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A MAYER-VIETORIS SEQUENCE FOR PICARD GROUPS, WITH APPLICATIONS TO INTEGRAL GROUP RINGS OF DIHEDRAL AND QUATERNION GROUPS

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In Memoriam Irving Reiner

0. Introduction

In this paper, we show how Mayer-Vietoris sequences can be constructed to permit the computation of Picard groups and outer automorphism groups of orders from fibre product diagrams. We then illustrate the use of these sequences by carrying out the computations for certain group rings. The idea that there might be such sequences was inspired by the use of pullback methods in the construction [20] by the second author and L.L. Scott, Jr. of a counterexample to the Zassenhaus conjecture.

We feel that the mathematics in the paper is very much in the spirit of our dear friend and teacher, the late Irving Reiner. We humbly dedicate this work to his memory.

Let R be a Dedekind domain with field of fractions K; for instance, the ring of algebraic integers in the algebraic number field K. Let Λ be an R-order in a separable K-algebra A. For an R-subalgebra T of the center $Z(\Lambda)$ of Λ, we denote by $\mathrm{Pic}_T(\Lambda)$ the group of isomorphism classes $[M]$ of invertible Λ-bimodules with $tm = mt$ whenever $t \in T$ and $m \in M$. This group was first studied in the setting of orders by Fröhlich [3]. We shall consider the subgroup $\mathrm{LFPic}_T(\Lambda)$ of $\mathrm{Pic}_T(\Lambda)$ consisting of those $[M]$ for which M is locally free on one side. By a result of Swan [23], we have

$$\mathrm{Pic}_T(\mathbf{Z}G) = \mathrm{LFPic}_T(\mathbf{Z}G),$$

where $\mathbf{Z}G$ denotes the integral group ring of the finite group G.

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Given central orthogonal idempotents e_1 and e_2 of A with $e_1 + e_2 = 1$, the order Λ can be written (cf., Lemma 1.1) as a fibre product

$$\begin{array}{ccc} \Lambda & \xrightarrow{\mathrm{pr}_1} & \Lambda e_1 = \Lambda_1 \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle \varphi_1} \\ \Lambda_2 = \Lambda e_2 & \xrightarrow{\varphi_2} & \overline{\Lambda} \end{array} \tag{0.1}$$

We shall denote by $\mathrm{Pic}_{e,T}(\Lambda)$ and $\mathrm{LFPic}_{e,T}(\Lambda)$ the subgroups of $\mathrm{Pic}_T(\Lambda)$ and $\mathrm{LFPic}_T(\Lambda)$ consisting of the isomorphism classes of those bimodules M in $\mathrm{Pic}_T(\Lambda)$ or $\mathrm{LFPic}_T(\Lambda)$ with $e_i m = me_i$ for every $m \in M$. It should be noted that

$$\mathrm{Pic}_{e,T}(\Lambda) \supseteq \mathrm{Picent}(\Lambda) =: \mathrm{Pic}_{Z(\Lambda)}(\Lambda).$$

Provided that $\mathrm{Ker}\,\varphi_i$ is characteristic in Λ_i for $i = 1, 2$, the maps in the fibre product diagram (0.1) give rise to homomorphisms

$$\mathrm{pr}_i\colon \mathrm{Pic}_{e,T}(\Lambda) \to \mathrm{Pic}_{T_i}(\Lambda_i), \tag{0.2}$$

and

$$\varphi_i\colon \mathrm{Pic}_{T_i}(\Lambda_i) \to \mathrm{Pic}_{\overline{T}}(\overline{\Lambda}),$$

where $T_i = Te_i$ and $\overline{T}$ is the image of T in the center of $\overline{\Lambda}$. Note that if T is the center $Z(\Lambda)$, it is often the case that T_i is properly contained in $Z(\Lambda_i)$. In general, $\mathrm{Pic}_{T_i}(\Lambda_i)$ is not abelian, and so φ_1 and φ_2 do *not* induce a group homomorphism

$$\varphi_1 \oplus \varphi_2\colon \mathrm{Pic}_{T_1}(\Lambda_1) \oplus \mathrm{Pic}_{T_2}(\Lambda_2) \to \mathrm{Pic}_{\overline{T}}(\overline{\Lambda}).$$

However, if we define

$$\mathrm{Pic}_T(\Lambda_1, \Lambda_2) = \{([M_1],[M_2])\colon [M_i] \in \mathrm{Pic}_{T_i}(\Lambda_i), \\ \varphi_1([M_1]) = \varphi_2([M_2])\}, \tag{0.3}$$

then it turns out that $\mathrm{Pic}_T(\Lambda_1, \Lambda_2)$ is a group.

We shall denote by $u(B)$ the group of units of the ring B, and put $u_Z(B) = u(Z(B))$. There is a group homomorphism

$$\Psi\colon u_Z(\overline{\Lambda}) \to \mathrm{Picent}(\Lambda) \tag{0.4}$$

defined by $\Psi(\bar{u}) = [\Lambda_{\bar{u}}]$, where

$$\Lambda_{\bar{u}} = \{(\lambda_1, \lambda_2) \in \Lambda_1 \oplus \Lambda_2 \colon \varphi_1(\lambda_1) = \bar{u}\varphi_2(\lambda_2)\}$$

We shall show in §1 that—provided each $\operatorname{Ker}\varphi_i$ is characteristic in Λ_i—the fibre product diagram gives rise to a Mayer-Vietoris sequence

$$(0.5) \qquad 1 \to u_Z(\bar{\Lambda})/\langle \varphi_1(u_Z(\Lambda_1)), \varphi_2(u_Z(\Lambda_2))\rangle \to \operatorname{Pic}_{e,T}(\Lambda) \to \operatorname{Pic}_T(\Lambda_1, \Lambda_2) \to 1,$$

where $\langle \varphi_1(u_Z(\Lambda_1)), \varphi_2(u_Z(\Lambda_2))\rangle$ is the subgroup of $u_Z(\bar{\Lambda})$ generated by $\varphi_1(u_Z(\Lambda_1))$ and $\varphi_2(u_Z(\Lambda_2))$. We have a similar sequence for locally free Picard groups. Let

$$\vartheta_T \colon \operatorname{LFPic}_{e,T}(\Lambda) \to \operatorname{Cl}(\Lambda),$$

where $\operatorname{Cl}(\Lambda)$ is the class group of locally free left Λ-ideals be the natural homomorphism [14], and put $\widetilde{\operatorname{Out}}_{e,T} = \ker(\vartheta_T)$. If Λ satisfies the Eichler condition, then $\widetilde{\operatorname{Out}}_{e,T}(\Lambda) = \operatorname{Out}_{e,T}(\Lambda)$ is the group of T-linear outer automorphisms of Λ that preserve e_1 and e_2. We write $\widetilde{\operatorname{Out}}_{e,C}(\Lambda)$ for $\widetilde{\operatorname{Out}}_{e,Z(\Lambda)}(\Lambda)$.

With or without the Eichler condition, we have

$$\operatorname{Out}_T(\Lambda) = \operatorname{Ker}\big(\operatorname{LFPic}_T(\Lambda) \xrightarrow{\vartheta} \operatorname{LF}_1(\Lambda)\big),$$

where $\operatorname{LF}_1(\Lambda)$ is the pointed set of isomorphism classes of locally free full left Λ-ideals, with $[\Lambda]$ as basepoint. Putting

$$\operatorname{Out}_{e,T}(\Lambda) = \operatorname{Ker}\big(\vartheta|_{\operatorname{LFPic}_{e,T}(\Lambda)}\big),$$

$$\operatorname{Out}_T(\Lambda_1, \Lambda_2) = \big\{(\alpha_1, \alpha_2) \colon \alpha_i \in \operatorname{Out}_{T_i}(\Lambda_i), \alpha_1 \equiv \alpha_2 \text{ in } \operatorname{Out}_{\bar{T}}(\bar{\Lambda})\big\}$$

and

$$(0.6) \qquad u_Z(\Lambda_1, \Lambda_2) = u_Z(\bar{\Lambda})\big/\langle \varphi_1(u_Z(\Lambda_1)), \varphi_2(u_Z(\Lambda_2))\rangle,$$

we obtain an exact sequence

$$(0.7) \qquad 1 \to u_Z(\Lambda_1, \Lambda_2) \to \operatorname{Out}_{e,T}(\Lambda) \xrightarrow{\varrho} \operatorname{Out}_T(\Lambda_1, \Lambda_2),$$

in which ϱ is surjective when $\bar{\Lambda}$ is commutative.

We shall apply these sequences to compute the central Picard groups and the central automorphism groups for the following classes of groups:

(1) Metacyclic groups of order pq, where p is an odd prime and $q|(p-1)$.
(2) Dihedral groups of order 2^{n+1}.
(3) Quaternion groups of order 2^{n+1}.

It should be noted that the groups in all these classes satisfy the Zassenhaus conjecture, i.e., for any normalized automorphism α of the group ring $\mathbf{Z}G$, there is an automorphism ρ of G such that $\alpha\rho$ is a central automorphism [17], [19], [22]. Note that all these groups have $O_{p'}(G) = 1$ for some prime p. Moreover, an arbitrary automorphism can be normalized by modifying it with an automorphism induced by an element of $\mathrm{Hom}(G, u(\mathbf{Z}))$. Hence, as indicated in [19], to describe $\mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}G)$ and the outer automorphism group, it suffices to describe $\mathrm{Picent}(\mathbf{Z}G)$ and the group $\mathrm{Out}_C(\mathbf{Z}G)$ of automorphisms that fix each element of the center.

In the first two cases, the computations have also been made by Endo, Miyata, and Sekiguchi [2]. Their method is less general than ours. They are able to show for these groups that $\mathrm{Picent}(\mathbf{Z}G)$ is isomorphic to the locally free class group of the center of $\mathbf{Z}G$. Then, they can apply known Mayer-Vietoris sequences for class groups. Some results on Picard groups of integral group rings of abelian groups were given by Bass and Murthy in [1].

In the discussion of our examples, the reader should keep in mind that the righthand term of the sequence (0.5) is contained in $\mathrm{Pic}_{T_1}(\Lambda_1) \oplus \mathrm{Pic}_{T_2}(\Lambda_2)$, while that of (0.7) is contained in $\mathrm{Out}_{T_1}(\Lambda_1) \oplus \mathrm{Out}_{T_2}(\Lambda_2)$. It may be necessary to refer to the subsequent sections to see how the groups explicitly given are embedded in the appropriate factors.

