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TORSION FREE CANCELLATION OVER ORDERS

BY

RICHARD G. SWAN

In memory of Irving Reiner

Let Λ be an order over a Dedekind ring R. We say that torsion free cancellation (hereafter abbreviated to TFC) holds for Λ if $X \oplus M \approx X \oplus N$ implies $M \approx N$ for lattices X, M, N over Λ; i.e. when X, M, N are finitely generated Λ modules torsion free over R. In [30], Wiegand developed a theory of torsion free cancellation over 1-dimensional commutative rings. Since the question is also of great interest for non-commutative orders, it is natural to ask whether Wiegand's results have non-commutative analogs. I will show here that this is indeed the case, at least when the quotient field of R is a global field. The main difference between the commutative and non-commutative case is due to the need to impose Eichler's condition on appropriate endomorphism rings.

I will also present some partial results on the case $\Lambda = ZG$ with G a finite group. The abelian case was discussed by Wiegand [30] who settled the question except for two special groups, the cyclic groups G_8 and C_9 of orders 8 and 9. It turns out that TFC holds also in these two cases.[1] For C_9 this can be deduced from Reiner's classification of ZC_{p^2} lattices [22]. The case C_8 requires a bit more work and will be discussed in §5. The final result for G abelian is that TFC holds for ZG if and only if $D(ZG) = 0$. However, I will show that this is no longer true in the non-abelian case. Note that Heitmann [30] has given an example of a commutative order with $D(\Lambda) = 0$ but without TFC.

Throughout this paper the term order will mean an order over a Dedekind ring R in a semisimple separable algebra over its quotient field K. Except for §1, I will also assume that K is a global field unless otherwise specified and will use the term "global order" to remind the reader of this assumption.

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[1]L. Levy has informed me that this result was obtained a few years ago (unpublished) by C. Odenthal who also showed that the Krull-Schmidt theorem holds for lattices over ZC_8.

1. General results

Let Λ be an order over a Dedekind ring R in a semisimple separable algebra A over the quotient field of K of R. As usual $C(\Lambda)$ will denote the projective class group of Λ and $D(\Lambda) = \ker[C(\Lambda) \to C(\Gamma)]$ where Γ is any maximal order of A containing Λ (e.g., see [28]). The following result of Endo and Miyata [7, Lemma 2.4] shows that $D(\Lambda) = 0$ is a necessary condition for TFC.

Recall that two lattices M and N over Λ are said to have the same genus (denoted $M \vee N$) if $M_v \approx N_v$ for all valuations v of K coming from R, where M_v denotes the completion of M at v [28].

PROPOSITION 1.1 [7]. *Let $\Lambda \subset \Gamma$ be R-orders in A. Let M and N be Λ-lattices of the same genus. Then*

$$M \oplus \Gamma N \approx \Gamma M \oplus N.$$

Here, as in [30], ΓM is the Γ submodule of KM generated by M or, equivalently, $\Gamma M = (\Gamma \otimes_\Lambda M)/\text{torsion}$.

Proof. By Roiter's lemma [27, Th. 3.1] choose an exact sequence $0 \to M \to N \to X \to 0$ where X has order prime to $|\Gamma : \Lambda|$. The sequence $0 \to M \to \Gamma M \oplus N \to \Gamma N \to 0$ is then split exact since this is so locally. In fact, locally either $\Gamma = \Lambda$ or $M = N$.

COROLLARY 1.2 (cf. [30, Th. 2.3]). *Let $\Lambda \subset \Gamma$ be R orders in A. Assume that* TFC *holds for Γ. Let M and N be Λ-lattices. Then there is a Λ lattice X with $X \oplus M \approx X \oplus N$ if and only if* (1) $M \vee N$ *and* (2) $\Gamma M \approx \Gamma N$.

Proof. (1) is necessary since cancellation holds locally by Krull-Schmidt. (2) is necessary by the hypothesis on Γ. The sufficiency follows from 1.1.

COROLLARY 1.3 (cf. [30, Cor. 2.4]). *If* TFC *holds for Λ, then $D(\Lambda) = 0$.*

Proof. Let $[P] - [Q] \in D(\Lambda)$ be non-zero, P and Q being locally free. Let $\Gamma \supset \Lambda$ be a maximal order. By definition of D, $[\Gamma P] = [\Gamma Q]$. Replacing P and Q by $P \oplus \Lambda^n$ and $Q \oplus \Lambda^n$ for some n we can assume that $\Gamma P \approx \Gamma Q$ and the result follows from 1.1.

Using an idea of Reiner [23, Proof of 40.22] (see also [19, Proof of 2.2]), we can give a sufficient condition for TFC but unfortunately this will usually involve looking at an infinite number of orders. We let Genus(M) be the set of isomorphism classes of Λ lattices having the same genus as M. In particular,

$\mathrm{Genus}(\Lambda) = \mathrm{LF}_1(\Lambda)$, the set of isomorphism classes of locally free Λ modules of rank 1.

PROPOSITION 1.4. $\mathrm{Genus}(M) \approx LF_1(\mathrm{End}_\Lambda(M))$.

A more general version of this result applicable to modules with torsion is given by Guralnick [13, Lemma 3.2], [12, Prop. 4.1].

Proof. As in [23, Proof of 40.22], we observe that if $M \vee N$, then $\mathrm{Hom}_\Lambda(M, N)$ is locally isomorphic to $\mathrm{Hom}_\Lambda(M, M) = \mathrm{End}_\Lambda(M)$ and so lies in $\mathrm{LF}_1(\mathrm{End}_\Lambda(M))$. Let $\Sigma = \mathrm{End}_\Lambda(M)$ acting on M from the right. If $P \in \mathrm{LF}_1(\Sigma)$ then clearly $M \otimes_\Sigma P \in \mathrm{Genus}(M)$. This gives maps $\mathrm{Genus}(M) \rightleftharpoons \mathrm{LF}_1(\Sigma)$ which are inverses since $M \otimes_\Sigma \mathrm{Hom}_\Lambda(M, N) \approx N$ and $P \approx \mathrm{Hom}_\Lambda(M, M \otimes_\Sigma P)$. It is enough to check these locally so we can assume that $M = N$ and $P = \Sigma$ in which case the maps are clearly isomorphisms.

The isomorphism of 1.4 is natural in the following sense.

PROPOSITION 1.5. *Let* $\Lambda \subset \Lambda'$, $M' = \Lambda' M$, $\Sigma = \mathrm{End}_\Lambda(M)$ *and* $\Sigma' = \mathrm{End}_{\Lambda'}(M')$. *Then*

$$\begin{tikzcd} \mathrm{Genus}(M) \arrow[r, "\approx"] \arrow[d] & \mathrm{LF}_1(\Sigma) \arrow[d] \\ \mathrm{Genus}(M') \arrow[r, "\approx"] & \mathrm{LF}_1(\Sigma') \end{tikzcd}$$

commutes.

Proof. We have to show that $\Lambda'(M \otimes_\Sigma P) \approx M' \otimes_{\Sigma'} P'$ with $P' = \Sigma' P$. It is enough to check this locally so we can assume that $P = \Sigma$. In this case the result is clear.

As usual we say that Λ satisfies locally free cancellation (LFC) if $X \oplus M \approx X \oplus N$ implies $M \approx N$ for locally free Λ modules X, M, and N. This is equivalent to $\mathrm{LF}_1(\Lambda) \to C(\Lambda)$ being an isomorphism, for instance by [26, Cor. A6]. The following result is also an immediate consequence of [13, Cor. 6.5].

COROLLARY 1.6. *Suppose that* LFC *holds for* $\mathrm{End}_\Lambda(M)$ *for all* Λ *lattices* M. *Then* TFC *holds for* Λ *if and only if* $D(\mathrm{End}_\Lambda(M) = 0$ *for all* Λ *lattices* M.

Proof. Let Γ be a maximal order containing Λ. We first observe that Γ satisfies TFC. Let $K\Gamma = A_1 \times \cdots \times A_n$ with the A_i simple. Then $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ where Γ_i is the image of Γ in A_i. Let $A_i = M_{n_i}(D_i)$ where D_i is a division algebra and choose a Γ_i lattice P_i such that KP_i is a simple A_i module. Then $\Delta_i = \mathrm{End}_{\Gamma_i}(P_i)$ is a maximal order in D_i. Since $\Delta_i = \mathrm{End}_\Lambda(P_i)$, we see that Δ_i satisfies LFC and hence TFC since all Δ_i lattices are locally free [23, Th. 18.10]. Therefore Γ_i also satisfies TFC since the category of Γ_i lattices is Morita equivalent to that of Δ_i lattices. It follows that Γ satisfies TFC so, by 1.2, TFC holds for Λ if and only if Genus$(M) \to$ Genus(ΓM) is injective for all Λ lattices M. By 1.5 this is equivalent to the injectivity of $\mathrm{LF}_1(\Sigma) \to \mathrm{LF}_1(\Sigma')$. But $\Sigma = \mathrm{End}_\Lambda(M)$ and $\Sigma' = \mathrm{End}_\Gamma(\Gamma M) = \mathrm{End}_\Lambda(\Gamma M)$ satisfy LFC by hypothesis so this map is just $C(\Sigma) \to C(\Sigma')$ with kernel $D(\Lambda)$ since Σ is maximal.

The following results show that it is sufficient to check the conditions of 1.6 for one module M in each genus.

PROPOSITION 1.7. *Let M and N be Λ lattices having the same genus. Then* $\mathrm{End}_\Lambda(M)$ *and* $\mathrm{End}_\Lambda(N)$ *are Morita equivalent.*

Proof. By Roiter's lemma [27, Th. 3.1] we can embed N in M with $M_v = N_v$ for all v such that Λ is not a maximal order. We now have $KM = KN$ and we can regard $\Gamma = \mathrm{End}_\Lambda(M)$ and $\Delta = \mathrm{End}_\Lambda(N)$ as orders in $\mathrm{End}_A(KM)$. Let $P = \Gamma\Delta$ as a left Γ and right Δ module. Then P is a projective generator over Γ and $\mathrm{End}_\Gamma(P) = \Delta$. It is sufficient to check these assertions locally. If Λ_v is maximal, so are Γ_v and Δ_v and the result is clear. In the remaining cases however, $P_v = \Gamma_v = \Delta_v$.

