i) = \delta_{i\lambda}.e(S_\lambda),$$

where $\delta_{i\lambda}$ is exactly the decomposition number $d_{i\lambda}$ defined in Section 2, and $\delta^*_{i\lambda}$ is the multiplicity of $\mathbf{Z}_i = (KY, \mathbf{X}_i)_{K\Lambda}$ as a component of $K(Y, Y_\lambda)_\Lambda \cong (KY, KY_\lambda)_{K\Lambda}$. But the functor

$$(KY, \ \ ): \text{mod } K\Lambda \to \text{mod } E(KY)^{\text{op}}$$

induces an equivalence of categories add $KY \to$ mod $E(KY)^{\text{op}}$, by the analog of Proposition (3.1) (all $E(KY)^{\text{op}}$-modules are projective, of course). From this follows at once that $\delta^*_{i\lambda}$ equals the multiplicity $d^*_{i\lambda}$ of $\mathbf{X}_i$ as component of KY_λ; also that $e(\mathbf{Z}_i) = e(\mathbf{X}_i)$. Therefore (4.4) is the required formula (4.3).

Remarks 1. If $Y =_\Lambda \Lambda$, we have $E(Y) \cong \Lambda^{\text{op}}$, and theorem (4.2) reverts to Brauer's theorem (1.3) in its original form.

2. If K is a splitting field for $K\Lambda$ and if F is a splitting field for $E(FY)$, then $e(\mathbf{X}_i) = 1$, $e(S_\lambda) = 1$ for all i, λ and hence (4.3) reduces to

$$(4.5) \qquad d^*_{i\lambda} = d_{i\lambda} \quad (i \in \{1,\ldots,s\}, \lambda \in \{1,\ldots,l\}).$$

3. Even in a case where Y is not a projective Λ-lattice, it may happen that some indecomposable component Y_λ of Y is projective. Then $Y_\lambda \cong \Lambda_\alpha$ for

some $\alpha \in \{1, \dots, a\}$ (see Section 1) and $d_{i\lambda}^* = \delta_{i\alpha}^*$ for $1 \le i \le s$, while for $s < i \le t$, $\delta_{i\alpha}^* = 0$, since $\mathbf{X}_i$ is not a component of $\Lambda_\alpha \cong Y_\lambda$. We may now use Brauer's theorem (1.3),

$$\delta_{i\alpha}^*.e(\mathbf{X}_i) = \delta_{i\alpha}.e(E_\alpha).$$

Comparing this with (4.3) we have a relation between decomposition numbers, namely

$$(4.6) \qquad \delta_{i\alpha}.e(E_\alpha) = d_{i\lambda}.e(S_\lambda) \quad \text{for all } i \in \{1, \dots, s\}.$$

In particular, if F is a splitting field for both $F\Lambda$ and $E(FY)$, then the λ-th column of the decomposition matrix $(d_{i\lambda})$ for $E(Y)$ coincides, as far as the rows $1, \dots, s$ are concerned, with the α-th column of the decomposition matrix $(\delta_{i\alpha})$ for Λ. The example in the next section provides a striking illustration of this phenomenon.

5. James's theorem

In this section we assume that $\operatorname{char} K = 0$, and that $\operatorname{char} F = p > 0$.

Let n, r be positive integers with $r \le n$, let E be a free R-module with basis $e_1, \dots, e_n$, and let $Y = E^{\otimes r}$ be the r-fold tensor product $E \otimes_R \cdots \otimes_R E$. Then Y can be regarded as right RG-lattice, where G is the symmetric group on $\{1, \dots, r\}$, acting by 'place permutations' [8, p. 28]. We shall use notations from [8] (with slight modifications) without further comment. However, since we start with a *right* RG-lattice Y, we must transpose 'left' and 'right' in Theorem (4.2), in order to apply it to the present case. This gives little trouble; the functor

$$(\ \ , Y)\colon \operatorname{mod} \Lambda^{\mathrm{op}} \to \operatorname{mod} E(Y)$$

takes the place of $(Y, \ \ )$, so that we regard (X, Y) as a left $E(Y)$-module, etc. We can identify $E(Y)$, $E(KY)$, $E(FY)$ with the corresponding Schur algebras $S_R(n, r)$, $S_K(n, r)$, $S_F(n, r)$. Since Y is a permutation RG-lattice, Y is F-endostable. The Weyl modules $\{V_{\lambda, K}\colon \lambda \vdash r\}$ [8, p. 65] form a full set of simple $S_K(n, r)$-modules, and the unique simple factor modules $\{F_{\lambda, F}\colon \lambda \vdash r\}$ of the 'characteristic p' Weyl modules $V_{\lambda, F}$ [8, p. 71] form a full set of simple $S_F(n, r)$-modules. The decomposition number $d_{\lambda\mu}$ (corresponding to $d_{i\lambda}$ in equation (4.3)) is the multiplicity of $F_{\mu, F}$ as a composition factor in $V_{\lambda, F}$. Moreover $e(F_{\mu, K}) = 1$, from the fact that $F_{\mu, F}$ is generated by its μ-weight space, which has dimension one [8, (5.4a), (5.4b), p. 71].