(A) *Metacyclic groups.* Let p be an odd prime and q a divisor of $p-1$. For an integer n, denote by C_n the cyclic group of order n. Let r have order q modulo p and set

$$G = C_p \rtimes C_q = \langle a, b \colon a^p = b^q = 1, bab^{-1} = a^r \rangle,$$

a subgroup of the 1-dimensional affine group over $\mathbf{Z}/p\mathbf{Z}$. If ζ_p is a primitive pth root of unity, we can interpret C_q as a subgroup of $\mathrm{Gal}_{\mathbf{Z}}(\mathbf{Z}[\zeta_p])$. Let S be the fixed ring in $\mathbf{Z}[\zeta_p]$ under C_q, and let π be the norm of $1-\zeta_p$ in S. For $e_1 = (1/p)\Sigma_{i=1}^{p} a^i$ and $e_2 = 1 - e_1$, $\mathbf{Z}G$ is the pullback

$$\begin{array}{ccc} \mathbf{Z}G & \xrightarrow{\mathrm{pr}_1} & \mathbf{Z}C_q \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \varphi_1} \\ \Lambda & \xrightarrow{\varphi_2} & \mathbf{F}_p C_q \end{array}$$

where

$$\Lambda = \begin{pmatrix} S & \cdots & \cdots & S \\ \pi S & \ddots & & S \\ \vdots & \ddots & \ddots & \vdots \\ \pi S & \cdots & \pi S & S \end{pmatrix}_{q \times q}.$$

Every $\sigma \in \mathrm{Hom}(G, \{\pm 1\})$ gives rise to an automorphism $\tilde{\sigma}$: $\mathbf{Z}G \to \mathbf{Z}G$, induced by $g \mapsto g\sigma(g)$. In (2.15) we prove that there are exact sequences

$$(0.8)\quad 1 \to U_q \to \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}G) \to \mathrm{Cl}(S)\mathrm{Gal}_{\mathbf{Z}}(S) \oplus \mathrm{Cl}(\mathbf{Z}C_q)\mathrm{Hom}(G, \{\pm 1\}) \to 1,$$

where U_q is an abelian group of order $(p-1)^{q-1}/(2, q)$;

$$(0.9)\qquad 1 \to U_q \to \mathrm{Picent}(\mathbf{Z}G) \to \mathrm{Cl}(S) \oplus \mathrm{Cl}(\mathbf{Z}C_q) \to 1,$$

and

$$(0.10)\quad 1 \to \tilde{U}_q \to \mathrm{Out}_{\mathbf{Z}}(\mathbf{Z}G) \to \mathrm{Cl}(S)_q\mathrm{Gal}_{\mathbf{Z}}(S) \oplus \mathrm{Hom}(G, \{\pm 1\}) \to 1,$$

where $\mathrm{Cl}(S)_q = \{(\mathscr{T}) \in \mathrm{Cl}(S)\colon (\mathscr{T})^q = 1\}$ and $\tilde{U}_q$ has order $(p-1)^{q-1}/q$. In both (0.8) and (0.10), $\mathrm{Gal}_{\mathbf{Z}}(S)$ should be interpreted as $\mathrm{Out}(G)$.

Remark. In the case where $q = 2$, the sequence (0.9) reduces to

$$1 \to C_{(p-1)/2} \to \mathrm{Picent}(\mathbf{Z}G) \to \mathrm{Cl}(S) \to 1,$$

which was previously obtained by Fröhlich, Reiner, and Ullom [3], [5]. They also determined $\mathrm{Out}_C(\mathbf{Z}G)$ in this case.

(B) *Dihedral 2-groups.* Let

$$D_n = \langle s_n, t_n\colon s_n^{2^n} = t_n^2 = 1,\ t_n s_n t_n = s_n^{-1}\rangle,$$

and let ζ_n be a primitive 2^n-th root of unity. Put $S_n = \mathbf{Z}[\zeta_n + \zeta_n^{-1}]$, the maximal real subfield of $\mathbf{Z}[\zeta_n]$. Then there are exact sequences

$$(0.11)\quad 0 \to V \to \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}D_n) \to \mathrm{Cl}(S_n) \times \mathrm{Picent}(\mathbf{Z}D_{n-1})\mathrm{Out}(D_n)\mathrm{Hom}(D_n, \{\pm 1\}) \to 1,$$

where V is Klein's 4-group,

$$(0.12)\qquad 0 \to V \to \mathrm{Picent}(\mathbf{Z}D_n) \to \mathrm{Cl}(S_n) \times \mathrm{Picent}(\mathbf{Z}D_{n-1}) \to 1,$$

and

$$(0.13)\quad 0 \to V \to \mathrm{Out}_{\mathbf{Z}}(\mathbf{Z}D_n) \to \mathrm{Cl}(S_n)_2 \times \mathrm{Out}_C(\mathbf{Z}D_{n-1})\mathrm{Out}(D_n)\mathrm{Hom}(D_n, \{\pm 1\}) \to 1.$$

Again, this was done for $n = 2$ by Fröhlich [3].

(C) *Quaternion 2-groups.* Let

$$H_n = \langle \sigma_n, \tau\colon \sigma_n^{2^n} = \tau^4 = 1,\ \sigma_n^{2^{n-1}} = \tau^2,\ \tau\sigma_n\tau^{-1} = \sigma_n^{-1} \rangle,$$

and let S be as above. Then

$$(0.14)\qquad \begin{aligned} \mathrm{Picent}(\mathbf{Z}H_n) &\cong \mathrm{Picent}(\mathbf{Z}D_n), \\ \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}H_n) &= \mathrm{Picent}(\mathbf{Z}H_n)\mathrm{Out}(H_n)\mathrm{Hom}(H_n, \{\pm 1\}),, \end{aligned}$$

and there is an exact sequence

$$(0.15)\qquad 0 \to C_2 \to \widetilde{\mathrm{Out}}_C(\mathbf{Z}H_n) \to \mathrm{Cl}(S_n)_2 \oplus \mathrm{Out}_C(\mathbf{Z}D_{n-1}) \to 0,$$

where $\widetilde{\mathrm{Out}}_C(\mathbf{Z}H_n)$ is the kernel of $\mathrm{Picent}(\mathbf{Z}H_n) \to \underline{\mathrm{Cl}}(\mathbf{Z}H_n)$. Once more, this was obtained by Fröhlich [3] for $n = 2$. For $n \leq 3$, $\widetilde{\mathrm{Out}}_{\mathbf{Z}}(\mathbf{Z}H_n) = \mathrm{Out}_{\mathbf{Z}}(\mathbf{Z}H_n)$.

1. Proof of the Mayer-Vietoris Sequence

As in the introduction, let e_1, e_2 be central idempotents with $e_1 + e_2 = 1$. We have claimed in the introduction that

$$\begin{array}{ccc} \Lambda & \xrightarrow{\mathrm{pr}_1} & \Lambda e_1 = \Lambda_1 \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \varphi_1} \\ \Lambda e_2 = \Lambda_2 & \xrightarrow{\varphi_2} & \overline{\Lambda} \end{array}$$

is a pullback. This will follow, once we prove the following.

LEMMA 1.1. $\overline{\Lambda} = \Lambda/((\Lambda e_1 \cap \Lambda) \oplus (\Lambda e_2 \cap \Lambda))$ *is isomorphic to* $\Lambda_i/(\Lambda \cap \Lambda_i)$, *for* $i = 1, 2$.

Proof. Multiplication by e_i gives an isomorphism $\overline{\Lambda} \cong \Lambda_i/(\Lambda \cap \Lambda_i)$, whence the statement.

We assume henceforth that

$$\operatorname{Ker} \varphi_i \textit{ is characteristic in } \Lambda_i.$$

We consider $\mathrm{LFPic}_{e,T}(\Lambda)$, consisting of the isomorphism classes of locally free invertible Λ-bimodules where the e_i and T act in the same way on each side. Let T_i be the image of T in Λ_i, $i = 1, 2$, and let $\overline{T}$ be the image of T in $\overline{\Lambda}$. Note that if $T = Z(\Lambda)$, it is not generally the case that $T_i = Z(\Lambda_i)$, where $Z(-)$ denotes the center. We have natural maps

$$\mathrm{pr}_i\colon \mathrm{LFPic}_{e,T}(\Lambda) \to \mathrm{LFPic}_{T_i}(\Lambda_i)$$

for $i = 1, 2$, given by $[M] \mapsto [Me_i]$. We note that $\overline{\Lambda}$ is a finitely generated R-torsion algebra. Hence, the map $\varphi_i\colon \Lambda_i \to \overline{\Lambda}$ factors through a semi-localization $\tilde{\Lambda}_i$, and hence, since we work with locally free invertible bimodules, the elements in $\mathrm{LFPic}_{\tilde{T}}(\tilde{\Lambda}_i)$ can be interpreted as automorphisms. Since $\operatorname{Ker} \varphi_i$ is characteristic, the same holds for the semi-localization, and hence we obtain a well defined map

$$\varphi_i\colon \mathrm{LFPic}_{T_i}(\Lambda_i) \to \mathrm{Pic}_{\overline{T}}(\overline{\Lambda}).$$

Let

$$\mathrm{LFPic}_T(\Lambda_1, \Lambda_2) = \big\{([M_1],[M_2])\colon M_i \in \mathrm{LFPic}_{T_i}(\Lambda_i) \text{ and } \varphi_1([M_1]) = \varphi_2([M_2]) \text{ in } \mathrm{LFPic}_{\overline{T}}(\overline{\Lambda})\big\}.$$

We note that $\mathrm{LFPic}_T(\Lambda_1, \Lambda_2)$ is a subgroup of $\mathrm{LFPic}_{T_1}(\Lambda_1) \times \mathrm{LFPic}_{T_2}(\Lambda_2)$. In fact, whenever $\varrho_i\colon G_i \to G$, $i = 1, 2$, are group homomorphisms, the map $\varrho\colon G_1 \times G_2 \to G$ defined by

$$(g_1, g_2) \mapsto \varrho_1(g_1) \cdot \varrho_2(g_2)^{-1}$$

may fail to be a group homomorphism, but $\varrho^{-1}(1)$ is a subgroup of $G_1 \times G_2$. Indeed, if (g_1, g_2) and (h_1, h_2) belong to this set, then $\varrho_1(g_1) = \varrho_2(g_2)$ and $\varrho_1(h_1) = \varrho_2(h_2)$. Multiplying these equations together, we obtain $\varrho_1(g_1h_1) = \varrho_2(g_2h_2)$, whence $(g_1h_1, g_2h_2) \in \varrho^{-1}(1)$. Taking inverses of both sides of the first equation gives $\varrho_1(g_1^{-1}) = \varrho_2(g_2^{-1})$, and $\varrho^{-1}(1)$ is a subgroup. In particular, $\mathrm{LFPic}_T(\Lambda_1, \Lambda_2)$ is a subgroup of $\mathrm{Pic}_{T_1}(\Lambda_1) \times \mathrm{Pic}_{T_2}(\Lambda_2)$. We then have a mapping

$$\Phi\colon \mathrm{LFPic}_{e,T}(\Lambda) \to \mathrm{LFPic}_T(\Lambda_1, \Lambda_2)$$

defined by

$$[M] \mapsto ([Me_1], [Me_2]),$$

which is, in fact, a group homomorphism.

LEMMA 1.2. $\operatorname{Im}\Phi = \mathrm{LFPic}_{e,T}(\Lambda_1, \Lambda_2)$.

Proof. Let $([M_1], [M_2]) \in \mathrm{LFPic}_T(\Lambda_1, \Lambda_2)$. Since $\varphi_1([M_1]) = \varphi_2([M_2])$, there is a $\overline{\Lambda}$-bimodule isomorphism $\varrho\colon \overline{M}_1 \to \overline{M}_2$, where $\overline{M}_i$ is the $\overline{\Lambda}$-bimodule $\varphi_i(M_i)$.

We now consider

$$M = \{(m_1, m_2) \in M_1 \times M_2\colon \varrho\varphi_1(m_1) = \varphi_2(m_2)\}.$$

Since φ_i and ϱ are bimodule homomorphisms, we conclude that M is a bimodule that is free on either side, and hence represents an element in $\mathrm{LFPic}_{e,T}(\Lambda)$; see [12, §2] or [13]. Clearly, $\mathrm{pr}_i M \cong M_i$.

This shows that Φ is surjective. It remains to consider the kernel of Φ. For this, we follow the idea of Reiner and Ullom [13, §5].

(1.3) For a ring B, let $u(B)$ be the units in B and let $u_Z(B)$ be the units in the center of B.