COROLLARY 1.8. *If M and N are as in* 1.7, *then* LFC *holds for* $\mathrm{End}_\Lambda(M)$ *if and only if* LFC *holds for* $\mathrm{End}_\Lambda(N)$.

This is clear since P is locally free of rank 1 over Γ and Δ so the Morita correspondence preserves locally free modules. The same result holds for TFC.

COROLLARY 1.9. *If M and N have the same genus then*

$$C(\mathrm{End}_\Lambda(M)) = C(\mathrm{End}_\Lambda(N)) \quad \textit{and} \quad D(\mathrm{End}_\Lambda(M)) = D(\mathrm{End}_\Lambda(N)).$$

This follows immediately from Matchett's theory of bimodule induced homomorphisms [18], [26, p. 149].

Remark. The condition TFC is left-right symmetric since $M \mapsto M^* = \mathrm{Hom}_R(M, R)$ gives an equivalence between the categories of left and right Λ lattices. This equivalence does not preserve locally free modules in general but for projective modules we can use instead the equivalence $P \mapsto P^\vee = \mathrm{Hom}_\Lambda(P, \Lambda)$ between the categories of left and right projective lattices. This preserves locally free modules showing that LFC is also left-right symmetric, and that $C(\Lambda)$ is the same for left and right modules. The same is true of

$D(\Lambda)$ since for P projective and $\Lambda \subset \Gamma$, we have

$$\begin{aligned} P^{\vee} \otimes_{\Lambda} \Gamma &= \mathrm{Hom}_{\Lambda}(P, \Lambda) \otimes_{\Lambda} \Gamma \xrightarrow{\approx} \mathrm{Hom}_{\Lambda}(P, \Gamma) \\ &= \mathrm{Hom}_{\Gamma}(\Gamma \otimes_{\Lambda} P, \Gamma) = (\Gamma \otimes_{\Lambda} P)^{\vee}, \end{aligned}$$

it being sufficient to check the isomorphism when $P = \Lambda$.

2. Orders over global fields

From now on I will assume that the quotient field K of R is a global field. Write $A = \prod M_{n_i}(D_i)$ where the D_i are division algebras.

DEFINITION. We say that A satisfies the *extended Eichler condition* (EEC) if each D_i satisfies Eichler's condition [28, p. 174].

Equivalently, EEC holds if and only if $\mathrm{End}_A(V)$ satisfies Eichler's condition for every finitely generated A module V. This is clear from the fact that $M_n(D)$ satisfies Eichler's condition for $n > 1$. Note that if $R = Z$, EEC just says that no D_i is a totally definite quaternion algebra.

As usual we say that Λ satisfies EEC if $K\Lambda = A$ does.

COROLLARY 2.1. *Suppose Λ satisfies* EEC. *Then Λ satisfies* TFC *if and only if $D(\mathrm{End}_{\Lambda}(M)) = 0$ for all Λ lattices M.*

This follows from 1.6 and Jacobinski's cancellation theorem [28].

The following considerations give an analogue of [30, Cor. 2.4]. Let Γ be a maximal order of A containing Λ and let $\mathcal{O}$ be the center of Γ. Suppose $\mathscr{A}$ is an ideal of $\mathcal{O}$ such that $\mathscr{A}\Gamma \subset \Lambda$. If M is a Λ lattice, write $\Gamma = \Gamma_M \times \Gamma_M'$ where $0 \times \Gamma_M'$ is the annihilator in Γ of ΓM. Then $\mathcal{O} = \mathcal{O}_m \times \mathcal{O}_M'$, $\mathscr{A} = \mathscr{A}_M \times \mathscr{A}_M'$ and $\mathcal{O}_M$ is isomorphic to the center of $\Delta = \mathrm{End}_{\Gamma}(\Gamma M)$. Since $\mathscr{A}\Gamma M \subset \Lambda M = M$, we see that $\mathscr{A}\Delta \subset \Sigma = \mathrm{End}_{\Lambda}(M)$. By Fröhlich's formula [28], $D(\Sigma) = U(\hat{\mathcal{O}}_M)/U^+(\mathcal{O}_M)\nu U(\hat{\Sigma})$, and $U(\hat{\Sigma}) \supset U(\hat{\Delta}, \mathscr{A}\hat{\Delta})$.

LEMMA 2.2 [28, Th. 15.1]. $\nu U(\hat{\Delta}, \mathscr{A}\hat{\Delta}) = U(\hat{\mathcal{O}}_M, \hat{\mathscr{A}}_M)$.

PROPOSITION 2.3. *For all Λ lattices M, $D(\mathrm{End}_{\Lambda}(M))$ is a quotient of $U(\mathcal{O}/\mathscr{A})/U^+(\mathcal{O})$.*

This group is an analogue of Wiegand's $E(R)$ [30].

COROLLARY 2.4. *If Λ satisfies* EEC *and $U^+(\mathcal{O}) \to U(\mathcal{O}/\mathscr{A})$ is onto, then Λ satisfies* TFC.

A version of this result applicable to non-global orders is given by Guralnick [13, Th. 5.3].

We now turn to the main results of this section. Let N be another Λ lattice. Clearly $\mathcal{O}_{M\oplus N} \supset \mathcal{O}_M$ so $\mathcal{O}_{M\oplus N} = \mathcal{O}_M \times \mathcal{O}'$ for some $\mathcal{O}'$.

LEMMA 2.5. *The following diagram commutes.*

$$\begin{array}{ccc} \mathrm{Aut}(\hat{M}) & \longrightarrow & \mathrm{Aut}(\hat{M} \oplus \hat{N}) \\ {\scriptstyle\nu}\downarrow & & {\scriptstyle\nu}\downarrow \\ U(\hat{\mathcal{O}}_M) & \longrightarrow & U(\hat{\mathcal{O}}_{M\oplus N}) \end{array}$$

Here the upper map sends α to $\alpha \oplus 1$ and the lower map sends u to $(\mathrm{u}, 1) \in \mathcal{O}_{\mathrm{M}} \times \mathcal{O}'$.

Proof. It is clearly sufficient to do the local case and it is enough to check the corresponding results for the algebra A rather than the order. After extending the groundfield to make the algebra split, the result reduces to the fact that $\det(f \oplus 1) = \det(f)$ over a field.

Since $U^+(\mathcal{O}) \subset U(\mathcal{O})$ is determined by the division rings D_i occurring in A, it clear that $U^+(\mathcal{O}_{M\oplus N}) = U^+(\mathcal{O}_M) \times U^+(\mathcal{O}')$. It follows from 2.5 that there is a well-defined map $D(\mathrm{End}_\Lambda(M)) \to D(\mathrm{End}_\Lambda(M \oplus N))$ which is induced by $U(\hat{\mathcal{O}}_M) \to U(\hat{\mathcal{O}}_{M\oplus N})$ sending u to $(u, 1) \in \hat{\mathcal{O}}_M \times \hat{\mathcal{O}}'$.

COROLLARY 2.6. *If* $\mathrm{Ann}_\Lambda(M \oplus N) = \mathrm{Ann}_\Lambda(M)$ *then* $D(\mathrm{End}_\Lambda(M)) \to D(\mathrm{End}_\Lambda(M \oplus N))$ *is onto.*

In this case $\mathcal{O}_{M\oplus N} = \mathcal{O}_M$ so the result is clear.

COROLLARY 2.7. *If* $M = \oplus M_i$ *then* $\oplus D(\mathrm{End}_\Lambda(M_i)) \to D(\mathrm{End}_\Lambda(M))$ *is onto.*

This follows from the obvious fact that $\prod U(\hat{\mathcal{O}}_{M_i}) \to U(\hat{\mathcal{O}}_M)$ is onto. In particular $D(\mathrm{End}_\Lambda(M))$ will be 0 if all $D(\mathrm{End}_\Lambda(M_i))$ are. This gives the following sufficient condition for TFC.

THEOREM 2.8. *Let* Λ *be a global order. If* Λ *satisfies* EEC *and if* $D(\mathrm{End}_\Lambda(M)) = 0$ *for all indecomposable* Λ *lattices* M, *then* Λ *satisfies* TFC.

Note that by 1.9 it is enough to consider one indecomposable lattice M in each genus.

In the commutative case this gives an improvement of one of Wiegand's main results [30, Th. 2.7] in the case of global orders.

COROLLARY 2.9. *Let Λ be a commutative global order. If $D(\Lambda) = 0$ and if every Λ lattice is a direct sum of ideals, then Λ satisfies* TFC.

Proof. Since EEC holds, we need only check that $D(\mathrm{End}_\Lambda(I)) = 0$ for ideals I of Λ. Let $J = \mathrm{Ann}_\Lambda(I)$. Then $(I \cap J)^2 = 0$ so $I \cap J = 0$ since $K\Lambda$ is a product of fields. Therefore I is an ideal of $\Sigma = \Lambda/J$. Since $K\Sigma$ is a product of fields and $\mathrm{Ann}_\Sigma(I) = 0$, we see that $KI = K\Sigma$ so $\mathrm{End}_\Lambda(I)$ is an order of $K\Sigma$ containing Σ. This implies that the maps $D(\Lambda) \to D(\Sigma) \to D(\mathrm{End}_\Lambda(I))$ are onto by Corollary 3.7.

COROLLARY 2.10. TFC *holds for ZC_9.*

This actually follows from Reiner's classification of modules over ZC_{p^2} [22]. However we can give a proof using Theorem 2.8 which only requires knowledge of the indecomposable ZC_9-modules [21]. With one exception these are isomorphic to ideals of ZC_9 and the argument of Corollary 2.9 can be applied. The exceptional module is $M = (Z \oplus E, S, 1 + \lambda)$ in the notation of [22]. This can be presented with generators e, f, g and relations

$$(x-1)e = 0, \quad (x^3 - 1)f = 0, \quad \Phi_9 g = e + (x-1)f$$

where $\Phi_9 = 1 + x^3 + x^6$. From this one easily sees that M is the pullback in the diagram

$$\begin{array}{ccc} M & \longrightarrow & I \\ \downarrow & & \downarrow \\ Z^2 & \longrightarrow & \mathscr{F}_3^{\,2} \end{array}$$

where $I = (3, x-1) \subset \Lambda = ZC_9/(N) = Z[x]/\Phi_3\Phi_9$ with N being the sum of the elements of the group. The maps send e, f, g to $0, \Phi_9, x - 1 \in I$ and to $(3,0), (0,1), (1,0) \in Z^2$. The right vertical map is just $I \to I/I^2 = \mathscr{F}_3^{\,2}$. Note that $I^2 = (x-1)I$. This follows from the identity $[(x-1)^2 + 3x]\Phi_9 = 0$ in Λ.