In [6], [9] and [11] it is proved (in three very different ways!) that, for any field k, a full set of indecomposable kG-components of $kY = (kE)^{\otimes r}$ can be labelled $U_{\lambda,k}$ ($\lambda \vdash r$) in such a way that for each pair λ, $\mu \vdash r$ with $\mu \rhd \lambda$ (see [10], p. 23 for the definition of the partial order $\trianglerighteq$) there exists a non-negative integer $a_{\lambda,\mu}(c)$ depending only on the characteristic c of k, so that

$$(5.1) \qquad M_{\lambda,k} \cong U_{\lambda,k} \oplus \sum_{\mu \rhd \lambda}^{\oplus} a_{\lambda,\mu}(c) U_{\mu,k},$$

for all $\lambda \vdash r$; here $M_{\lambda,k}$ is the permutation kG-module $k_{G_\lambda}^{G}$ where G_λ is the Young subgroup [10, p. 16] corresponding to λ. It is clear from the Krull-Schmidt theorem that the indecomposable kG-modules $U_{\lambda,k}$ are determined up to isomorphism by these equations (5.1); therefore $U_{\lambda,k}$ is isomorphic to the module denoted V_λ in [9], p. 12, and also to James's $I_{\lambda,k}$ (see [11], Theorem 3.1(i); note that James's fields K and F are our F and K, respectively!).

It is proved in [9], Remark 6, pp. 14–16, that the simple $GL_n(k)$-module (or $S_k(n,r)$-module) $F_{\lambda,k}$ is associated by the Brauer-Fitting theorem to the components of kY of type $U_{\lambda,k}$, which means precisely that

$$(5.2) \qquad (U_{\lambda,k}, kY)_{kG} / \mathrm{rad}(U_{\lambda,k}, kY)_{kG} \cong F_{\lambda,k}.$$

(James proves an equivalent result in [11], but a little less directly.)

By 'idempotent lifting' we find a full set $\{Y_\lambda: \lambda \vdash r\}$ of indecomposable RG-components of $Y = E^{\otimes r}$ such that $FY_\lambda \cong U_{\lambda,F}$ ($\lambda \vdash r$). Equations (5.2) give

$$(FY_\lambda, FY)_{FG} / \mathrm{rad}(FY_\lambda, FY)_{FG} \cong F_{\lambda,F},$$

and so our labelling Y_λ is compatible (see (3.8)) with the labelling of the simple $E(FY) = S_F(n,r)$-modules $F_{\lambda,F}$.

Now take $k = K$ in (5.1) and (5.2). Equations (5.1) show that the (simple) KG-module $U_{\lambda,K}$ has character ζ^λ in standard notation (see [10], §2.2). So we may take $U_{\lambda,K}$ to be the Specht module S_K^λ over K [10, p. 396]. Another classical result says that $e(S_K^\lambda) = 1$ [3, Exercise 3, p. 206]. The full set $\{S_K^\lambda: \lambda \vdash r\}$ of simple KG-modules corresponds to $\{\mathbf{X}_1, \ldots, \mathbf{X}_t\}$ in our general notation, so that (definition) $d_{\lambda\mu}^*$ is the multiplicity of S_K^μ in KY_λ. All the S_K^λ appear as components of KY, so that $s = t$ in the notation of Section 4; but we must be sure to label the simple $E(KY) = S_K(n,r)$-modules $\mathbf{Z}_\lambda$ so that

$$\mathbf{Z}_\lambda \cong (S_K^\lambda, KY)_{KG}$$

(this corresponds to $\mathbf{Z}_i = (KY, \mathbf{X}_i)_{K\Lambda}$ in Section 4). Fortunately (5.2) gives

$$(S_K^\lambda, KY)_{KG} \cong (U_{\lambda,K}, KY)_{KG} \cong F_{\lambda,K} \cong V_{\lambda,K}.$$

So we may take $\mathbf{Z}_\lambda = V_{\lambda, K}$, which means that the $d_{\lambda\mu}$ have the meaning announced earlier in this section, and Theorem (4.2) gives James's Theorem 3.4(ii) [11] namely

$$d^*_{\lambda\mu} = d_{\lambda\mu} \quad \text{for all } \lambda, \mu \vdash r.$$

Finally we may recover an earlier theorem of James involving the decomposition numbers $\delta_{\lambda\mu}$ for G, namely

$$\delta_{\lambda\mu} = d_{\lambda\mu}$$

for all $\lambda \vdash r$, and all column p-regular $\mu \vdash r$ (see [11], Section 1). For it can be shown that Y_μ (or, what comes to the same thing, FY_μ) is projective if and only if μ is column p-regular; now we may apply Remark 3 of the last section.

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