Define

$$\Psi\colon u_Z(\overline{\Lambda}) \to \mathrm{LFPicent}(\Lambda) \tag{1.4}$$

as follows: for $\bar{u} \in u_Z(\overline{\Lambda})$, let

$$\Lambda_{\bar{u}} = \{(x_1, x_2)\colon x_i \in \Lambda_i, \varphi_1(x_1) = \bar{u}\varphi_2(x_2)\}.$$

Since $\bar{u}$ is central, $\Lambda_{\bar{u}}$ is a Λ-bimodule. In fact, let $(\lambda_1, \lambda_2) \in \Lambda$, i.e., $\varphi_1\lambda_1 = \varphi_2\lambda_2$; then we have

$$(\lambda_1, \lambda_2)(x_1, x_2) = (\lambda_1 x_1, \lambda_2 x_2),$$

and

$$\begin{aligned}\varphi_1(\lambda_1 x_1) &= \varphi_1(\lambda_1)\varphi_1(x_1)\\ &= \varphi_2(\lambda_2)\bar{u}\varphi_2(x_2)\\ &= \bar{u}\varphi_2(\lambda_2 x_2).\end{aligned}$$

Hence, $\Lambda_{\bar{u}}$ is a left Λ-module, and one shows similarly that it is a right Λ-module as well. By [12, §2] or [13, 4.20], $\Lambda_{\bar{u}}$ is locally free, and is hence an invertible bimodule. Moreover, if $z \in Z(\Lambda)$, then

$$z(x_1, x_2) = (x_1, x_2)z \quad \text{for } (x_1, x_2) \in \Lambda_{\bar{u}}.$$

By [13, 4.20], Ψ is a group homomorphism. We point out also that $\Lambda_{\bar{u}} e_i = \Lambda e_i$, for $i = 1, 2$.

LEMMA 1.5. $\operatorname{Ker} \Psi = \langle \varphi_1(u_Z(\Lambda_1)), \varphi_2(u_Z(\Lambda_2)) \rangle$.

Proof. Assume $\Lambda_{\bar{u}} \cong \Lambda$ as bimodules. Since $\Lambda_{\bar{u}}$ and Λ are contained in A, there must exist a central element γ in A with $\Lambda_{\bar{u}}\gamma = \Lambda$, and so $\Lambda_{\bar{u}} e_i \gamma = \Lambda_i$. In particular, $\gamma_i = e_i \gamma$ is a central unit of Λ_i, $i = 1, 2$. The equation $\Lambda_{\bar{u}}\gamma = \Lambda$ now shows that $\bar{u} = \varphi_1(\gamma_1)^{-1}\varphi_2(\gamma_2)$. Conversely, assume that $\bar{u} = \varphi_1(\gamma_1)^{-1}\varphi_2(\gamma_2)$, for central units γ_i of Λ_i. Then

$$\Lambda \cong \Lambda(\gamma_1^{-1}, \gamma_2^{-1}) = \{(x_1, x_2) : x_i \in \Lambda_i,\ \varphi_1(x_1) = \bar{u}\varphi_2(x_2)\} = \Lambda_{\bar{u}}.$$

The lemma now follows.

THEOREM 1.6. *Under the assumptions announced at the beginning of this section, there is an exact sequence*

$$\begin{aligned} 1 \to u_Z(\bar{\Lambda}) \big/ \langle \varphi_1(u_Z(\Lambda_1)), \varphi_2(u_Z(\Lambda_2)) \rangle &\xrightarrow{\Psi} \operatorname{LFPic}_{e,T}(\Lambda) \\ &\xrightarrow{\Phi} \operatorname{LFPic}_T(\Lambda_1, \Lambda_2) \to 1. \end{aligned}$$

Proof. Let $[M] \in \operatorname{LFPic}_{e,T}(\Lambda)$ lie in the kernel of Φ. Then Me_i is isomorphic to Λ_i, and hence M must be central. We may thus assume that $M \subset \Lambda$ is a two sided Λ-ideal. Thus, $Me_i = \Lambda_i \gamma_i$, for a central element γ_i of Λ_i, with $\varphi_i(\gamma_i) \in u(\bar{\Lambda})$. Hence, $\bar{M} = \varphi_1(Me_1) = \varphi_2(Me_2)$ can be chosen to be all of $\bar{\Lambda}$. Consequently, $\varphi_1(\gamma_1) = \bar{u}\varphi_2(\gamma_2)$ for some $\bar{u} \in u_Z(\bar{\Lambda})$. Replacing M with the isomorphic bimodule $M(\gamma_1^{-1}, \gamma_2^{-1})$ gives

$$M = \{(\lambda_1, \lambda_2) : \lambda_1 \in \Lambda_i,\ \lambda_1 = \bar{u}\lambda_2\}.$$

Hence, $\operatorname{Im}\Psi \supset \operatorname{Ker}\Phi$. Since it is clear that $\Phi\Psi = 0$, the proof is complete.

Remark 1.7 (1) Because of the uncanonical behavior of the center, there is no analogous sequence for the central Picard group Picent.

(2) For locally free class groups, Reiner and Ullom [14, (5.6)] established the exact sequence

$$\begin{aligned} 1 \to u(\bar{\Lambda}) \big/ \langle \varphi_1(u(\Lambda_1)), \varphi_2(u(\Lambda_2)) \rangle &\xrightarrow{\Psi} \operatorname{LFCl}(\Lambda) \\ &\to \operatorname{LFCl}(\Lambda_1) \oplus \operatorname{LFCl}(\Lambda_2) \to 1, \end{aligned}$$

if the Eichler condition is satisfied. In the absence of the Eichler condition, they still get an exact sequence for class groups. However, if one wants to describe the group of outer automorphisms of Λ, then for the natural map ϑ: $\mathrm{LFPic}_T(\Lambda) \to \mathrm{LFCl}(\Lambda)$, we have $\mathrm{Ker}\,\vartheta = \mathrm{Out}_T(\Lambda)$ if and only if stably free Λ-lattices are free. In particular, this is assured when Eichler's condition holds. The remedy for this is to consider, as did Swan in [25], $\mathrm{LF}_1(\Lambda)$, the *pointed set* of isomorphism classes of locally free left Λ-lattices. For pointed sets, the notions "kernel" and "exact sequence" make sense. Whether or not Λ satisfies the Eichler condition, the kernel of the natural map ϑ_T: $\mathrm{LFPic}_T(\Lambda) \to \mathrm{LF}_1(\Lambda)$ is

$$\mathrm{Ker}\,\vartheta_T = \mathrm{Out}_T(\Lambda). \tag{1.8}$$

Examination of the Reiner and Ullom proof of the sequence for class groups reveals that they actually prove:

THEOREM. *There is an exact sequence of pointed sets*

$$\begin{aligned} 1 &\to \varphi_1(u(\Lambda_1)) \backslash u(\overline{\Lambda})/\varphi_2(u(\Lambda_2)) \to \mathrm{LF}_1(\Lambda) \\ &\to \mathrm{LF}_1(\Lambda_1) \times \mathrm{LF}_1(\Lambda_1) \to 1, \end{aligned}$$

where $\varphi_1(u(\Lambda_1)) \backslash u(\overline{\Lambda})/\varphi_2(u(\Lambda_2))$ *denotes the pointed set of doubled cosets of* $u(\overline{\Lambda})$ *with respect to the subgroups* $\varphi_1(u(\Lambda_1))$ *and* $\varphi_2(u(\Lambda_2))$.

Now let us put

$$\mathrm{Out}_{e_i,T}(\Lambda) = \mathrm{Ker}\vartheta|_{\mathrm{LFPic}_{e_i,T}(\Lambda)} \tag{1.9}$$

and

$$\mathrm{Out}_T(\Lambda_1, \Lambda_2) = \left\{(\alpha_1, \alpha_2) \colon \alpha_i \in \mathrm{Out}_{T_i}(\Lambda_i),\ \alpha_1 \equiv \alpha_2 \text{ in } \mathrm{Out}_{\overline{T}}(\overline{\lambda})\right\}.$$

Further, we set

$$\tilde{u}_Z(\Lambda_1, \Lambda_2) = \left\{\bar{u} \in u_Z(\overline{\Lambda}) \colon \exists \lambda_i \in u(\Lambda_i) \text{ with } \bar{u} = \varphi_1(\lambda_1)\varphi_2(\lambda_2)\right\}.$$

LEMMA 1.10. $\tilde{u}_Z(\Lambda_1, \Lambda_2)$ *is a group.*

Proof. Let $u = \varphi_1(\lambda_1)\varphi_2(\lambda_2)$ and $v = \varphi_1(\mu_1)\varphi_2(\mu_2)$ be two of its elements. As v lies in $Z(\overline{\Lambda})$, we have $uv^{-1} = \varphi_1(\lambda_1)\varphi_1(\mu_1^{-1})\varphi_2(\mu_2^{-1})\varphi_2(\lambda_2) \in \tilde{u}_Z(\Lambda_1, \Lambda_2)$. Hence, $\tilde{u}_Z(\Lambda_1, \Lambda_2)$ is a group.

We now put

$$u_Z(\Lambda_1, \Lambda_2) = u_Z(\overline{\Lambda})\big/\langle \varphi_1(u_Z(\Lambda_1)), \varphi_2(u_Z(\Lambda_2))\rangle.$$

Then $u_Z(\Lambda_1, \Lambda_2)$ is the kernel of the well defined mapping,

$$(1.11)\qquad \kappa\colon u_Z(\overline{\Lambda})\big/\big\langle \varphi_1(u_Z(\Lambda_1)), \varphi_2(u_Z(\Lambda_2))\big\rangle \\ \to \varphi_1(u(\Lambda_1)) \setminus u(\overline{\Lambda})/\varphi(u(\Lambda_2)),$$

that sends a coset in $u_Z(\overline{\Lambda})$ to the corresponding double coset.

THEOREM 1.12. *We have a commutative diagram with exact rows and columns*:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & u_Z(\Lambda_1, \Lambda_2) & \longrightarrow & \mathrm{Out}_{e_i, T}(\Lambda) & \xrightarrow{\varrho} & \mathrm{Out}_T(\Lambda_1, \Lambda_2) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & u_Z(\overline{\Lambda})/\varphi_1(u_Z(\Lambda_1))\varphi_2(u_Z(\Lambda_2)) & \longrightarrow & \mathrm{LFPic}_{e_i, T}(\Lambda) & \longrightarrow & \mathrm{LFPic}_T(\Lambda_1, \Lambda_2) & \to & 1 \\
 & & \downarrow{\scriptstyle\kappa} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \varphi_1(u(\Lambda_1)) \setminus u(\overline{\Lambda})/\varphi_2(u(\Lambda_2)) & \longrightarrow & \mathrm{LF}_1(\Lambda) & \longrightarrow & \mathrm{LF}_1(\Lambda_1) \times \mathrm{LF}_1(\Lambda_2) & \to & 1
\end{array}$$

Moreover, if κ *is surjective, e.g., if* $\overline{\Lambda}$ *is commutative, then* ϱ *is also surjective.*

Proof. This follows from the remarks above and some routine diagram chasing.

Remark 1.13. If Λ satisfies the Eichler condition, then in (1.12), $\mathrm{LF}_1(—)$ should be replaced by LFC(—), and $\varphi_1(u(\Lambda_1)) \setminus u(\overline{\Lambda})/\varphi(u(\Lambda_2))$ is a group (cf., [13, §5]).

2. Metacyclic groups of order pq

Let p be a fixed odd prime, and let q be a divisor of $p-1$. For an integer n, we denote by C_n the cyclic group of order n. In this section, G is the semidirect product of C_p and C_q, with C_q acting in a fixed-point-free manner on C_p. Let a be a generator for C_p, and b be a generator for C_q. Denote by ζ_p a primitive p-th root of unity, and view C_q as a subgroup of $\mathrm{Gal}_{\mathbf{Z}}(\mathbf{Z}[\zeta_p])$. Let S be the fixed ring $\mathbf{Z}[\zeta_p]^{C_q}$, and π the norm $N_{\mathbf{Z}[\zeta_p]/S}(1-\zeta_p)$.