Any endomorphism of M induces one of this diagram and endomorphisms of I and Z^2 define an endormorphism of M if and only if they agree on $\mathscr{F}_3^{\,2}$. Now $\mathrm{End}_\Lambda(I) = \Lambda_1 = \Lambda + \Lambda(x-1)\Phi_9/3$ because it is easy to check that $\Lambda_1(x-1) = I$ using the identity just mentioned, so $\Lambda'(x-1) \subset I$ implies $\Lambda' \subset \Lambda_1$. The elements of Λ induce scalar multiplications on $\mathscr{F}_3^{\,2}$ while $(x-1)\Phi_9/3$ induces the matrix

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix};$$

so the elements of $\mathrm{End}(Z^2)$ which match endomorphisms from Λ_1 on $\mathscr{F}_3^{\,2}$ are those of

$$\Lambda_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a \equiv d, c \equiv 0 \bmod 3 \right\}.$$

Therefore we have a cartesian diagram of epimorphisms

$$\begin{array}{ccc} \mathrm{End}(M) & \longrightarrow & \Lambda_1 \\ \downarrow & & \downarrow \\ \Lambda_2 & \longrightarrow & \bar{\Lambda} \end{array}$$

with

$$\bar{\Lambda} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in M_2(\mathscr{F}_3) \right\}.$$

Note that $C(\Lambda_1) = 0$ by Corollary 3.7 applied to $ZC_9 \to \Lambda \to \Lambda_1$. It is now an easy exercise to show that $C(\mathrm{End}_\Lambda(M)) = 0$ using the Mayer-Vietoris sequence [25].

3. Bass orders

In [30, Th. 2.7] Wiegand shows that $D(\Lambda) = 0$ implies TFC for 1-dimensional commutative Bass rings. Using the results of §2 we can prove a similar result in the non-commutative case. Recall that an R-order Λ is called Gorenstein if $\Lambda^* = \mathrm{Hom}_R(\Lambda, R)$ is projective over Λ or equivalently if Λ^* is a generator [5, 37.9]. These conditions turn out to be left-right symmetric [5]. The order is called a *Bass order* if it and all larger orders in the same algebra are Gorenstein.

THEOREM 3.1. *If Λ is a global Bass order satisfying* EEC *and if $D(\Lambda) = 0$, then Λ satisfies* TFC.

Proof. Let M be a Λ lattice. If $J = \mathrm{Ann}_\Lambda(M)$ then Λ/J satisfies all the hypotheses of 3.1. In fact $A = K\Lambda = A_1 \times A_2$ where $KJ = 0 \times A_2$ so $\Lambda_1 = \Lambda/J$ is an order in A_1. If Λ_2 is the image of Λ in A_2 then $\Lambda_1 \times \Lambda_2$ is a Bass order containing Λ and so Λ_1 is Bass. By 3.7, $D(\Lambda_1) = 0$. We can therefore assume that M is faithful. We can also replace Λ by $o_l(M) = \{a \in A | aM \subset M\}$ so that $\Lambda = o_l(M)$. By a result of Faddeev [5, 37.12] there is an epimorphism $M^k \to \Lambda^*$ for some k. Since Λ^* is a generator we get an epimorphism $M^n \to \Lambda$ so $M^n \approx \Lambda \oplus N$ for some N. By 2.6 we see that $D(\mathrm{End}_\Lambda(M^n) = 0$. But

$$\mathrm{End}_\Lambda(M^n) = M_n(\mathrm{End}_\Lambda(M))$$

is Morita equivalent to $\mathrm{End}_\Lambda(M)$ so $D(\mathrm{End}_\Lambda(M)) = 0$ and the theorem follows from 2.1.

Remark. Unfortunately this result is of no use in deciding when ZG has TFC for non-commutative G since ZG is a Bass order if and only if G is cyclic of squarefree order. In fact if ZG is Bass then $\Lambda = ZG/(N)$ is Gorenstein where N is the sum of the elements of G. But $\Lambda^* \approx I$, the augmentation ideal of ZG. If there is an epimorphism $I^m \to \Lambda$ we get $(I/I^2)^m \to \Lambda/I$, i.e. $(G/[G,G])^m \to Z/gZ$ where g is the order of G. This implies that $G/[G,G]$ has an element of order g so G is cyclic. The result is well known in this case [2], [11].

In [2, Cor. 7.3] Bass shows that an indecomposable torsionless module over a 1-dimensional commutative Bass ring is projective over its endomorphism ring. A converse to this for domains is given by Handelman [14]. I do not know to what extent Bass' result holds in the non-commutative case. A related result over complete DVR's is given in [5, 37.13]. In any case, hereditary orders have this property.

LEMMA 3.2. *If Λ is a hereditary order and M is a Λ lattice then M is projective over* $\mathrm{End}_\Lambda(M)$.

Proof. As in the proof of 3.1 we can assume M is faithful and that $\Lambda = o_l(M)$. Then M will be a generator by Faddeev's result since hereditary orders are Bass orders. Therefore M is now a projective generator and the result follows from standard Morita theory [1], [5], [23].

The following is an analogue of 3.1 for the property just considered.

THEOREM 3.3. *Let Λ be a global order which satisfies* EEC *and has $D(\Lambda) = 0$. Suppose that every indecomposable Λ lattice is projective over its endomorphism ring. Then Λ satisfies* TFC.

By Theorem 2.8 it is enough to show that $D(\mathrm{End}_\Lambda(M)) = 0$ for indecomposable Λ lattices M. This follows from $D(\Lambda) = 0$ and the following general result which does not require K to be a global field.

PROPOSITION 3.4. *Let M be a lattice over an order Λ such that M is projective over $\Sigma =$* $\mathrm{End}_\Lambda(M)$. *Then there are epimorphisms $C(\Lambda) \to C(\Sigma)$ and $D(\Lambda) \to D(\Sigma)$.*

Proof. If M is a left Λ module we regard it as a right Σ module and define C and D using right modules. This is possible by the remark at the end of §1. Choose a maximal order Γ containing Λ.

Define a map $C(\Lambda) \to C(\Sigma)$ by sending $\xi = [P] - [Q]$ to $[P \otimes_\Lambda M] - [Q \otimes_\Lambda M]$. These modules are projective by the hypothesis and the image lies in $C(\Sigma) \subset K_0(\Sigma)$ since $P \otimes_\Lambda M$ and $Q \otimes_\Lambda M$ are locally isomorphic. Let $\Delta = \operatorname{End}_\Gamma(\Gamma M) \supset \Sigma$ and define $C(\Gamma) \to C(\Delta)$ similarly using ΓM.

LEMMA 3.5. *The following diagram commutes*:

$$\begin{array}{ccc} C(\Lambda) & \longrightarrow & C(\Sigma) \\ \downarrow & & \downarrow \\ C(\Gamma) & \longrightarrow & C(\Delta). \end{array}$$

Proof. Let $\xi = [P] - [Q] \in C(\Lambda)$. By Roiter's lemma [27, Th. 3.1] we can find a sequence

$$0 \to Q \to P \to X \to 0$$

where $X_v = 0$ whenever $\Lambda_v \neq \Gamma_v$. Write $\xi = [X]_\Lambda \in C(\Lambda)$. Since $Q \otimes_\Lambda \Gamma$ is torsion free,

$$0 \to Q \otimes_\Lambda \Gamma \to P \otimes_\Lambda \Gamma \to X \otimes_\Lambda \Gamma \to 0$$

is exact. But $X \otimes_\Lambda \Gamma = X$ so the image of ξ in $C(\Gamma)$ is $[X]_\Gamma$. Similarly

$$0 \to Q \otimes_\Lambda M \to P \otimes_\Lambda M \to X \otimes_\Lambda M \to 0$$

is exact so ξ maps to $[X \otimes_\Lambda M]$ in $C(\Sigma)$ which in turn goes to $[X \otimes_\Lambda M]$ in $C(\Delta)$. Now $[X]_\Gamma \in C(\Gamma)$ goes to $[X \otimes_\Gamma \Gamma M] \in C(\Delta)$ but $X \otimes_\Gamma \Gamma M \approx X \otimes_\Lambda M$ since $X_v = 0$ when $(\Gamma M)_v \neq M_v$.

It follows that our map induces a map $D(\Lambda) \to D(\Sigma)$.

LEMMA 3.6. *The maps $C(\Lambda) \to C(\Sigma)$ and $D(\Lambda) \to D(\Sigma)$ are onto.*

Proof. As in the proof of 3.5 we can represent an element $\eta = [P] - [Q] \in C(\Sigma)$ by $\eta = [Y]_\Sigma$ where $0 \to Q \to P \to Y \to 0$ and $Y_v = 0$ whenever $\Lambda_v \neq \Gamma_v$. Let $X = Y \otimes_\Sigma M^\vee$ where $M^\vee = \operatorname{Hom}_\Lambda(M, \Lambda)$. Then

$$0 \to Q \otimes_\Sigma M^\vee \to P \otimes_\Sigma M^\vee \to X \to 0$$

is exact since $M_v^\vee$ is projective whenever $P_v \neq Q_v$ and $\xi = [X]_\Lambda \in C(\Lambda)$. The image of ξ in $C(\Sigma)$ is

$$[X \otimes_\Lambda M] = \left[Y \otimes_\Sigma M^\vee \otimes_\Lambda M\right].$$

Now $M^\vee \otimes_\Lambda M \to \Sigma$ by $f \otimes m \to \varphi$, where $\varphi(x) = f(x)m$, becomes an isomorphism locally at each v such that $\Lambda_v = \Gamma_v$ since then M_v is projective. It follows that $Y \otimes_\Sigma M^\vee \otimes_\Lambda M \xrightarrow{\approx} Y \otimes_\Sigma \Sigma = Y$.

It remains to show that $\xi \in D(\Lambda)$ if $\eta \in D(\Sigma)$. The image ξ' of ξ in $C(\Gamma)$ is given by $[X]_\Gamma$ as above and

$$X = Y \otimes_\Delta (\Gamma M)^\vee$$

since $\Gamma M_v = M_v$ whenever $Y_v \neq 0$. Since Γ is maximal, $(\Gamma M)^\vee$ is projective over Γ so the functor$-\otimes_\Delta (\Gamma M)^\vee$ defines a map $C(\Lambda) \to C(\Gamma)$ as above. This sends the image η' of η to ξ'; but $\eta' = 0$ so $\xi' = 0$ also.

Proposition 3.4 can be regarded as a generalization of the following well known result of Fröhlich [9, §2 III]. This result is stated for C but the same proof works for D (cf. Cor. 2.6).

COROLLARY 3.7. *Let Λ and Γ be orders over R and let $\Lambda \to \Gamma$ be an R-algebra morphism such that $K\Lambda \to K\Gamma$ is onto. Then there are epimorphisms $C(\Lambda) \to C(\Gamma)$ and $D(\Lambda) \to D(\Gamma)$.*

We choose $M = \Gamma$ and note that $\mathrm{End}_\Lambda(M) = \mathrm{End}_\Gamma(M) = \Gamma$ because $K\Lambda \to K\Gamma$ is onto.

4. A patching method

In order to use Theorem 2.8 we must first find the indecomposable Λ lattices. The method considered in this section only requires knowledge of the indecomposable lattices over certain quotients of Λ. Suppose $K\Lambda = A = A_1 \times A_2$ and let Λ_i be the image of Λ under the projection on A_i. Write $\Lambda_i = \Lambda/I_i$ and let $\bar{\Lambda} = \Lambda/(I_1 + I_2)$. Then we have a cartesian diagram

(1)
$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda_1 \\ \downarrow & & \downarrow \\ \Lambda_2 & \longrightarrow & \bar{\Lambda} \end{array}$$

If M is a Λ lattice let

$$M_i = \Lambda_i M = (M/I_i M)/\text{torsion}.$$

Write $M_i = M/M_i'$ and let $\bar{M} = M/(M_1' + M_2')$. Since $M_1' \cap M_2' = 0$ we have a cartesian diagram

(2)
$$\begin{array}{ccc} M & \longrightarrow & M_1 . \\ \downarrow & & \downarrow{\scriptstyle f_1} \\ M_2 & \xrightarrow[f_2]{} & \bar{M} \end{array}$$

Note that in contrast to the usual Milnor patching of projective modules, $\overline{M}$ is not determined by M_1 and M_2 alone.

Let N be another Λ lattice and construct an analogous diagram

(3)
$$\begin{array}{ccc} N & \longrightarrow & N_1. \\ \downarrow & & \downarrow g_1 \\ N_2 & \xrightarrow[g_2]{} & \overline{N} \end{array}$$

We can assume that Λ_1 and Λ_2 satisfy TFC since this is clearly a necessary condition for Λ to satisfy TFC. Suppose that $X \oplus M \approx X \oplus N$ for some Λ lattice X. It follows that $X_i \oplus M_i \approx X_i \oplus N_i$ and so $M_i \approx N_i$. Similarly $\overline{M} \approx \overline{N}$ since the Krull-Schmidt theorem holds for the finite ring $\overline{\Lambda}$. Fix such isomorphisms and replace (3) by the isomorphic diagram

(4)
$$\begin{array}{ccc} N & \longrightarrow & M_1. \\ \downarrow & & \downarrow g_1 \\ M_2 & \xrightarrow{g_2} & \overline{M} \end{array}$$

Let $\hat{M} = \prod_{v \in S} M_v$ denote the completion of M at a finite set S of valuations including all v for which $\overline{\Lambda}_v \neq 0$. Since the Krull-Schmidt theorem holds in the complete case, $\hat{M} \approx \hat{N}$ so the completions of (2) and (4) are isomorphic. Therefore there is a commutative diagram

(5)
$$\begin{array}{ccccc} \hat{M}_1 & \xrightarrow{f_1} & \overline{M} & \xleftarrow{f_2} & \hat{M}_2 \\ \approx\downarrow & & \approx\downarrow & & \approx\downarrow \\ \hat{M}_1 & \xrightarrow{g_1} & \overline{M} & \xleftarrow{g_2} & M_2. \end{array}$$

If this can be refined to

(6)
$$\begin{array}{ccccc} M_1 & \xrightarrow{f_1} & \overline{M} & \xleftarrow{f_2} & M_2 \\ \approx\downarrow & & \approx\downarrow & & \approx\downarrow \\ M_1 & \xrightarrow{g_1} & \overline{M} & \xleftarrow{g_2} & M_2 \end{array}$$

it will follow that (2) and (4) are isomorphic and that $M \approx N$. It is convenient to split this lifting problem into two parts as follows. Enlarge (5) and (6) to

(7)
$$\begin{array}{ccccccccc}
\hat{M}_1 & \xrightarrow{\eta_1} & M_1/I_2M_1 & \xrightarrow{\bar{f}_1} & \overline{M} & \xleftarrow{\bar{f}_2} & M_2/I_1M_2 & \xleftarrow{\eta_2} & \hat{M}_2 \\
\approx\downarrow & & \approx\downarrow & & \approx\downarrow & & \approx\downarrow & & \approx\downarrow \\
\hat{M}_1 & \xrightarrow{\eta_1} & M_1/I_2M_1 & \xrightarrow{\bar{g}_1} & \overline{M} & \xleftarrow{\bar{g}_2} & M_2/I_1M_2 & \xleftarrow{\eta_2} & \hat{M}_2
\end{array}$$

and

(8)
$$\begin{array}{ccccccccc}
M_1 & \xrightarrow{\eta_1} & M_1/I_2M_1 & \xrightarrow{\bar{f}_1} & \overline{M} & \xleftarrow{\bar{f}_2} & M_2/I_1M_2 & \xleftarrow{\eta_2} & M_2 \\
\approx\downarrow & & \approx\downarrow & & \approx\downarrow & & \approx\downarrow & & \approx\downarrow \\
M_1 & \xrightarrow{\eta_1} & M_1/I_2M_1 & \xrightarrow{g} & \overline{M} & \xleftarrow{g} & M_2/I_1M_2 & \xleftarrow{\eta_2} & M_2
\end{array}$$

where η_1 and η_2 are the canonical quotient maps. Here is a simple case in which this approach obviously succeeds.

PROPOSITION 4.1. *Suppose for all Λ_i lattices M_i, $i = 1, 2$, that, with $I_1' = I_2$, $I_2' = I_1$,*

$$\mathrm{Im}[\mathrm{Aut}(M_i) \to \mathrm{Aut}(M_i/I_i'M_i)] = \mathrm{Im}\left[\mathrm{Aut}\left(\hat{M}_i\right) \to \mathrm{Aut}(M_i//I_i'M_i)\right].$$

Then if TFC *holds for Λ_1 and Λ_2, it also holds for Λ.*

In fact, it is sufficient to verify the hypothesis for indecomposable M_i using an easy generalization of standard results on elementary transformations [1]: Suppose that we are given a decomposition $M = \oplus M_i$. For given i, j with $i \neq j$ and f: $M_j \to M_i$ we define $e_{ij}(f) = 1 + \bar{f} \in \mathrm{Aut}(M)$ where $\bar{f}$: $M \to M_j \xrightarrow{f} M_i \to M$. Clearly $e_{ij}(f)^{-1} = e_{ij}(-f)$. Let $E(M)$ be the subgroup of $\mathrm{Aut}(M)$ generated by all $e_{ij}(f)$ for all $i \neq j$ and all f. It depends, of course, upon the decomposition $M = \oplus M_i$.

LEMMA 4.2. *Suppose Λ/I is finite. Then* $\mathrm{Im}[E(M) \to \mathrm{Aut}(M/IM)] = \mathrm{Im}[E(\hat{M}) \to \mathrm{Aut}(M/IM)]$.

Here, as above, $M = \oplus M_i$ is given and $\hat{M} = \prod_{v \in S} M_v$ for a finite set S of valuations.

Proof. Since $\mathrm{Hom}_\Lambda(M_j, M_i)^\wedge = \mathrm{Hom}_{\hat{\Lambda}}(\hat{M}_j, \hat{M}_i)$ we can approximate each of a finite set of maps f_v: $\hat{M}_{j_v} \to \hat{M}_{i_v}$ by g_v: $M_{j_v} \to M_{i_v}$ choosing g so close to f

that the induced maps $M_j/IM_j \to M_i/IM_i$ are the same. Therefore $\prod e_{i_\nu j_\nu}(f_\nu)$ and $\prod e_{i_\nu j_\nu}(g_\nu)$ have the same effect on M/IM.

If $M = \oplus M_i$ as above, define $D(M) \subset \mathrm{Aut}(M)$ to be the set of "diagonal" automorphisms; i.e. those preserving the summands so $\delta = \oplus \delta_i$ where δ_i: $M_i \approx M_i$.

PROPOSITION 4.3. *If each* $\mathrm{End}(M_i)$ *is semilocal then* $\mathrm{Aut}(M) = D(M)E(M) = E(M)D(M)$.

This is an analogue of a theorem of Bass [1] stating that $GL_n(A) = E_n(A)D_n(A)$ for a semilocal ring A. The same proof works in general, replacing matrices over A by matrices whose i, j component is a map $M_j \to M_i$. The same remark applies to the following which is an analogue of a result of Vaserstein [29].

PROPOSITION 4.4. *Let* f: $M \to N$ *and* g: $N \to M$. *If* $1 + gf \in \mathrm{Aut}(M)$ *then* $(1 + fg) \in \mathrm{Aut}(N)$ *and*

$$\begin{pmatrix} 1 + gf & 0 \\ 0 & (1 + fg)^{-1} \end{pmatrix} \in E(M \oplus N).$$

Vaserstein's proof works with no essential change. Explicitly, the matrix is

$$e_{21}(-fu)e_{12}(g)e_{21}(f)e_{12}(-ug)$$

where $u = (1 + gf)^{-1}$ and $(1 + fg)^{-1} = 1 - fug$.

As usual [29], we deduce a form of the Whitehead lemma.