Let $e_1 = (1/p)\sum_{i=1}^{p} a^i$, and put $e_2 = 1 - e_1$. Then from [5], it follows that the group ring $\mathbf{Z}G$ is the pullback

$$(2.1)\qquad \begin{array}{ccc}
\mathbf{Z}G & \xrightarrow{\mathrm{pr}_1} & \mathbf{Z}C_q \\
\downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \varphi_1} \\
\Lambda & \xrightarrow{\varphi_2} & \mathbf{F}_p C_q
\end{array}$$

where

$$\Lambda = \begin{pmatrix} S & \cdots & \cdots & S \\ \pi S & \ddots & & S \\ \vdots & \ddots & \ddots & \vdots \\ \pi S & \cdots & \pi S & S \end{pmatrix}_{q\times q} .$$

is isomorphic to the twisted group ring $S \circ C_q$. See also [15], and note that (2.1) follows easily by looking at the following commutative diagram with exact rows, in which $I(C_q)G$ is the $\mathbf{Z}G$-module induced from the augmentation ideal $I(C_q)$ of $\mathbf{Z}C_q$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(C_q)G & \longrightarrow & \mathbf{Z}G & \longrightarrow & \mathbf{Z}C_q & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle e_2} & & \downarrow & & \\ 0 & \longrightarrow & I(C_q)G & \longrightarrow & \Lambda & \longrightarrow & \mathbf{F}_pC_q & \longrightarrow & 0 \end{array}$$

Since Λ is an order in a simple algebra and $\mathbf{Z}C_q$ is commutative, we have

$$\text{(2.2)} \qquad \operatorname{Pic}_{\mathbf{Z}}(\mathbf{Z}G) = \operatorname{Pic}_{e,\mathbf{Z}}(\mathbf{Z}G).$$

We note that $\operatorname{Ker}\varphi_2 \cong \operatorname{Ker}\operatorname{pr}_1 = I(C_q)\cdot G$ generates the radical of Λ at p, since $(b-1)\cdot\Lambda$ generates the radical of Λ at p. In particular, it is characteristic in Λ. We have $\operatorname{Ker}\varphi_2 = p\cdot\mathbf{Z}C_q$, and hence this is also characteristic. Thus, we can apply (0.5) to conclude that we have an exact sequence

$$\text{(2.3)} \qquad u(Z(\Lambda)) \oplus u(\mathbf{Z}C_q) \to u(\mathbf{F}_2C_q) \to \operatorname{Pic}_{\mathbf{Z}}(\mathbf{Z}G) \to \operatorname{Pic}_{\mathbf{Z}}(\Lambda) \oplus \operatorname{Pic}_{\mathbf{Z}}(\mathbf{Z}C_q).$$

LEMMA 2.4. $\operatorname{Pic}_{\mathbf{Z}}(\Lambda) = \operatorname{Cl}(S)\langle\tilde{\omega}\rangle\langle\tau\rangle$, *where $\tilde{\omega}$ is conjugation with*

$$\omega = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ \pi & & & 0 \end{pmatrix}_{q\times q}$$

and has order q as an automorphism, and τ generates $\operatorname{Gal}_{\mathbf{Z}}(S)$.

Proof. From [3, Theorems 2 and 6], [14, §9] we have the exact sequences

$$\text{(2.5)} \qquad 0 \to \operatorname{Picent}(\Lambda) \to \operatorname{Pic}_{\mathbf{Z}}(\Lambda) \to \operatorname{Aut}_{\mathbf{Z}}(S)$$

and

$$0 \to \mathrm{Cl}(S) \to \mathrm{Picent}(\Lambda) \to \mathrm{Picent}(\hat{\Lambda}_p) \to 0, \tag{2.6}$$

where $\hat{\Lambda}_p = \hat{\mathbf{Z}}_p \otimes_{\mathbf{Z}} \Lambda$ and $\hat{\mathbf{Z}}_p$ is the ring of p-adic integers. Now, $\mathrm{Aut}_{\mathbf{Z}}(S) = \mathrm{Gal}_{\mathbf{Z}}(S) = \langle \tau \rangle$. Since πS is the unique prime ideal of S above p, we have $\tau(\pi S) = \pi S$, whence the right-most mapping in (2.5) is a surjection, and (2.5) splits. Moreover, $\mathrm{Picent}(\hat{\Lambda}_p)$ is generated by conjugation with ω. Since πS is principal, it follows that (2.6) splits, and the proof is complete.

In this connection, we point out that a detailed study of Picard groups of hereditary orders is presented in [18].

LEMMA 2.7. $\mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}C_q) = \mathrm{Cl}(\mathbf{Z}C_q)\mathrm{Aut}(C_q)\mathrm{Hom}(C_q, \{\pm 1\})$, *where* $\mathrm{Hom}(C_q, \{\pm 1\})$ *induces automorphisms as described before* (0.8).

Proof. The sequence

$$0 \to \mathrm{Picent}(\mathbf{Z}C_q) \to \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}C_q) \to \mathrm{Aut}_{\mathbf{Z}}(\mathbf{Z}C_q) \to 0$$

is exact, since $\mathbf{Z}C_q$ is commutative [3, Theorem 2], and it splits, since

$$\mathrm{Aut}_{\mathbf{Z}}(\mathbf{Z}C_q) = \mathrm{Aut}(C_q)\mathrm{Hom}(C_q, \{\pm 1\})$$

by the result of Higman's thesis [10]. But $\mathrm{Picent}(\mathbf{Z}C_q) = \mathrm{Cl}(\mathbf{Z}C_q)$, whence the lemma follows.

Remark 2.8. For an order Λ, one must distinguish carefully between the class group $\mathrm{Cl}(\Lambda)$ and the locally free class group $\mathrm{LFCl}(\Lambda)$, made up from modules in the genus of Λ. A similar distinction must be observed between $\mathrm{Picent}(\Lambda)$ and $\mathrm{LFPicent}(\Lambda)$, the latter being made up from invertible two-sided Λ-ideals that are locally free on one side. However, these distinctions disappear in the case of group rings, thanks to a theorem of Swan [23].

LEMMA 2.9. $\mathrm{Pic}_{\mathbf{Z}}(\Lambda, \mathbf{Z}C_q) = \mathrm{Cl}(S)\langle\tau\rangle\mathrm{Cl}(\mathbf{Z}C_q)\mathrm{Hom}(G, \{\pm 1\})$.

Proof. Recall that

$$\mathrm{Pic}_{\mathbf{Z}}(\Lambda, \mathbf{Z}C_q) = \{([M_1],[M_2]) \colon [M_1] \in \mathrm{Pic}_{\mathbf{Z}}(\Lambda), [M_2] \in \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}C_q) \text{ and } \varphi_1(M_1) \cong \varphi_2(M_2) \text{ as bimodules}\}.$$

We note that

$$\mathbf{F}_p C_q \cong \prod_{i=1}^{q} \mathbf{F}_p,$$

since $\mathbf{F}_p$ contains the q-th roots of unity. Since the map $\varphi_2\colon \Lambda \to \mathbf{F}_pC_q$ is just reduction modulo $\omega\Lambda$, we have $\overline{\Lambda} = \Lambda/\omega\Lambda = \mathbf{F}_pC_q$. The kernel of

$$\varphi_1\colon \mathrm{Pic}_{\mathbf{Z}}(\Lambda) \to \mathrm{Pic}_{\mathbf{F}_p}(\overline{\Lambda})$$

is $\mathrm{CL}(S) \cdot \langle \tau \rangle$. Indeed, since $\overline{\Lambda}$ is artinian, its locally free ideals are free, whence $\mathrm{Cl}(S)$ lies in the kernel. The subgroup $\langle \tau \rangle$ is in the kernel because $\mathrm{Gal}_{\mathbf{Z}}(S)$ acts trivially modulo πS.

For the same reasons, the kernel of

$$\varphi_1\colon \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}C_q) \to \mathrm{Pic}_{\mathbf{F}_p}(\overline{\Lambda})$$

is $\mathrm{Cl}(\mathbf{Z}C_q)$.

Consequently,

$$\mathrm{Im}\,\varphi_2 = \varphi_2(\langle \tilde{\omega} \rangle)$$

and

$$\mathrm{Im}\,\varphi_1 = \varphi_1\big(\mathrm{Aut}(C_q)\mathrm{Hom}(G, \{\pm 1\})\big).$$

We must now find the equalizer of φ_1 and φ_2. Note that conjugation with ω on $\overline{\Lambda}$ permutes the q copies of $\mathbf{F}_p$ in $\mathbf{F}_pC_q$ cyclically. Thus, we can view ω as the q-cycle $(1, \dots, q)$, where i represents the i-th copy of $\mathbf{F}_p$.

Claim. *The mapping on $\mathbf{F}_pC_q$ induced by conjugation with ω is not induced by a group automorphism of C_q.*

Proof. ω acts as a q-cycle on $\prod_{i=1}^q \mathbf{F}_p$ and so fixes no component. On the other hand, every group automorphism has the trivial module in its fixed-point set. This proves the claim.

Let q be even. Then $a \mapsto a$, $b \mapsto -b$ induces an automorphism ι of $\mathbf{Z}G$. Let ι_q be the map it induces on $\mathbf{F}_pC_q$. In the regular representation of C_q on $\mathbf{F}_pC_q \cong \prod_{i=1}^q \mathbf{F}_p$, the element b is represented by $(f_q^1, \dots, f_q^q)$, where f_q is an element of order q in $\mathbf{F}_p^\times$. Hence,

$$\iota_q\big((f_q^i)\big) = \big(-f_q^i\big).$$

But we have $f_q^j = -f_q^i$ if and only if $f_q^{j-i} = -1 = f_q^{q/2}$, i.e., if and only if $j - i \equiv \frac{1}{2}q \bmod q$. It follows that ι_q coincides with the map induced on $\mathbf{F}_pC_q$ by conjugation with $\omega^{q/2}$ on Λ. But, it is equally clear that ι_q is induced by the nontrivial element of $\mathrm{Hom}(C_q, \{\pm 1\})$ or by that of $\mathrm{Hom}(G, \{\pm 1\})$. Thus, $\mathrm{Hom}(G, \{\pm 1\})$ and $\tilde{\omega}^{q/2}$ have the same image as automorphisms of $\mathbf{F}_pC_q$, whence (2.9) follows.

Now we must find the images in $\mathbf{F}_p C_q$ of the units of $\mathbf{Z}C_q$ and of $S = Z(\Lambda)$.

LEMMA 2.10. *The image of* $\varphi_1\colon u(\mathbf{Z}C_q) \to \bar{\Lambda} = \mathbf{F}_p C_q = \prod_{i=1}^q \mathbf{F}_p$ *is* $\pm C_q$. *The image of the generator b of* C_q *in* $\prod_{i=1}^q \mathbf{F}_p$ *is* $(f_q^i)_{1 \le i \le q}$, *where* $f_q \in \mathbf{F}_p^\times$ *has order q. Hence, the image of* φ_1 *is generated by* $\pm(f_q^i)_{1 \le i \le q}$.

Proof. This is immediate from the remarks above, since the only units of $\mathbf{Z}C_q$ are the $\pm b^i$, $1 \le i \le q$, by [10].

Remark 2.11. It should be noted that $\operatorname{Im}\varphi_1$ always has order $2q$.

LEMMA 2.12. *The image of* $\varphi_2\colon u_Z(\Lambda) \to \bar{\Lambda} = \prod_{i=1}^q \mathbf{F}_p$ *is a diagonal copy* ΔV_q *of the image* V_q *of* $u(S)$ *in* $S/\pi S = \mathbf{F}_p$.

Proof. This is clear, since φ_2 is just reduction modulo

$$\pi\Lambda = \begin{pmatrix} \pi S & S & \cdots & S \\ \vdots & \ddots & \ddots & \vdots \\ \pi S & & \ddots & S \\ \pi S & & \cdots & \pi S \end{pmatrix}_{q\times q}.$$

Remarks 2.13. (1) V_q is a subgroup of $\mathbf{F}_p^\times$ of order at least $(p-1)/q$. For, since p is prime, the elements

$$\tau_i = \frac{1-\zeta_p^i}{1-\zeta_p} = 1 + \zeta_p + \cdots + \zeta_p^{i-1}, \qquad 2 \le i < p,$$

are units in $\mathbf{Z}[\zeta_p]$. Then $[i]$ is the image of τ_i in $\mathbf{F}_p$, and if $[i_0]$ generates $\mathbf{F}_p^\times$, the norm of τ_{i_0} in S is a unit whose image is $[i_0]^q$, so $|V_q| \ge (p-1)/q$.