COROLLARY 4.5. *If* $f \in \mathrm{Aut}(M)$ *then*

$$\begin{pmatrix} f & 0 \\ 0 & f^{-1} \end{pmatrix} \in E(M \oplus N).$$

Combining 4.2 and 4.3 gives us the following result.

COROLLARY 4.6. *Let* Λ/I *be finite and let* $M = \oplus M_i$. *If*

$$\mathrm{Im}\left[\mathrm{Aut}(M_i) \to \mathrm{Aut}(M_i/IM_i)\right] = \mathrm{Im}\left[\mathrm{Aut}(\hat{M}_i) \to \mathrm{Aut}(M_i/IM_i)\right]$$

for all i *then the same holds for* M.

Proof. Since $\operatorname{End}(\hat{M}_i)$ is semilocal, 4.3 shows that $\operatorname{Aut}(\hat{M}) = D(\hat{M})E(\hat{M})$. By 4.2, $E(\hat{M})$ and $E(M)$ have the same image. The analogous statement for $D(M)$ is the hypothesis of 4.6.

This shows that it is enough to assume the hypothesis of 4.1 for indecomposable modules. In fact, by the next result, it will usually suffice to look at one indecomposable lattice in each genus.

LEMMA 4.7. *If Λ satisfies* EEC, *Λ/I is finite, and M is a Λ lattice satisfying*

$$(*) \quad \operatorname{Im}[\operatorname{Aut}(M) \to \operatorname{Aut}(M/IM)] = \operatorname{Im}[\operatorname{Aut}(\hat{M}) \to \operatorname{Aut}(M/IM)]$$

then any Λ lattice in the same genus as M also satisfies $(*)$.

Proof. Let $\Sigma = \operatorname{End}(M)$. Then, as in the proof of 4.2, Σ and $\hat{\Sigma} = \operatorname{End}(\hat{M})$ have the same image $\bar{\Sigma}$ in $\operatorname{End}(M/IM)$. Write $\bar{\Sigma} = \Lambda/J$. Since $U(\hat{\Sigma}) \to U(\bar{\Sigma})$ is onto, the condition $(*)$ just says that $U(\Sigma) \to U(\bar{\Sigma})$ is onto. By EEC and [28, Th. 10.2],

$$U(\bar{\Sigma})/U(\Sigma) = \nu U(\hat{\Sigma})/\nu U(\hat{\Sigma}, \hat{J})[\nu U(\hat{\Sigma}) \cap U^{+}(\mathcal{O})]$$

where $\mathcal{O}$ is the integral closure of R in the center of $K\Sigma$. Now suppose that $N \vee M$. In the construction used in the proof of 1.6 we can assume that $M_v = N_v$ whenever $I_v \neq \Lambda_v$. Let $\Delta = \operatorname{End}(N)$. We have the same ring $\mathcal{O}$ for Σ and Δ. If $I_v = \Lambda_v$ then $J_v = \Sigma_v$ while if Λ_v is maximal $\nu U(\Sigma_v) = U(\mathcal{O}_v)$. For the remaining v, $M_v = N_v$. Therefore the above formula shows that $U(\bar{\Sigma})/U(\Sigma) = U(\bar{\Delta})/U(\Delta)$.

Two simple examples to which the result applies are ZC_4 and ZV where $V = C_2 \times C_2$. We use the two diagrams

$$\begin{tikzcd} ZC_4 \arrow[r] \arrow[d] & Z[\zeta_4] \arrow[d] \\ ZC_2 \arrow[r] & \mathcal{F}_2 C_2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} ZV \arrow[r] \arrow[d] & ZC_2 \arrow[d] \\ ZC_2 \arrow[r] & \mathcal{F}_2 C_2. \end{tikzcd}$$

The method also applies to ZC_{2p} for odd primes p. We use the diagram

$$\begin{tikzcd} ZC_{2p} \arrow[r] \arrow[d] & ZC_p \arrow[d] \\ ZC_p \arrow[r] & \mathcal{F}_2 C_p. \end{tikzcd}$$

By [20], the indecomposable ZC_p modules are Z, ideals $\mathcal{A}$ of $Z[\zeta_p]$, and $P \in LF_1(ZC_p)$ with endormorphism rings Z, $Z[\zeta_p]$, and ZC_p. It will suffice for TFC to have $Z[\zeta_p]^* \to (Z[\zeta_p]/(2))^*$ and $(ZC_p)^* \to (\mathscr{F}_2C_p)^*$ onto. These conditions hold if $p = 2$. If p is odd, the diagram

$$\begin{array}{ccc} ZC_p & \longrightarrow & Z[\zeta_p] \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathscr{F}_p^* \end{array}$$

shows that $U(Z[\zeta_p], \mathscr{P}) \subset ZC_p$ and $\mathscr{F}_2C_p = \mathscr{F}_2 \times Z[\zeta_p]/(2)$ where $\mathscr{P}$ is the prime ideal over p. Therefore it will suffice for TFC to have $U(Z[\zeta_p], \mathscr{P}) \to (Z[\zeta_p]/(2))^*$ onto. This is so for $p = 3$, 5 and 7 so we get a new proof of Wiegand's result that ZG has TFC for $G = V, C_4, C_6, C_{10}$, and C_{14} [30, §5].

5. The cyclic group of order 8

The method developed in §4 can be used to settle the cancellation problem for ZC_8.

THEOREM 5.1. *ZC_8 satisfies* TFC.

COROLLARY 5.2. *If G is a finite abelian group the ZG satisfies* TFC *if and only if $D(ZG) = 0$.*

This follows from the work of Wiegand [30, §5], 2.10, and 5.1. The groups in question were classified by Cassou-Noguès [4]. They are $C_2 \times C_2, C_p$ for p a prime, and C_n for $n \leq 14$, $n \neq 12$. In §8 I will show that the condition $D(ZG) = 0$ does not suffice for TFC in the non-abelian case.

To prove 5.1 we consider the diagram

$$\begin{array}{ccc} ZC_8 & \longrightarrow & Z[\zeta_8] \\ \downarrow & & \downarrow \\ ZC_4 & \longrightarrow & \mathscr{F}_2C_4 \end{array}$$

noting that $\mathscr{F}_2C_4 = Z[\zeta_8]/(2)$. As in §4 we construct a diagram of the form (7) and try to produce one of the form (8).

LEMMA 5.3. *Let M be a $Z[\zeta_8]$ lattice. Then* $\mathrm{Aut}(M) \to \mathrm{Aut}(M/2M)$ *is onto.*

Proof. Since M is free, this map is just $GL_n(Z[\zeta_8]) \to GL_n(B)$ where $B = Z[\zeta_8]/(2)$. Since B is semilocal,

$$GL_n(B)/E_n(B) = B^*$$

so it is enough to show that $Z[\zeta_8]^* \to B^*$ is onto. Now

$$B = \mathscr{F}_2 C_4 = \mathscr{F}_2[x]/(x^4 - 1) = \mathscr{F}_2[\lambda]/\lambda^4$$

where $\lambda = x - 1$. One easily checks that B^*/C_4 has order 2 and is generated by $1 + \lambda^2 + \lambda^3 = 1 + x + x^{-1}$. This is the image of $1 + \zeta_8 + \zeta_8^{-1} = 1 + \sqrt{2} \in Z[\zeta_8]^*$.

LEMMA 5.4. *Let B be an artinian ring, P a finitely generated projective B module, and let $P \to X$ be an epimorphism of B modules. Then every automorphism of X lifts to an automorphism of P.*

Proof. Let π: $Q \to X$ be a projective cover [1, III 2.12], [5, §6C]. Then $P \approx Q \oplus P'$ with the given map being

$$\pi \circ pr_1 : P \to Q \to X.$$

Any automorphism of X lifts to a map $Q \to Q$ which is onto and hence an automorphism. We extend it to P by the identity on P'.

COROLLARY 5.5. *Let M be a $Z[\zeta_8]$ lattice and let X be a quotient of $M/2M$. Then any automorphism of X lifts to one of M.*

We now consider the corresponding question for ZC_4. At the same time I will discuss the case of ZV where $V = C_2 \times C_2$ is the *four group*. This case will be needed in §6.

LEMMA 5.6. *Let G be a finite p-group and let $\Delta = ZG + Z(N/p) \subset QG$. Then any ZG lattice has the form $P \oplus M$ where P is projective and M is a Δ lattice.*

Proof. We first observe that $\mathscr{F}_p G$ has a unique minimal non-zero ideal $(N) = (\mathscr{F}_p G)^G$. If N annihilates M/pM then M is clearly a Δ module since $NM \subset pM$. If this is not the case, let $x \in M/pM$ be such that $Nx \neq 0$ and define a monomorphism $\mathscr{F}_p G \to M/pM$ by sending 1 to x. Lift this map to a map $ZG \to M$. The kernel J of this map is a Z-direct summand so J/pJ is contained in $\mathscr{F}_p G$ and maps to 0 in M/pM. Therefore J/pJ, and hence J itself, is 0. Let $M' \approx ZG$ be the image of $ZG \to M$ and let $M'' = M \cap QM'$. Then M'' has the same rank as M'. The composition $M'/pM' \to M''/pM'' \to M/pM$ is injective so it follows by comparing ranks that $M'/pM' \xrightarrow{\approx} M''/pM''$ is an isomorphism and therefore $M'' = M' + pM''$. This implies that M''/M' has order prime to p and therefore M'' is projective over ZG [27], [5]. Since M'' is a Z-direct summand of M, it is also a ZG direct summand since projective ZG modules are weakly injective [3] (or by 5.9 below). Therefore $M = M'' \oplus M_1$ and we are done by induction on the rank.

As an immediate corollary we see that for a p-group G, a ZG lattice M with M/pM free must be projective.

LEMMA 5.7. *Let $|G| = 4$ and let Δ be as in 5.6. If M is a Δ lattice and $\hat{M}$ is its 2-adic completion then*

$$\operatorname{Im}[\operatorname{Aut}(M) \to \operatorname{Aut}(M/2M)] = \operatorname{Im}[\operatorname{Aut}(\hat{M}) \to \operatorname{Aut}(M/2M)].$$

Proof. Let $\Sigma = \operatorname{End}_\Delta(M)$. Then the image of Σ in $\operatorname{End}_\Delta(M/2M)$ is $\bar{\Sigma} = \Sigma/2\Sigma$. Since $\hat{\Sigma}^* \to \bar{\Sigma}^*$ is onto we must show that $\bar{\Sigma}^*/\Sigma^* = 0$. By [28, Th. 10.2], $\bar{\Sigma}^*/\Sigma^* = \nu U(\hat{\Sigma})/\nu U(\hat{\Sigma}, 2\hat{\Sigma})[U^+(\mathcal{O}) \cap \nu U(\hat{\Sigma})]$. Since TFC holds for ZG, $D(\Sigma) = 0$ by 1.6 and Fröhlich's formula [28, Th. 10.6] shows that

$$U(\hat{\mathcal{O}}) = \nu U(\hat{\Sigma}) U^+(\mathcal{O})$$

so

$$\nu U(\hat{\Sigma})/\left[U^+(\mathcal{O}) \cap \nu U(\hat{\Sigma})\right] = U(\hat{\mathcal{O}})/U^+(\mathcal{O}).$$

Therefore

$$\bar{\Sigma}^*/\Sigma^* = U(\hat{\mathcal{O}})/U^+(\mathcal{O})\nu U(\hat{\Sigma}, 2\hat{\Sigma}).$$

Let $\Gamma \supset \Delta$ be the maximal order of QG. For $G = V$, $\Gamma = Z^4$ while for $G = C_4$, $\Gamma = Z \times Z \times Z[i]$. The conductor $\mathbf{f}$ of $\Delta \subset \Gamma$ is easily seen to be $(2Z)^4$ for $G = V$ and $(2) \times (2) \times \mathcal{P}$ for $G = C_4$ where $\mathcal{P} = (1 + i)$ is the prime ideal of $Z[i]$ over 2. Let $\Xi = \operatorname{End}_\Gamma(\Gamma M)$. Then $\mathbf{f}\Xi \subset \Sigma$ so

$$U(\hat{\Sigma}, 2\hat{\Sigma}) \supset U(\hat{\Sigma}, 2\mathbf{f}\hat{\Sigma}) = U(\hat{\Xi}, 2\mathbf{f}\hat{\Xi}).$$

Now $\nu U(\hat{\Xi}, 2\mathbf{f}\hat{\Xi}) = U(\hat{\mathcal{O}}, 2\mathbf{f}\hat{\mathcal{O}})$ where $\mathcal{O}$ is the center of Ξ. This is immediate here since Ξ is a product of matrix algebras over Z and $Z[i]$. It follows that $\bar{\Sigma}^*/\Sigma^*$ is a quotient of

$$U(\hat{\mathcal{O}})/U(\hat{\mathcal{O}}, 2\mathbf{f}\hat{\mathcal{O}})U^+(\mathcal{O}) = U(\mathcal{O}/2\mathbf{f}\mathcal{O})/U(\mathcal{O}).$$

But this is 0. Since $\mathcal{O} = Z^r \times Z[i]^s$ for some r and s, it is sufficient to check the cases $\mathcal{O} = Z$, $\mathbf{f} = 2Z$ and $\mathcal{O} = Z[i], \mathbf{f} = \mathcal{P}$.

LEMMA 5.8. *Let M be a ZG lattice where $|G| = 4$. Let α be an automorphism of $M/2M$. Then α lifts to an automorphism of M in the following cases:*

(1) *$M/2M$ is not free and α lifts to an automorphism of the 2-adic completion $\hat{M}$.*

(2) *$M/2M$ is free and* $\det(\alpha) \in G \subset (\mathcal{F}_2 G)^*$.

Proof. In case (2), $\mathrm{Aut}(M) \to \mathrm{Aut}(M/2M)$ is just $GL_n(ZG) \to GL_n(\mathscr{F}_2G)$ since in this case M is projective (and hence free) by the remark following 5.6. Since $\mathscr{F}_2G$ is local $GL_n(\mathscr{F}_2G)/E_n(\mathscr{F}_2G) = (\mathscr{F}_2G)^*$ and the result follows since the image of ZG^* in $(\mathscr{F}_2G)^*$ is G.

In case (1), write $M = (ZG)^n \oplus M'$ by 5.6 where M' is a Δ lattice. By 4.3, $\mathrm{Aut}(\hat{M}) = D(\hat{M})E(\hat{M})$ with respect to this decomposition and the image of $E(\hat{M})$ is the same as that of $E(M)$ by 4.2. If $\delta \in D(\hat{M})$, the image of $\delta' = \delta|M'$ lifts to $\mathrm{Aut}(M')$ by 5.7. By 4.5, $\delta'' = \delta|(\hat{Z}G)^n$ can be modified modulo $E((\hat{Z}G)^n) \subset E(\hat{M})$ so that

$$\delta'' = (\beta, 1, \ldots, 1) \quad \text{where } \beta \in (\hat{Z}G)^*.$$

If the image $\bar{\beta}$ of β in $(\mathscr{F}_2G)^*$ lies in G we can lift it to $(ZG)^*$. Since $(\mathscr{F}_2G)^*/G = Z/2Z$ is generated by $1 + N$, it will suffice to show that $(1 + N, 1) \in \mathrm{Aut}(\mathscr{F}_2G \oplus M'/2M')$ lifts to $\mathrm{Aut}(ZG \oplus M')$.

By assumption, $M' \neq 0$ so $\overline{M}' = M'/2M' \neq 0$. Since $\mathscr{F}_2G$ is local with residue field $\mathscr{F}_2$ we can find an epimorphism θ: $\overline{M}' \to \mathscr{F}_2$. Let g: $\overline{M}' \to \mathscr{F}_2G$ by $g(m) = \theta(m)N$. Choose $z \in \overline{M}'$ with $\theta(z) = 1$ and define f: $\mathscr{F}_2G \to \overline{M}'$ by $f(1) = z$. Then $1 + gf = 1 + N$ while $1 + fg = 1$ because M' is a Δ lattice and so N annihilates $\overline{M}'$. By 4.4,

$$(1 + N, 1) \in E\left(\mathscr{F}_2G \oplus \overline{M}'\right).$$

This lifts to $E(ZG \oplus M')$ by the following standard result.

LEMMA 5.9. *Let L and M be ZG lattice one of which is projective. Then*

$$\mathrm{Hom}_{ZG}(L, M) \to \mathrm{Hom}_{ZG}(L/nL, M/nM)$$

is onto for all integers n.

Proof. We can lift to a Z-homomorphism h: $L \to M$ since L is free over Z. A projective ZG module has a Z endomorphism θ with

$$\sum_{\sigma \in G} \sigma\theta\sigma^{-1} = 1 \quad [3]$$

and we use $\Sigma\sigma\theta h\sigma^{-1}$ or $\Sigma\sigma h\theta\sigma^{-1}$ as our lift.

We can now prove Theorem 5.1. Consider a diagram of the form (7) with $\Lambda_1 = Z[\zeta_8]$ and $\Lambda_2 = ZG$ with $G = C_4$. There are three cases to consider. For the first two, the only fact we need about Λ_1 is that

$$\mathrm{Im}[\mathrm{Aut}(M_1) \to \mathrm{Aut}(M_1/2M_1)] = \mathrm{Im}\left[\mathrm{Aut}\left(\hat{M}_1\right) \to \mathrm{Aut}(M_1/2M_1)\right]$$

which shows that we can produce the required map on the left hand side of the diagram (8).

Case 1. If $M_2/2M_2$ is not free we can produce a diagram of the form (8) by applying 5.8(1).

Case 2. Suppose $M_2/2M_2$ is free but $\bar{f}_2$ is not an isomorphism. Let b be a non-zero element of $\ker \bar{f}_2$ which is fixed by G. Write $b = Nc$ for some c which is necessarily a part of a base for $M_2/2M_2$ since $B = \mathscr{F}_2G$ is local. Write

$$M_2/2M_2 = B \times \cdots \times B$$

with $c = (1, 0, \dots, 0)$. If the map α: $M_2/2M_2 \xrightarrow{\approx} M_2/2M_2$ in (7) has $\det(\alpha) \in G$ we can complete the right end of (8) by 5.8(2). If not, replace α by $\beta = \alpha \circ (1 + N, 1, \dots, 1)$ which will then have $\det(\beta) \in G$ while the resulting diagram will still commute.

Case 3. Finally, suppose that $M_2/2M_2$ is free and that $\bar{f}_2$ is an isomorphism. In this case we let $\theta = \bar{g}_2\bar{f}_2^{-1}$, form the diagram

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] & \overline{M} \arrow[d, "\theta"', "\approx"] & M_2 \arrow[l, "f_2"'] \arrow[d, "1"] \\
M_1 \arrow[r, "g_1"] & \overline{M} & M_2 \arrow[l, "g_2"']
\end{tikzcd}$$

and complete to a diagram of the form (8) using 5.5.

6. The dihedral group of order 8

This case can be treated by the same method which was used for C_8. As in [26] I will use the notation D_8 to denote the dihedral group of order 8 (sometimes denoted by D_4).

THEOREM 6.1. *ZD_8 satisfies* TFC.

Let $D = D_8 = \langle x, y \colon x^4 = y^2 = 1, yxy = x^{-1} \rangle$. We will apply the method of §4 to the diagram

$$\begin{tikzcd}
ZD \arrow[r] \arrow[d] & \Lambda \arrow[d] \\
ZV \arrow[r] & \mathscr{F}_2V
\end{tikzcd}$$

where V is the four group and $\Lambda = ZD/(x^2+1) = \Lambda'_4$ in the notation of [26]. Note that $\Lambda/2\Lambda = \mathcal{F}_2 V$. We first determine the Λ lattices. By [26, Lemma 3.2]

$$\Lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(Z) \mid a \equiv d, c \equiv 0 \pmod 2 \right\}.$$

LEMMA 6.1. *The Λ lattices M with QM simple are*

$$P = \begin{pmatrix} Z \\ Z \end{pmatrix} \quad \textit{and} \quad Q = \begin{pmatrix} Z \\ 2Z \end{pmatrix}.$$

Proof. Since $QM = \begin{pmatrix} Q \\ Q \end{pmatrix}$ we can assume that $M \subset \begin{pmatrix} Z \\ Z \end{pmatrix}$ and that the greatest common divisor of all the entries in M is 1. Writing the elements of M as rows for convenience we easily see that $(a, b) \in M$ implies that $(2a, 0), (b, 0), (0, 2a), (0, 2b) \in M$. Therefore $(2,0), (0,2) \in M$. If an odd b occurs then $M = P$ and otherwise $M = Q$.