(2) In general, $|V_q| > (p-1)/q$. For instance, $|V_2| = p-1$ since the elements $\sigma_i = (\zeta_p^{-i} - \zeta_p^i)/(\zeta_p^{-1} - \zeta_p)$ are units in S, for $2 \le i \le p$, and the image of σ_i is $[i]$.

(3) We note that $[-1]$ is always in V_q.

(4) Galovich, Reiner and Ullom [6] have shown that in fact, $|V_q| = (p-1)(2,q)/q$. For a regular prime p, i.e., one that does not divide the class number of $\mathbf{Z}[\zeta_p]$, Galovich [7, §2] has described V_q in some detail.

We are now in a position to prove the main result for $\operatorname{Pic}_{\mathbf{Z}}(\mathbf{Z}G)$. We introduce some more notation. Let $U_q = \prod_{i=1}^q \mathbf{F}_p^\times/(\Delta V_q \cdot \langle (f_q^i)\rangle)$. Then U_q is just

$$u(\bar{\Lambda})/(\operatorname{Im}\varphi_2(u(S))) \cdot (\operatorname{Im}\varphi_1(u(\mathbf{Z}C_q))).$$

In fact, since $[-1] \in V_q$ and $\Delta V_q \cap \langle (f_q^i) \rangle = 1$, we have, in light of (2.13, 4):

LEMMA 2.14.

$$|U_q| = \frac{(p-1)^q}{|V_q| \cdot q} = \frac{(p-1)^{q-1}}{(2,q)}.$$

THEOREM 2.15. *We have the exact sequence*

$$1 \to U_q \to \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}G) \to \big(\mathrm{Cl}(S)\langle \tau \rangle \oplus \mathrm{Cl}(\mathbf{Z}C_q)\big)\mathrm{Hom}(G, \{\pm 1\}) \to 1.$$

The proof is just an application of (0.5), together with (2.4), (2.7), (2.9), (2.11), and (2.12).

Remarks 2.16. (1) Since there is an exact sequence

$$1 \to \mathrm{Picent}(\mathbf{Z}G) \to \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}G) \to \mathrm{Aut}_{\mathbf{Z}}(Z(\mathbf{Z}G)),$$

we conclude that there is an exact sequence

$$0 \to U_q \to \mathrm{Picent}(\mathbf{Z}G) \to \mathrm{Cl}(S) \oplus \mathrm{Cl}(\mathbf{Z}C_q) \to 0.$$

Note that the elements of U_q give rise to central bimodules, by means of (0.4).

(2) For the dihedral groups of order $2p$, we get the exact sequence

$$0 \to C_{(p-1)/2} \to \mathrm{Picent}(\mathbf{Z}G) \to \mathrm{Cl}(S) \to 0,$$

as given in [14, p. 38].

(3) Thanks to (0.4) and (0.5), the exact sequence allows the explicit description of the bimodules forming $\mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}G)$.

We now turn to the description of the group of outer automorphisms of $\mathbf{Z}G$. Since $\mathbf{Z}G$ satisfies Eichler's condition, we must compute the kernel of

$$\mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}G) \xrightarrow{\vartheta} \mathrm{Cl}(\mathbf{Z}G),$$

cf., (0.9). The fibre product sequence, (0.6) and (0.7) give rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_q & \longrightarrow & \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}G) & \longrightarrow & \mathrm{Cl}(S)\langle \tau \rangle \oplus \mathrm{Cl}(\mathbf{Z}C_q) \cdot \mathrm{Hom}(G, \{\pm\}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \vartheta'} & & \downarrow{\scriptstyle \vartheta} & & \downarrow{\scriptstyle \vartheta''} & & \\
0 & \longrightarrow & \tilde{U}_q & \longrightarrow & \mathrm{Cl}(\mathbf{Z}G) & \longrightarrow & \mathrm{Cl}(\Lambda) \oplus \mathrm{Cl}(\mathbf{Z}C_q) & \longrightarrow & 0,
\end{array}$$

where $\tilde{U}_q$ is cyclic of order $q/(q,2)$ by [6], [14, 7.7]. Note that $\overline{\Lambda}$ is commutative, whence ϑ' is surjective, and the sequence of kernels is exact. We want to find the kernel of ϑ. Recall that

$$\mathrm{Cl}(S)_q = \{(\mathscr{T}) \in \mathrm{Cl}(S) \colon (\mathscr{T})^q = 1\}.$$

THEOREM 2.17. *We have exact sequences*

$$0 \to \operatorname{Ker} \vartheta' \to \operatorname{Out}_{\mathbf{Z}}(\mathbf{Z}G) \to \mathrm{Cl}(S)_q \cdot \langle \tau \rangle \cdot \operatorname{Hom}(G, \{\pm 1\}) \to 0$$

and

$$0 \to \operatorname{Ker} \vartheta' \to \operatorname{Out}_C(\mathbf{Z}G) \to \mathrm{Cl}(S)_q \to 0,$$

where $\operatorname{Out}_C(\mathbf{Z}G)$ *is the group of automorphisms of* $\mathbf{Z}G$ *fixing the center elementwise, modulo inner automorphisms.*

Remarks 2.18. (1) $\tilde{U}_q$ is just U_q modulo the image of the units in Λ, where we recall that U_q was defined by the units in the center of Λ. Hence,

$$|\operatorname{Ker} \vartheta'| = \frac{|U_q|}{q/(q,2)} = \frac{(p-1)^{q-1}}{q},$$

by (2.14).

(2) For $q = 2$, the second exact sequence is that of [5, (4.3)].

Proof. We first compute the kernel of ϑ''. Since $\langle \tau \rangle$ and $\operatorname{Hom}(G, \{\pm 1\})$ come from automorphisms, they surely lie in the kernel of ϑ''. As C_q is abelian, $\vartheta''\colon \mathrm{Cl}(\mathbf{Z}C_q) \to \mathrm{Cl}(\mathbf{Z}C_q)$ is an isomorphism. By [21], we can identify $\mathrm{Cl}(\Lambda)$ with $\mathrm{Cl}(S)$, and with this done, it is shown in [18] that $\operatorname{Ker} \vartheta'' = \mathrm{Cl}(S)_q$. Since $\langle \tau \rangle$ and $\operatorname{Hom}(G, \{\pm 1\})$ do not come from central automorphisms, the theorem follows.

3. The dihedral 2-groups

Let

$$D_n = \langle s_n, t \mid s_n^{2^n} = t^2 = 1,\ t s_n t = s_n^{-1} \rangle \tag{3.1}$$

be the dihedral group of order 2^{n+1}, and let c_n be the central involution $s_n^{2^{n-1}}$.

Easy computations show the following:

LEMMA 3.2. (1) *For $n > 1$, the conjugacy class sums in $\mathbf{Z}D_n$ are* 1, c, *and for* $1 \leq i \leq 2^{n-1}$, *the class sums*

$$K_i^n = s_n^i + s_n^{-i}, \qquad 1 \leq i \leq 2^{n-1} - 1,$$
$$K_t^n = t\left(1 + s_n^2 + s_n^4 + \cdots + s_n^{2^n - 2}\right) = t(1 + c_n)\left(1 + s_n^2 + \cdots + s_n^{2^{n-1}-2}\right),$$
$$K_{ts_n}^n = K_t^n \cdot s_n.$$

Hence, there are $2^{n-1} + 3$ *classes.*
(2) *Under the natural projection* pr_n: $\mathbf{Z}D_n \to \mathbf{Z}D_{n-1}$, *we have*

$$\begin{aligned} 1 &\mapsto 1, \\ c_n &\mapsto 1, \\ K_i^n &\mapsto K_i^{n-1} \quad \text{for } i \neq 2^{n-2}, \\ K_{2^{n-2}}^n &\mapsto 2c_{n-1}, \\ K_t^n &\mapsto 2K_t^{n-1}, \\ K_{ts_n}^n &\mapsto 2K_{ts_{n-1}}^{n-1}. \end{aligned}$$

Note that each K_i^{n-1} is hit twice.
(3) *Every automorphism of D_n stabilizing the conjugacy classes is inner, and*

$$\mathrm{Out}(D_n) \cong C_{2^{n-2}} \cdot C_2.$$

For $n > 1$, we let ζ_n be a primitive 2^n-th root of unity. Set

$$\omega_n = \begin{cases} \zeta_n + \zeta_n^{-1} & \text{for } n > 2, \\ 2 & \text{for } n = 2, \end{cases}$$

and $S_n = \mathbf{Z}[\omega_n]$. Observe that $S_2 = \mathbf{Z}$. In order to compute $\mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}D_n)$, we express $\mathbf{Z}D_n$ as a fibre product.

LEMMA 3.3. *The group ring $\mathbf{Z}D_n$ is a pullback*

$$\begin{array}{ccc} \mathbf{Z}D_n & \xrightarrow{\mathrm{pr}_n} & \mathbf{Z}D_{n-1} \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \varphi_1} \\ \Lambda_n & \xrightarrow{\varphi_2} & \mathbf{F}_2 D_{n-1}, \end{array}$$

where

$$\Lambda_n = \begin{pmatrix} S_n \diagdown & S_n \\ & \omega_n \\ \omega_n S_n & S_n \end{pmatrix} = \left\{ \begin{pmatrix} s_1 & s_2 \\ \omega_n s_3 & s_1 + \omega_n s_4 \end{pmatrix}, s_i \in S_n \right\}.$$

Proof. Let $e_1 = (1 + c_n)/2$ and $e_2 = (1 - c_n)/2$. Then e_1 and e_2 are central orthogonal idempotents with $e_1 + e_2 = 1$, and we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(c_n)\mathbf{Z}D_n & \longrightarrow & \mathbf{Z}D_n & \xrightarrow{\cdot e_1} & \mathbf{Z}D_{n-1} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \cdot e_2} & & \downarrow & & \\ 0 & \longrightarrow & I(c_n)\mathbf{Z}D_n & \longrightarrow & \Lambda_n & \longrightarrow & \mathbf{F}_2 D_{n-1} & \longrightarrow & 0. \end{array}$$

Hence, we get the fibre product diagram. That Λ_n has the asserted structure is shown in [16].

We note in addition that c_n acts as -1 on Λ_n, so that

(3.4) $$\varphi_2 \colon \Lambda_n \to \mathbf{F}_2 D_{n-1}$$

is reduction modulo 2. It follows that here also, $\operatorname{Ker} \varphi_2$ is characteristic in Λ_n, and $\operatorname{Ker} \varphi_1$ is characteristic in $\mathbf{Z}D_{n-1}$. Hence, we can apply our Mayer-Vietoris sequence.

We first compute $\operatorname{Pic}_{\mathbf{Z}}(\Lambda_n)$. Let $\tilde{\omega}_n$ denote conjugation by

$$\bar{\omega}_n = \begin{pmatrix} 0 & 1 \\ \omega_n & 0 \end{pmatrix}$$

on Λ_n. This is a central automorphism of order 2 of Λ_n.

LEMMA 3.5. $\operatorname{Pic}_{\mathbf{Z}}(\Lambda_n) = \operatorname{Cl}(S_n)\langle \tilde{\omega}_n \rangle \operatorname{Gal}_{\mathbf{Z}}(S_n)$.