LEMMA 6.2. (1) $\mathrm{Ext}^1_\Lambda(P, P) = \mathrm{Ext}^1_\Lambda(Q, Q) = Z/2Z$.
(2) $\mathrm{Ext}^1_\Lambda(P, Q) = \mathrm{Ext}^1_\Lambda(Q, P) = 0$.
(3) $\mathrm{Ext}^1_\Lambda(P, \Lambda) = \mathrm{Ext}^1_\Lambda(Q, \Lambda) = 0$.

Proof. This follows easily from the resolutions

$$0 \to P \to \Lambda \to P \to 0, \quad 0 \to Q \to \Lambda \to Q \to 0$$

in which the maps are

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 2a & 0 \\ 2b & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} b \\ d \end{pmatrix}$$

and

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix}.$$

LEMMA 6.3. *The indecomposable Λ lattices are P, Q, and Λ.*

Proof. We show that any Λ lattice M is a direct sum of these modules by induction on the rank. By mapping QM onto a simple $Q\Lambda$ module we get an exact sequence $0 \to N \to M \to P \to 0$ or a similar sequence with Q in place of P. Let $N = P^a \oplus Q^b \oplus \Lambda^c$. Then $\mathrm{Ext}^1_\Lambda(P, N) = (Z/2Z)^a$ by 7.2. If the class of the extension is non-zero, we can reduce it to $(1, 0, \dots, 0)$ by elementary transformations on $(Z/2Z)^a$. These lift to automorphisms of P^a since all endomorphisms of $Z/2Z$ lift. It then follows that $M \approx P^{a-1} \oplus Q^b \oplus \Lambda^{c+1}$.

LEMMA 6.4. *Let M be a Λ lattice and $\hat{M}$ its 2-adic completion. Then*

$$\operatorname{Im}[\operatorname{Aut}(M) \to \operatorname{Aut}(M/2M)] = \operatorname{Im}[\operatorname{Aut}(\hat{M}) \to \operatorname{Aut}(M/2M)].$$

Proof. By 4.6 it is enough to do this for indecomposable M. If $M = P$ or Q then $\operatorname{Aut}(M) = Z^*$ and $\operatorname{Aut}(\hat{M}) = \hat{Z}^*$ have image 1 in $\operatorname{Aut}(M/2M)$. If $M = \Lambda$ we must show that $\Lambda^* \to (\mathscr{F}_2 V)^*$ is onto. Now $(\mathscr{F}_2 V)^*$ is generated by V and $1 + x + y$ and V is clearly the image of $D \subset \Lambda^*$. Finally $1 - x + y$ is a unit of Λ with inverse $1 + x - y$ since $xy = -yx$ and $x^2 = -1$ in Λ.

LEMMA 6.5. *Let M be a Λ lattice and let $\overline{M} = M/2M$. Given a decomposition $\overline{M} = X \oplus Y$ with $X \approx \mathscr{F}_2 V$ we can find a decomposition $M = S \oplus T$ with $S \approx \Lambda$, $S/2S \approx X$ and $T/2T \approx Y$.*

Proof. Let $M = P^a \oplus Q^b \oplus \Lambda^c$. As observed in the proof of 5.6, $\overline{\Lambda} = \mathscr{F}_2 V$ has a unique minimal ideal (N). This must annihilate $\overline{P}$ and Q since these have dimension 2 over $\mathscr{F}_2$. Let $x = (p_1, \dots, p_a, q_1, \dots, q_b, r_1, \dots, r_c)$ in $\overline{M}$ generate X. Since $Nx \neq 0$, some $Nr_i \neq 0$, say for $i = c$. Then r_c generates $\overline{\Lambda}$. By elementary transformations of M we can reduce x to $(0, \dots, 0, r_c)$. These transformations involve only maps from $\overline{\Lambda}$ and therefore lift to M. Therefore we can choose a decomposition of $M = P^a \oplus Q^b \oplus \Lambda^c$ so that $x = (0, \dots, 0, r)$. Let S be the last summand and L the sum of the remaining ones. Then $M = S \oplus L$ and $S/2S = X$. The map

$$M \to \overline{M} = X \oplus Y$$

restricts to

$$(\varphi, \psi)\colon L \to X \oplus Y.$$

Now $\operatorname{Ext}^1_\Lambda(L, S) = 0$ by 6.2 and 6.3 since $S \approx \Lambda$. Therefore the exact Ext sequence for

$$0 \to S \xrightarrow{2} S \to X \to 0$$

shows that $\operatorname{Hom}_\Lambda(L, S) \to \operatorname{Hom}_\Lambda(L, X)$ is onto so that we can lift $\varphi\colon L \to X$ to a map $\theta\colon L \to S$. Let ε be the elementary transformation of $M = S \oplus L$ determined by $-\theta$. Then we can choose $T = \varepsilon(L)$ since $M = S \oplus T$ and the map $M \to \overline{M}$ sends T to Y.

COROLLARY 6.6. *Let M and $\overline{M} = X \oplus Y$ be as in 6.5. If α is any automorphism of X then*

$$\alpha \oplus 1\colon X \oplus Y \approx X \oplus Y$$

lifts to an automorphism of M.

This follows from the fact that $\Lambda^* \to (\mathscr{F}_2 V)^*$ is onto as we have observed in the proof of 6.4.

We can now prove Theorem 6.1 by the same method used to prove Theorem 5.1 at the end of §5. We take $\Lambda_1 = \Lambda$ and $\Lambda_2 = ZV$. The first two cases are identical with those of §5. Only case 3 needs to be modified since the analogue of 5.5 does not hold here. In this case $M_2/2M_2$ is free and $\bar{f}_2$ is an isomorphism. Therefore M is also free and we can write

$$M_1/2M_1 = W \oplus \overline{M}$$

with $\bar{f}_1$ being the projection on $\overline{M}$. Let $\theta\colon \overline{M} \to \overline{M}$ be $(1 + N, 1, \ldots, 1)$ with respect to some base of $\overline{M}$ and let $\varphi\colon M_1/2M_1 \to M_1/2M_1$ be $\theta \oplus 1\colon \overline{M} \oplus W \to \overline{M} \oplus W$. By 6.6 this lifts to an automorphism $\sigma\colon M_1 \approx M_1$.

Let $\alpha\colon M_2/2M_2 \overset{\approx}{\to} M_2/2M_2$ be the map occurring in the diagram (7). If $\det(\alpha) \in V$ we can complete diagram (8) just as in §5 using 5.8(2). If $\det(\alpha) \notin V$ replace the diagram (7) by the top and bottom lines of

$$\begin{array}{ccccccccc}
\hat{M}_1 & \longrightarrow & M_1/2M_1 & \xrightarrow{\bar{f}_1} & \overline{M} & \xleftarrow{\bar{f}_2} & M_2/2M_2 & \longleftarrow & \hat{M}_2 \\
{\scriptstyle\sigma}\downarrow & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\theta}\downarrow & & {\scriptstyle\psi}\downarrow & & {\scriptstyle\approx}\downarrow \\
\hat{M}_1 & \longrightarrow & M_1/2M_1 & \xrightarrow{\bar{f}_1} & \overline{M} & \xleftarrow{\bar{f}_2} & M_2/2M_2 & \longleftarrow & \hat{M}_2 \\
{\scriptstyle\approx}\downarrow & & {\scriptstyle\approx}\downarrow & & {\scriptstyle\approx}\downarrow & & {\scriptstyle\approx}\downarrow{\scriptstyle\alpha} & & {\scriptstyle\approx}\downarrow \\
\hat{M}_1 & \longrightarrow & M_1/2M_1 & \xrightarrow{\bar{g}_1} & \overline{M} & \xleftarrow{\bar{g}_2} & M_2/2M_2 & \longleftarrow & \hat{M}_2
\end{array}$$

where $\psi = \bar{f}_2^{-1}\theta\bar{f}_2$ and the lower two lines are the original diagram (7). We can now complete diagram (8) by using 6.4 to fill in on the left and 5.8(2) on the right. Since $\det(\alpha) \notin V, \det(\alpha\psi) = \det(\alpha)\det(\theta) \in V$ as required.

7. Dihedral groups

If ZG satisfies TFC for a finite group G then $D(ZG) = 0$ by 1.3. Endo and Hironaka [6] have shown that if $D(ZG) = 0$ then G is either abelian, dihedral, or one of A_4, S_4, or A_5. Further restrictions on the order of G in the dihedral case are given by Endo and Miyata [8]. I will give here a few positive results on the TFC problem for dihedral groups. The method of §4 requires some knowledge of the indecomposable modules over reasonable quotients of ZG. Because of this, I have only been able to handle the dihedral groups of order $2p$ and $4p$ at the present time, p being a prime. We can, of course, assume p is odd because of Theorem 6.1.

Note. I will continue to use the notation of [26] in which D_n denotes the dihedral group of order n.

The following is a special case of a theorem of Klingler [16, Th. 11.4] which shows that cancellation holds for *all* finitely generated ZD_{2p}-modules (possibly with torsion). Using the method of §4 we can give a short proof in the torsion free case.

THEOREM 7.1. *ZD_{2p} satisfies* TFC *for all primes p.*

Remark. Theorem 7.1 would follow from Lee's classification of lattices over ZD_{2p} [17] provided the last sentence of [17, Th. 3.2] is altered to read "up to ZG isomorphism" rather than "up to $Z_{2p}G$ isomorphism". The theorem is true in this form. Conversely, one can use Theorem 7.1 to deduce this classification from Lee's results in the local case and her classification of indecomposable lattices. In fact, by 1.2 we see that if TFC holds, a module M is determined by its genus and by ΓM where $\Gamma \supset ZD_{2p}$ is a maximal order. The genus of ΓM is determined by M. Since $\Gamma = Z \times Z \times M_2(R_p)$ where $R_p = Z_p[\zeta_p + \zeta_p^{-1}]$, we see that ΓM is determined by its genus and an ideal class of R_p.