Proof. Since $\omega_n S_n$ is the unique maximal ideal of S_n lying over 2, it is preserved by every Galois automorphism. As in §1, in the exact sequence

$$0 \to \operatorname{Picent}(\Lambda_n) \to \operatorname{Pic}_{\mathbf{Z}}(\Lambda_n) \to \operatorname{Aut}_{\mathbf{Z}}(S_n)$$

the right-hand mapping is surjective and splits; see [19]. Moreover, if $\hat{\Lambda}_n$

denotes the 2-adic completion of Λ, the exact sequence

$$0 \to \mathrm{Cl}(S_n) \to \mathrm{Picent}(\Lambda_n) \to \mathrm{Picent}(\hat{\Lambda}_n) \to 0$$

is split, again by [18]. Since Picent($\hat{\Lambda}_n$) is generated by $\tilde{\omega}_n$, the result follows (See also [16].)

LEMMA 3.6. *The map* Picent($\mathbf{Z}D_n$) $\to$ Picent($\mathbf{Z}D_{n-1}$) *is surjective.*

Proof. We use Fröhlich's localization sequence

$$0 \to \mathrm{Cl}(Z(\mathbf{Z}D_n)) \to \mathrm{Picent}(\mathbf{Z}D_n) \to \mathrm{Picent}(\hat{\mathbf{Z}}_2 D_n) \to 0.$$

Since no outer automorphisms of D_n stabilize the conjugacy classes (cf., (3.2, 3)), the results in [19] and [2, Theorem 3.3] imply that $\mathrm{Cl}(Z(\mathbf{Z}D_n)) \cong \mathrm{Picent}(\mathbf{Z}D_n)$. Hence, it remains to show that pr_n: $\mathrm{Cl}(Z(\mathbf{Z}D_n)) \to \mathrm{Cl}(Z(\mathbf{Z}D_{n-1}))$ is surjective. By (3.2), $\mathrm{pr}_n(Z(\mathbf{Z}D_n))$ is a subring of finite index in $Z(\mathbf{Z}D_{n-1})$. We first note that

$$\mathrm{Cl}(Z(\mathbf{Z}D_n)) \to \mathrm{Cl}(\mathrm{Im}\,\mathrm{pr}_n(Z(\mathbf{Z}D_n)))$$

is surjective. Indeed, we have a fibre product diagram of the form

$$\begin{array}{ccc} Z(\mathbf{Z}D_n) & \longrightarrow & \mathrm{Im}\,\mathrm{pr}_n(Z(\mathbf{Z}D_n)) \\ \downarrow & & \downarrow \\ \Omega & \longrightarrow & \bar{\Omega} \end{array}$$

from (0.1). Then the Meyer-Vietoris sequence for class groups (0.7) shows that

$$\mathrm{Cl}(Z(\mathbf{Z}D_n)) \to \mathrm{Cl}(\mathrm{Im}\,\mathrm{pr}_n(Z(\mathbf{Z}D_n)))$$

is surjective, since $\mathrm{Cl}(\bar{\Omega}) = 0$. (We remark that the Mayer-Vietoris sequence for class groups requires no special conditions on the $\mathrm{Ker}\,\varphi_i$.) On the other hand, for any pair of orders $\Lambda \subset \Lambda'$ in the same algebra, $\mathrm{Cl}(\Lambda) \to \mathrm{Cl}(\Lambda')$ is surjective (cf., [14, p. 13]). Hence, pr_n is surjective, and the lemma follows.

Now, we recall from [19]:

LEMMA 3.7. *We have a split exact sequence*

$$0 \to \mathrm{Picent}(\mathbf{Z}D_n) \to \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}D_n) \to \mathrm{Out}(D_n) \cdot \mathrm{Hom}(D_n, \{\pm 1\}) \to 0.$$

We also note:

LEMMA 3.8. (1) *For* $n > 2$, $\operatorname{Aut}_{\mathbf{Z}}(S_n) = \langle \tau_n \rangle$, *where* τ_n *is determined by* $\zeta_n \mapsto \zeta_n^5$. *We have* $\tau_n^{2^{n-2}} = 1$.

(2) $\operatorname{Out}(D_n) = \langle \tau(n) \rangle = \langle \tau(n) \rangle \times \langle \iota \rangle$, *where*

$$\tau(n)\colon s_n \mapsto s_n^5, \qquad t \mapsto t,$$

$$\iota\colon s_n \mapsto s_n, \qquad t \mapsto ts_n.$$

Proof. For part (1), note that by [9, p. 388], $\operatorname{Aut}_{\mathbf{Z}}(\mathbf{Z}[\zeta_n])$ is isomorphic to the unit group $u(\mathbf{Z}/2^n\mathbf{Z})$. By [9, p. 40], $u(\mathbf{Z}/2^n\mathbf{Z}) = \langle -1 \rangle \times \langle 5 \rangle$ is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2^{n-2}\mathbf{Z}$. Since S_n is the fixed ring of $\langle -1 \rangle$, (1) now follows from Galois theory.

For part (2), we adapt the discussion of [11, p. 169]. The involutions in D_n are $s_n^{2^{n-1}}$, which is central, and the elements ts_n^k $(0 \le k \le 2^n - 1)$, which are not central. If φ is an automorphism of D_n, it must take s_n to an element of order 2^n; the only such elements are the s_n^r, where r is a unit of $\mathbf{Z}/2^n\mathbf{Z}$. The element $\varphi(t)$ must be a noncentral involution, hence equal to ts_n^k, for some k. All such choices of r and k do, in fact, give automorphisms. Hence, if we define $\varphi_{r,k}$ by $\varphi_{r,k}(s_n) = s_n^r$ and $\varphi_{r,k}(t) = ts_n^k$, then $\operatorname{Aut}(D_n) = \{\varphi_{r,k}\}$, and we have

$$\varphi_{r,k} \circ \varphi_{r',k'} = \varphi_{rr',k+rk'}.$$

It follows that $\operatorname{Aut}(D_n)$ is isomorphic to the split extension of $u(\mathbf{Z}/2^n\mathbf{Z})$ by the additive group $\mathbf{Z}/2^n\mathbf{Z}$, i.e., to the holomorph of $\mathbf{Z}/2^n\mathbf{Z}$. It is easily checked that conjugation by s_n^k is $\varphi_{1,2k}$, while conjugation by t is $\varphi_{-1,0}$. From this and the structure of $u(\mathbf{Z}/2^n\mathbf{Z})$, (2) easily follows.

LEMMA 3.9. *For* $n \ge 3$, $\tau_n^{2^{n-3}}$ *is the identity modulo* $2S_n$.

Proof. τ_n induces an automorphism of order 2^{n-2} on D_n stabilizing the augmentation ideal $I(s_n^{2^{n-1}})$ of the subring $\mathbf{Z}\langle s_n^{2^{n-1}} \rangle$ and hence inducing τ_n on Λ_n. On D_{n-1}, τ_n induces τ_{n-1}, which has order 2^{n-3} on $\mathbf{Z}D_{n-1}$. Since τ_n has order 2^{n-2} on $\mathbf{Z}D_n$, it must have order 2^{n-2} on Λ_n. Because of the fibre product diagram, $\tau^{2^{n-3}} = 1$ on $\mathbf{F}_2 D_{n-1}$. Then, the exact sequence

$$0 \to 2\Lambda_n \to \Lambda_n \to \mathbf{F}_2 D_{n-1} \to 0$$

shows that $\tau_n^{2^{n-3}}$ is the identity modulo $2S_n$, as claimed.

LEMMA 3.10. *For $n > 2$, $\mathrm{Pic}_{\mathbf{Z}}(\Lambda_n, \mathbf{Z}D_{n-1})$ is the pullback of the diagram*

$$\begin{array}{ccc} & & \mathrm{Picent}(\mathbf{Z}D_{n-1}) \cdot \langle \tau_{n-1}, \iota \rangle \cdot \mathrm{Hom}(D_n, \{\pm 1\}) \\ & & \downarrow \\ \mathrm{Cl}(S_n) \cdot \langle \tilde{\omega}_n \rangle \langle \tau_n \rangle & \xrightarrow{\Upsilon} & \langle \tau_{n-1}, \iota \rangle \end{array},$$

where $\Upsilon(\tau_n) = \tau_{n-1}$ and $\Upsilon(\tilde{\omega}_n) = \iota$.

Proof. We look at the map

$$\begin{array}{ccc} \mathrm{Pic}_{\mathbf{Z}}(\Lambda_n) & \xrightarrow{\varphi_2} & \mathrm{Pic}(\mathbf{F}_2 D_{n-1}) \\ \| & & \\ \mathrm{Cl}(\Lambda_n) \cdot \langle \tilde{\omega}_n \rangle \langle \tau_n \rangle. & & \end{array}$$

This map has kernel containing $\mathrm{Cl}(\Lambda_n)\langle \tau_n^{2^{n-3}} \rangle$, by (3.9). On the other hand, τ_n on Λ_n is—modulo conjugation with units—induced from the group automorphism $\tau(n)$ of D_n, and since group automorphisms of D_n show up in $\mathbf{F}_2 D_{n-1}$, we conclude that $\mathrm{Cl}(\Lambda_n) \cdot \langle \tau_n^{2^{n-2}} \rangle$ is precisely the kernel. We continue the proof by showing

CLAIM. *Modulo inner automorphisms of Λ_n, $\tilde{\omega}_n$ induces the map ι on $\mathbf{F}_2 D_{n-1}$.*

Proof. Since ι comes from a group automorphism of D_n, it is enough to show that ι induces a central automorphism of Λ_n, which then must be conjugation with ω_n, modulo inner automorphisms. We establish this by an inertia group argument. Let χ be an irreducible character of $\mathbf{C} \otimes_{\mathbf{Z}} \Lambda_n$. Then $\chi = \psi \uparrow_{\langle s_n \rangle}^{D_n}$, for some character ψ of $\langle s_n \rangle$. Hence, the inertia group of χ is $\langle s_n \rangle$. But the only conjugacy classes K of D_n that are moved by ι lie outside $\langle s_n \rangle$. Hence, $\chi(K) = 0$, and so ι is central. This proves the claim.

Hence, the image of φ_2 is $\langle \tau_{n-1}, \iota \rangle$. On the other hand,

$$\begin{array}{ccc} \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}D_{n-1}) & \xrightarrow{\varphi_1} & \mathrm{Pic}(\mathbf{F}_2 D_{n-1}) \\ \| & & \\ \mathrm{Picent}_{\mathbf{Z}}(\mathbf{Z}D_{n-1})\langle \tau_{n-1}, \iota \rangle \mathrm{Hom}(D_n, \{\pm 1\}) & & \end{array}$$

has kernel $\mathrm{Picent}(\mathbf{Z}D_{n-1})\mathrm{Hom}(D_n, \{\pm 1\})$ and image $\langle \tau_{n-1}, \iota \rangle$. This completes the proof.

Remarks 3.11. (1) As a subset of $\mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}D_n)$, the pullback of

$$\begin{array}{ccc} \langle \tau_n, \tilde{\omega}_n \rangle & \longrightarrow & \langle \tau_{n-1}, \iota \rangle \\ & & \uparrow \\ & & \langle \tau_{n-1}, \iota \rangle \end{array}$$

is just $\langle \tau_n, \iota \rangle$; i.e., the outer automorphism group of D_n. Thus, we have the epimorphism

$$\begin{array}{c} \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}D_n) \cong \mathrm{Picent}(\mathbf{Z}D_n) \cdot \mathrm{Out}(D_n)\mathrm{Hom}(D_n \cdot \{\pm 1\}) \\ \downarrow \\ \mathrm{Pic}(\Lambda_n, \mathbf{Z}D_{n-1}) \cong \mathrm{Cl}(S_n)\mathrm{Picent}(\mathbf{Z}D_{n-1})\mathrm{Out}(D_n)\mathrm{Hom}(D_n, \{\pm 1\}). \end{array}$$

(2) This also holds for D_2, as is easily checked.

We must now determine the kernel of $\mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}D_n) \to \mathrm{Pic}_{\mathbf{Z}}(\Lambda_n, \mathbf{Z}D_{n-1})$. That is to say, we must find the images in $\mathbf{F}_2 D_{n-1}$ of the units of $Z(\Lambda_n) = S_n$ and of $Z(\mathbf{Z}D_{n-1})$.

LEMMA 3.12. *For $n > 1$, the subgroup*

$$G_n = \left\{ 1 + \sum_{i=1}^{2^{n-1}-1} \alpha_i K_i^n, \quad \alpha_i \in \mathbf{F}_2 \right\}$$

of the group of units of the center of $\mathbf{F}_2 D_n$ is generated by

$$\left\{ 1 + K_i^n, 1 \le i \le 2^{n-1} - 1, i \text{ odd} \right\}.$$

Proof. We use induction on n.
For $n = 2$, $G_2 = \{1 + \alpha K_1^2\}$, and the statement is clear.
Now let $n > 2$. Then $|G_n| = 2^{2^{n-1}} - 1$. We consider the subgroup

$$H_n = \left\{ 1 + \sum \alpha_i (K_i^n)^2 \colon \alpha_i \in \mathbf{F}_2 \right\}.$$

We note that

$$(K_i^n)^2 = (s_n^i + s_n^{-1})^2 = s_n^{2i} + s_n^{-2i} = K_{2i}^n$$

in $\mathbf{F}_2 D_n$. Since $\langle s_n^2, t \rangle$ is D_{n-1}, we may use induction to conclude that since $(K_{2^{n-2}}^n)^2 = 0$, $H_n \cong G_n$ is generated by $\{1 + (K_i^n)^2 \colon i \text{ is odd}\}$. Now, the index $[G_n : H_n]$ is 2^{n-2}. The units $1 + K_{2j+1}^n$, for $0 \le j \le 2^{n-2} - 1$ are independent modulo H_n, and each gives rise to a cyclic group of order 2. As $K_{2i}^n = (K_i^n)^2$, the result follows.

LEMMA 3.13. *The group $u(Z(\mathbf{F}_2 D_n))$ of units in the center of $\mathbf{F}_2 D_n$ has order $2^{2^{n-1}+3}$. There is an isomorphism*

$$u(Z(\mathbf{F}_2 D_n))/\langle c_n \rangle \cong G_n \times C_t \times C_{ts_n},$$

where

$$C_t = \langle 1 + K_t^n \rangle \quad \text{and} \quad C_{ts_n} = \langle 1 + K_{ts_n}^n \rangle.$$

Proof. $\langle c_n \rangle$ is a normal subgroup of order 2, and

$$u(Z(\mathbf{F}_2 D_n))/\langle c_n \rangle = \left\{1 + \textstyle\sum \alpha_i K_i^n + \beta K_t^n + \gamma K_{ts_n}^n \colon \alpha_i, \beta, \gamma \in \mathbf{F}_2\right\}$$

has order $2^{2^{n-1}+1}$. Moreover,

$$1 + K_t^n = 1 + t\left(1 + s_n^2 + \cdots + s_n^{2^{n-1}-1} + c_n\left(1 + s_n^2 + \cdots + s_n^{2^{n-1}-1}\right)\right)$$

has square equal to the identity, and so does $1 + K_{ts_n}^n$. The lemma now follows.

LEMMA 3.14. *For $n > 2$ there are units in S_n that map onto the elements $1 + K_i^{n-1}$, for odd i.*

Proof. For $n > 2$, we claim that $1 + \omega_n = 1 + \zeta_n + \zeta_n^{-1}$ is a unit. For, since

$$\omega_n^2 = \zeta_n^2 + \zeta_n^{-2} + 2 = \omega_{n-1} + 2,$$

we have

$$(1 + \omega_n)(\omega_n - 1) = \omega_n^2 - 1 = \omega_{n-1} + 1.$$

For $n = 2$, $\omega_{n-1} = 0$, and hence our claim follows by induction. Now, S_n is a cyclic Galois extension of $\mathbf{Z}$ with Galois group of order 2^{n-2} generated by τ_n. Hence, for each j with $0 \le j \le 2^{n-2} - 1$, there exists $k(j)$ such that

$$\zeta_n^{2j+1} + \zeta_n^{-2j-1} = \zeta_n^{\tau_n^{k(j)}} + \left(\zeta_n^{-1}\right)^{\tau_n^{k(j)}}.$$

The lemma now follows.

Remark 3.15. Note that G_{n-1} has generators $1 + K_{2j+1}^{n-1}$, for $0 \le j \le 2^{n-3} - 1$, and we have constructed the units $1 + \omega_n^{\tau_n^k}$, for $1 \le k \le 2^{n-2}$. On the other hand, we need only 2^{n-3} elements. The explanation is that $\tau_n^{2^{n-3}} \equiv 1 \bmod 2$, by (3.9).

LEMMA 3.16. *Let $\overline{u(Z(\Lambda_n))}$ and $\overline{u(Z(\mathbf{Z}D_{n-1}))}$ denote the images of the central unit groups $u(Z(\Lambda_n))$ and $u(Z(\mathbf{Z}D_{n-1}))$ in $u(Z(\mathbf{F}_2D_{n-1}))$. Then for $n > 2$,*

$$u(Z(\mathbf{F}_2D_{n-1}))/\langle\overline{u(Z(\Lambda_n))}\cdot\overline{u(Z(\mathbf{Z}D_{n-1}))}\rangle \cong C_t \times C_{ts_{n-1}}.$$

Proof. By (3.14), $G_n \times \langle c_{n-1}\rangle$ comes from central units in Λ_n and $\mathbf{Z}D_{n-1}$. The image of S_n in $\mathbf{F}_2D_n$ is just $\mathbf{F}_2[\overline{\omega}_n]$, so no element in $C_t \times C_{ts_{n-1}}$ is hit by a unit in S_n. Hence, it suffices to show that noting in $C_t \times C_{ts_{n-1}}$ is hit by a unit in $\mathbf{Z}D_{n-1}$.

CLAIM. *For $n > 1$, if*

$$x = \alpha + \beta c_n + \sum \gamma_i K_i^n + \delta K_t + \varepsilon K_{ts_n}^n$$

is a central unit, then $\delta = \varepsilon = 0$.

Proof. We use induction on n.
For $n = 2$, the image $\bar{x}$ of x in $\mathbf{Z}D_1$ must be unit. However,

$$\bar{x} = \alpha + \beta + 2\gamma\bar{s} + 2\delta K_t + 2\varepsilon K_{ts_n}.$$

Since $\mathbf{Z}D_1$ is commutative and has no units of infinite order, the only units are the $\pm g$, $g \in D_1$. Hence, $\delta = \varepsilon = 0$.
For $n > 2$, consider again the image $\bar{x}$ of x in $\mathbf{Z}D_{n-1}$; it has the form

$$\bar{x} = \alpha' + \beta' c_{n-1} + \sum \gamma' K_i^{n-1} + 2\delta K_t + 2\varepsilon K_{ts_{n-1}}.$$

By induction, $\delta = \varepsilon = 0$, which proves both the claim and the lemma.
Combining all this, we get a complete inductive description of $\mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}D_n)$.

THEOREM 3.17. *For $n > 2$ there are exact sequences*

$$0 \to C_t \times C_{ts_{n-1}} \to \mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}D_n) \to (\mathrm{Cl}(S_n) \times \mathrm{Picent}(\mathbf{Z}D_{n-1})) \cdot \mathrm{Out}(D_n) \cdot \mathrm{Hom}(D_n, \{\pm 1\})$$

and

$$0 \to C_t \times C_{ts_{n-1}} \to \mathrm{Picent}(\mathbf{Z}D_n) \to \mathrm{Cl}(S_n) \times \mathrm{Picent}(\mathbf{Z}D_{n-1}).$$

The case where $n = 2$ needs separate treatment, but that has been done for us by Fröhlich [3, Theorem 18]:

THEOREM 3.18. *Picent($\mathbf{Z}D_2$) and $\mathrm{Out}_C(\mathbf{Z}D_2)$ are each of order 2.*

Remark 3.19. The unit group of $\mathbf{Z}D_1$ is $\pm D_1$, and hence is elementary abelian of order 8. Its image in $u(\mathbf{F}_2D_1)$ has order 4 and index 2. Since $u(\mathbf{Z})$ has trivial image in $u(\mathbf{F}_2D_1)$, Picent($\mathbf{Z}D_1$) is generated by $1 + s_1 + t$ in $\mathbf{F}_2D_1$.

We now turn to the description of the automorphism group of $\mathbf{Z}D_n$. From [19] we have the split exact sequence

$$\begin{aligned}(3.20)\qquad & 0 \to \mathrm{Out}_C(\mathbf{Z}D_n) \\ & \to \mathrm{Out}_{\mathbf{Z}}(\mathbf{Z}D_n) \to \mathrm{Out}(D_n)\cdot \mathrm{Hom}(D_n, \{\pm 1\}) \to 0.\end{aligned}$$

Hence, it is enough to describe $\mathrm{Out}_C(\mathbf{Z}D_n) = \mathrm{Ker}(\mathrm{Picent}(\mathbf{Z}D_n) \to \mathrm{Cl}(\mathbf{Z}D_n))$. It was shown by Fröhlich, Keating and Wilson [4] that

$$(3.21)\qquad \mathrm{Cl}(\mathbf{Z}D_n) \cong \mathrm{Cl}(S_n) \oplus \mathrm{Cl}(\mathbf{Z}D_{n-1}).$$

Hence, we have the commutative diagram with exact rows

$$(3.22)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & C_t \times C_{ts_n} & \longrightarrow & \mathrm{Picent}(\mathbf{Z}D_n) & \longrightarrow & \mathrm{Cl}(S_n) \oplus \mathrm{Picent}(\mathbf{Z}D_{n-1}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \kappa_n} & & \downarrow{\scriptstyle \kappa' \oplus k_{n-1}} & & \\ & & 0 & \longrightarrow & \mathrm{Cl}(\mathbf{Z}D_n) & \longrightarrow & \mathrm{Cl}(\Lambda_n) \oplus \mathrm{Cl}(\mathbf{Z}D_{n-1}) & \longrightarrow & 0, \end{array}$$

whence an exact sequence

$$1 \to C_t \times C_{ts_n} \to \mathrm{Out}_C(\mathbf{Z}D_n) \to \mathrm{Ker}\,\kappa' \oplus \mathrm{Ker}\,\kappa_{n-1} \to 1$$

of kernels.

LEMMA 3.23. $\mathrm{Ker}\,\kappa' = \mathrm{Cl}(S_n)_2 = \{(\mathscr{T}) \in \mathrm{Cl}(S_n)\colon \mathscr{T}^2$ *is principal*$\}$.

Proof. Since the Schur index of $\mathbf{Q}\Lambda_n$ is trivial, one has $\mathrm{Cl}(\Lambda_n) = \mathrm{Cl}(\Gamma) = \mathrm{Cl}(S_n)$, for any maximal order $\Gamma \supset \Lambda_n$, by [4]. The result now follows as in [18].

We thus have:

THEOREM 3.24. *There are exact sequences*

$$0 \to C_t \times C_{ts_n} \to \mathrm{Out}_C(\mathbf{Z}D_n) \to \mathrm{Cl}(S_n) \oplus \mathrm{Out}_C(\mathbf{Z}D_{n-1}) \to 0$$

and

$$0 \to C_t \times C_{ts_n} \to \mathrm{Out}(\mathbf{Z}D_n)$$
$$\to (\mathrm{Cl}(S_n) \oplus \mathrm{Out}_C(\mathbf{Z}D_{n-1})) \cdot \mathrm{Out}(D_n) \cdot \mathrm{Hom}(D_n, \{\pm 1\}) \to 0.$$

4. The quaternion 2-groups

Let

$$(4.1) \qquad H_n = \left\langle \sigma_n, \tau \colon \sigma_n^{2^n} = \tau^4 = 1, \sigma_n^{2^{n-1}} = \tau^2, \tau\sigma_n\tau^{-1} = \sigma^{-1} \right\rangle$$

be the generalized quarternion group of order 2^{n+1}, and put $\gamma_n = \sigma_n^{2^{n-1}} = \tau^2$, the central involution. Easy computations show:

LEMMA 4.2. (1) *For $n > 1$, the conjugacy class sums of $\mathbf{Z}H_n$ are*

$$1, \gamma_n,$$
$$\mathcal{K}_i^n = \left(\sigma_n^i + \sigma_n^{-i}\right), \quad 1 \le i \le 2^{n-1},$$
$$\mathcal{K}_\tau^n = \tau\left(1 + \sigma_n^2 + \cdots + \sigma_n^{2^n-2}\right)$$
$$\mathcal{K}_{\tau\sigma_n}^n = \tau\sigma_n\left(1 + \sigma_n^2 + \cdots + \sigma_n^{2^n-2}\right)$$

(2) *There is a natural homomorphism $H_n \to D_{n-1}$, given by $\sigma_n \mapsto s_{n-1}$, $\tau \to t$. Its kernel is $\langle\gamma_n\rangle$, and it maps the class sums as follows:*

$$1 \mapsto 1,$$
$$\gamma_n \mapsto 1,$$
$$\mathcal{K}_i^n \mapsto K_i^{n-1} \quad \textit{for } i \neq 2^{n-2},$$
$$\mathcal{K}_{2^{n-2}}^n \mapsto 2c_{n-1},$$
$$\mathcal{K}_\tau^n \mapsto 2K_t^{n-1},$$
$$\mathcal{K}_{\tau\sigma_n}^n \mapsto 2K_{ts_{n-1}}^{n-1}.$$

(3) *Every automorphism of H_n stabilizing the conjugacy classes is inner, and we have*

$$\mathrm{Out}(H_n) \cong C_{2^{n-2}} \cdot C_2.$$

LEMMA 4.3. *We have a fibre product diagram*

$$\begin{array}{ccc} \mathbf{Z}H_n & \xrightarrow{\psi_1} & \mathbf{Z}D_{n-1} \\ \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \varphi_1} \\ \Gamma_n & \xrightarrow{\varphi_2} & \mathbf{F}_2 D_{n-1}, \end{array}$$

where (*cf.*, [25]) $\Gamma_n = \mathbf{Z}\langle \zeta_n, j\rangle = \mathbf{Z}[\zeta_n] \oplus \mathbf{Z}[\zeta_n]j$, *with* $j^2 = -1$ *and* $ja = \bar{a}j$, *for* $a \in \mathbf{Z}[\zeta_n]$.

We note that as in the previous situations, $\operatorname{Ker}\varphi_1 = 2 \cdot \mathbf{Z}D_{n-1}$ and $\operatorname{Ker}\varphi_2 = 2 \cdot \Gamma_n$ are characteristic in $\mathbf{Z}D_{n-1}$ and Γ_n, respectively. Hence, the Mayer-Vietoris sequence can be applied.

THEOREM 4.4. $\operatorname{Picent}(\mathbf{Z}H_n) \cong \operatorname{Picent}(\mathbf{Z}D_n)$, *for* $n > 1$.

Proof. For a prime p and any p-group G, we have the exact sequence

$$0 \to \operatorname{Cl}(Z(\mathbf{Z}G)) \to \operatorname{Picent}(\mathbf{Z}G) \to \operatorname{Out}_C(G) \to 1,$$

where $\operatorname{Out}_C(G) = \operatorname{Aut}_C(G)/\operatorname{Inn}(G)$, and $\operatorname{Aut}_C(G)$ is the group of automorphisms of G that stabilize the conjugacy classes [19]. Because of (3.2) and (4.2),

$$\operatorname{Picent}(\mathbf{Z}H_n) \cong \operatorname{Cl}(Z(\mathbf{Z}H_n))$$

and

$$\operatorname{Picent}(\mathbf{Z}D_n) \cong \operatorname{Cl}(Z(\mathbf{Z}D_n)).$$

The result will follow if we prove:

CLAIM 4.5. $Z(\mathbf{Z}D_n) \cong Z(\mathbf{Z}H_n)$.

Proof. It follows from (3.2) and (4.2) that the maps

$$\varphi_1 \colon \mathbf{Z}D_n \to \mathbf{Z}D_{n-1}, \quad \psi_1 \colon \mathbf{Z}H_n \to \mathbf{Z}D_{n-1},$$

satisfy

$$\operatorname{Im}\left(\varphi_1|_{Z(\mathbf{Z}D_n)}\right) = \operatorname{Im}\left(\psi_1|_{Z(\mathbf{Z}H_n)}\right).$$

Moreover, the maps

$$\varphi_2\colon \mathbf{Z}D_n \to \Lambda_n, \quad \psi_1\colon \mathbf{Z}H_n \to \Gamma_n$$

satisfy

$$\operatorname{Im}\left(\varphi_2|_{Z(\mathbf{Z}D_n)}\right) = \operatorname{Im}\left(\psi_2|_{Z(\mathbf{Z}H_n)}\right) = S_n.$$

For, S_n is the center of Λ_n and of Γ_n, and it is generated by

$$\omega_1 = \varphi_2\left(s_n + s_n^{-1}\right) = \psi_2\left(\sigma_n + \sigma_n^{-1}\right).$$

It follows that the centers of $\mathbf{Z}H_n$ and $\mathbf{Z}D_n$ are both pullbacks of

$$\begin{array}{ccc} \operatorname{Im}\varphi_1|_{Z(\mathbf{Z}D_n)} & = & \operatorname{Im}\psi_1|_{Z(\mathbf{Z}H_n)} \\ & & \downarrow \\ S_n & \to & Z(\mathbf{F}_2 D_{n-1}), \end{array}$$

and hence are naturally isomorphic. This proves both claim and lemma.

THEOREM 4.6. *For $n > 2$, there is an exact sequence*

$$1 \to C_2 \to \operatorname{Out}_C(\mathbf{Z}H_n) \to \operatorname{Out}_C(\Gamma_n) \oplus \operatorname{Out}_C(\mathbf{Z}D_{n-1}) \to 1.$$

Moreover, $\operatorname{Out}_C(\mathbf{Z}H_2) = 1$.

Proof. From [24], we get the exact sequence of pointed sets

$$(4.7) \quad 1 \to C_2 \cong D(\mathbf{Z}H_n) \to \operatorname{LF}_1(\mathbf{Z}H_n) \to \operatorname{Cl}(\mathbf{Z}D_{n-1}) \times \operatorname{LF}_1(\Gamma_n) \to 1.$$

Using (1.12) and the description of Picent($\mathbf{Z}H_n$), we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C_2 & \longrightarrow & \operatorname{LF}_1(\mathbf{Z}H_n) & \longrightarrow & \operatorname{Cl}(\mathbf{Z}D_{n-1}) \times \operatorname{LF}_1(\Gamma_n) & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \vartheta'} & & \uparrow{\scriptstyle \vartheta} & & \uparrow{\scriptstyle \vartheta_{n-1}\times\vartheta''} & & \\ 1 & \longrightarrow & C_\tau \times C_{\tau\sigma_n} & \longrightarrow & \operatorname{Picent}(\mathbf{Z}H_n) & \longrightarrow & \operatorname{Picent}(\mathbf{Z}D_{n-1}) \times \operatorname{Cl}(S_n) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & X & \longrightarrow & \operatorname{Out}_C(\mathbf{Z}H_n) & \longrightarrow & \operatorname{Out}_C(\mathbf{Z}D_{n-1}) \times \operatorname{Cl}_{\Gamma_n}(S_n) & \longrightarrow & 1. \end{array}$$

$\mathrm{Out}_C(\mathbf{Z}D_{n-1})$ is described in §3. We have put

$$\mathrm{Cl}_{\Gamma_n}(S_n) = \{(\mathcal{T}) \in \mathrm{Cl}(S_n) \colon \mathcal{T}\Gamma_n \text{ is a principal ideal}\}.$$

We do not know whether $\mathrm{Out}_C(\Gamma_n) \cong \mathrm{Cl}(S_n)_2$. It remains to determine the image of ϑ'. It was shown by Swan [25] that the projective module generating C_2 in (4.7) can be chosen as

$$\Sigma_n = 3\mathbf{Z}H_n + I(\mathbf{Z}H_n),$$

where $I(\mathbf{Z}H_n)$ is the augmentation ideal. Since Σ_n is surely an invertible bimodule, it follows that ϑ' is surjective. Hence, the desired result follows from (1.11), except for the case $n = 2$, where $C_\tau \times C_{\tau\sigma_n}$ is replaced by a cyclic group of order 2, which gives Fröhlich's result that $\mathrm{Out}_C(\mathbf{Z}H_2) = 1$. This completes the proof.

THEOREM 4.8. $\mathrm{Pic}_{\mathbf{Z}}(\mathbf{Z}H_n) \cong \mathrm{Picent}(\mathbf{Z}H_n)\mathrm{Out}(H_n)\mathrm{Hom}(H_n, \{\pm 1\})$.

Proof. Let M be an invertible bimodule. Localizing at 2, we obtain from M an automorphism α of $\hat{\mathbf{Z}}_2 H_n$. By [19], α modulo conjugation by units of $\hat{\mathbf{Z}}_2 H_n$ is of the form $\alpha = \varrho\nu$, where $\varrho \in \mathrm{Out}(H_n)$ and $\nu \in \mathrm{Hom}(H_n, \{\pm 1\})$. In particular, α is in fact a global automorphism and ${}_{\alpha^{-1}}M$ is a central bimodule.

This also proves:

COROLLARY 4.9. $\mathrm{Out}(\mathbf{Z}H_n) = \mathrm{Out}_C(\mathbf{Z}H_n) \cdot \mathrm{Out}(H_n) \cdot \mathrm{Hom}(H_n, \{\pm 1\})$.

Concluding remarks 4.10. (1) For H_n, $n \geq 4$, there remains the question of whether $\widetilde{\mathrm{Out}}_C(\mathbf{Z}H_n) = \mathrm{Out}_C(\mathrm{ZH_n})$. The answer is *yes* if and only if whenever $\mathcal{T}$ is an ideal of S_n such that

$$\mathcal{T}\mathbf{Z}H_n \oplus \mathbf{Z}H_n \cong \mathbf{Z}H_n \oplus \mathbf{Z}H_n$$

as left $\mathbf{Z}H_n$-modules, $\mathcal{T}\mathbf{Z}H_n$ is a principal ideal. There can never be such an isomorphism as bimodules. There is some evidence in [22] to suggest that

$$\widetilde{\mathrm{Out}c}(\mathbf{Z}H_n) = \mathrm{Out}_C(\mathbf{Z}H_n).$$

We have been unable to find an example of a $\mathbf{Z}$-order Λ and an invertible bimodule M with a left Λ-module isomorphism $M \oplus \Lambda \cong \Lambda \oplus \Lambda$ with M not left Λ-free.

(2) Let K_n denote the field of fractions of S_n. K_n is totally real, and for $n > 2$, it has an even number of embeddings into the real field. The central K_n-division algebra $A_n = K_n\Gamma_n$ has local invariant $1/2$ at each of these

embeddings, since $-1 < 0$ is fixed by all of them. Now, A_n clearly splits at all finite primes of S_n except possibly at $\mathfrak{p} = \omega_n S_n$, which is the only prime of S_n over the rational prime 2. But then Hasse's description [8] of the Brauer group of K_n implies that A_n must split at $\mathfrak{p}$ as well. Hence, a nontrivial division algebra split at all finite primes "occurs in nature". This also shows an example of a group algebra with trivial local Schur indices, but nontrivial global ones, and reminds us not to neglect the infinite primes when using the Hasse principal for quadratic forms.

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