Proof of 7.1. Let

$$D_{2p} = \langle x, y \colon x^p = y^2 = 1,\ yxy = x^{-1} \rangle$$

and consider the diagram

$$\begin{array}{ccc} ZD_{2p} & \longrightarrow & \Lambda \\ \downarrow & & \downarrow \\ ZC_2 & \longrightarrow & \mathscr{F}_pC_2 \end{array}$$

where $\Lambda = ZD_{2p}/(\Phi_p(x)) = \Lambda'_p$ in the notation of [26]. Let J be the kernel of the map $\Lambda \to \mathscr{F}_pC_2$. In the following we will assume that p is odd.

LEMMA 7.2. *If M is any Λ lattice, then* $\mathrm{Aut}(M) \to \mathrm{Aut}(M/JM)$ *is onto.*

Proof. By [26, Lemma 8.1],

$$\Lambda \approx \begin{pmatrix} R & R \\ P & R \end{pmatrix}$$

where $R = Z[\zeta_p + \zeta_p^{-1}]$ and P is the prime ideal of R over p. Furthermore, Λ is hereditary and the indecomposable Λ lattices are

$$S = \begin{pmatrix} R \\ R \end{pmatrix}, \qquad T = \begin{pmatrix} R \\ P \end{pmatrix}, \qquad \mathscr{A}S, \mathscr{A}T$$

for ideals $\mathscr{A}$ of R [17, §1]. One checks easily that

$$J = \begin{pmatrix} P & R \\ P & P \end{pmatrix}$$

and $S/JS \approx \mathscr{F}_p \times 0, T/JT \approx 0 \times \mathscr{F}_p$ over $\mathscr{F}_pC_2 = \mathscr{F}_p \times \mathscr{F}_p$. Since all Λ lattices are projective, all elementary transformations of M/JM lift to M. Therefore it is sufficient to check 7.2 for indecomposable modules M. But $\mathrm{End}_\Lambda(\mathscr{A}S) = \mathrm{End}_\Lambda(\mathscr{A}T) = R$ and $R^* \to \mathscr{F}_p^*$ is onto as required.

If W is a module over $\mathscr{F}_pC_2 = \mathscr{F}_p \times \mathscr{F}_p$, write

$$W = W' \times W'' \quad \text{where } W' = (1,0)W \text{ and } W'' = (0,1)W.$$

If $\alpha \in \mathrm{Aut}(W)$ then $\alpha = (\alpha', \alpha'')$ and we define $\det'(\alpha) = \det(\alpha')$ and $\det''(\alpha) = \det(\alpha'')$.

LEMMA 7.3. *Let M be a ZC_2 lattice and let p be an odd prime. Let*

$$\alpha \in \mathrm{Aut}_{ZC_2}(M/pM).$$

If $\det'(\alpha) = \det''(\alpha) = 1$ *then α lifts to* $\mathrm{Aut}_{ZC_2}(M)$.

Proof. Clearly α is a product of elementary transformations. These lift to the p-adic completion of M since Z_pC_2 is hereditary and the result follows from 4.2.

LEMMA 7.4. *Let V and W be finite dimensional vector spaces and let f: $V \to W$, g: $V \to W$ be epimorphisms. Then there is a commutative diagram*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\alpha"', "\approx"] & W \arrow[d, "\theta"', "\approx"] \\ V \arrow[r, "g"] & W \end{tikzcd}$$

where $\det(\alpha) = 1$.

Proof. If f is an isomorphism, then so is g and we choose $\alpha = 1$ and $\theta = gf^{-1}$. Otherwise, let

$$V = U_1 \oplus W_1 = U_2 \oplus W_2$$

where $U_1 = \ker f$ and $U_2 = \ker g$, choose $\theta = 1$ and let $\alpha = (\beta, \gamma)$ where $g\gamma = f$ and β is chosen so that $\det(\alpha) = 1$.

We can now prove Theorem 7.1 along the same lines as the proof of 5.1 and 6.1. Consider a diagram of the form (7) with $\Lambda_1 = \Lambda$, and $\Lambda_2 = ZC_2$. Applying Lemma 7.4 to each of the two components of f_2 and g_2 we obtain a diagram

$$\begin{array}{ccccc} M_1/JM_1 & \longrightarrow & \overline{M} & \xleftarrow{f_2} & M_2/pM_2 \\ & & {\scriptstyle\theta}\downarrow{\scriptstyle\approx} & & {\scriptstyle\alpha}\downarrow{\scriptstyle\approx} \\ M_1/JM_1 & \longrightarrow & \overline{M} & \xleftarrow{g_2} & M_2/pM_2 \end{array}$$

in which $\det'(\alpha) = \det''(\alpha) = 1$. By 7.3, α lifts to an automorphism of M_2. There is no difficulty in lifting θ to an automorphism of M_1/JM_1, for example by 5.4. This in turn lifts to an automorphism of M_1 by 7.2.

THEOREM 7.5. *Let p be a prime. Then D_{4p} satisfies* TFC *if and only if $D(ZD_{4p}) = 0$.*

Proof. Let

$$D_{4p} = \langle x, y \colon x^{2p} = y^2 = 1, yxy = x^{-1} \rangle.$$

We consider the diagram

$$\begin{array}{ccc} ZD_{4p} & \longrightarrow & ZD_{2p} \\ \downarrow & & \downarrow \\ ZD_{2p} & \longrightarrow & \mathscr{F}_p D_{2p} \end{array}$$

where x^p goes to $+1$ and -1 in the two ZD_{2p} terms.

LEMMA 7.6. *Let $D = D_{2p}$ with p an odd prime. If $D(ZD_{4p}) = 0$ then $\mathrm{Aut}_{ZD}(M) \to \mathrm{Aut}_{ZD}(M/2M)$ is onto for all ZD lattices M.*

This clearly implies 7.5 by 4.1 and 7.1. To prove 7.6 we need only consider indecomposable M by 4.6 and, by 4.7, we need only consider one module in each genus. We begin by recalling Lee's classification of the genera of indecomposable ZD lattices [17]. Write $D = \langle x, y \colon x^p = y^2 = 1, yxy = x^{-1} \rangle$.

We describe the ZD modules by giving a ZC_p module where $C_p = \langle x \rangle$ and specifying the action of y. We assume p is odd. The labelling is as in [L]. The lattices are as follows:

(s_1) Z with y acting as 1,
(s_2) Z with y acting as -1,
(l) ZC_2 where $C_2 = \langle y \rangle$,
(r_1) $Z[\zeta_p]$ with $y =$ complex conjugation,
(r_2) $Z[\zeta_p]$ with $y = -$complex conjugation,
(u_1) ZC_p with $y \cdot f(x) = -f(x^{-1})$,
(u_2) ZC_p with $y \cdot f(x) = f(x^{-1})$,
(v_1) $Z \oplus ZC_p$ with $y \cdot (a, f(x)) = (a, -f(x^{-1}) + aN)$ where $N = \Sigma x^i$,
(v_2) $Z \oplus ZC_p$ with $y \cdot (a, f(x)) = (-a, f(x^{-1}) + aN)$,
(t) ZD.

It is straightforward to verify that these modules represent the extensions considered by Lee [17]. The endomorphism rings are easily computed to be as follows:

(s) Z,
(l) ZC_2,
(r) $R = Z[\zeta_p + \zeta_p^{-1}]$,
(u) $ZRC_p^+ = \{f(x) \in ZC_p\colon f(x) = f(x^{-1})\}$,
(v) the ring E described below,
(t) ZD.

The ring E can be described as the pullback in the diagram

$$\begin{tikzcd} E \arrow[r, "\delta"] \arrow[d, "\alpha"'] & ZC_p^+ \arrow[d, "\bar{\varepsilon}"] \\ Z \arrow[r] & \mathscr{F}_2 \end{tikzcd}$$

where $\bar{\varepsilon}$ is induced by the augmentation ε. E acts on $Z \oplus ZC_p$ by the matrix

$$\begin{pmatrix} \alpha & 0 \\ cN & \delta \end{pmatrix}$$

where $2c = \alpha - \varepsilon(\delta)$ for (v_1) and $2c = \varepsilon(\delta) - \alpha$ for (v_2).

When reduced modulo 2 these modules become:

(s) $\mathscr{F}_2$,
(l) $\mathscr{F}_2C_2$,
(r) $Z[\zeta_p]/(2)]$
(u) $\mathscr{F}_2C_p = \mathscr{F}_2 \oplus Z[\zeta_p]/(2)$,
(v) $\mathscr{F}_2C_2 \oplus Z[\zeta_p]/(2)$,
(t) $\mathscr{F}_2D$

and the endomorphism rings of these modules are:

(s) $\mathscr{F}_2$,

(ℓ) $\mathscr{F}_2C_2$,
(r) $(Z[\zeta_p]/(2))^+ = R/2R$,
(u) $\mathscr{F}_2C_2 \times R/2R$,
(v) $\mathscr{F}_2C_2 \times R/2R$,
(t) $\mathscr{F}_2D$.

For the equality listed under (r), note that $Z[\zeta_p]$ is Z free with base $\zeta, \zeta^2, \dots, \zeta^{p-1}$ permuted by y.

It is clear that $\operatorname{Aut}(M) \to \operatorname{Aut}(M/2M)$ is onto for (s) and (l). For (r) this map is $R^* \to (R/2R)^*$. By [26, Lemma 11.1], the cokernel of this map is $D(ZD_{4p})$ which is 0 by hypothesis. For (u) we have a cartesian diagram

$$\begin{tikzcd} ZC_p^+ \arrow[r] \arrow[d] & R \arrow[d] \\ Z \arrow[r] & \mathscr{F}_p. \end{tikzcd}$$

This shows that $U(R, P) \subset U(ZC_p^+)$ where $R/P = \mathscr{F}_p$. By [25, §7], $R^* = U(R, P)U(R, 2)$ so $U(R, P)$ maps onto $(R/2R)^*$ if R^* does. For (v) we have $E^* = Z^* \times (ZC_p^+)^*$ and, after a bit of calculation, it is easily checked that the map is onto. The summand $Z[\zeta_p]/(2)$ is best represented by $(x - x^{-1})\mathscr{F}_2C_p$. The element of E with $\alpha = -1, \delta = 1$ maps onto the non-trivial unit of $\mathscr{F}_2C_p$. Finally, the Mayer-Vietoris sequence for our cartesian diagram for ZD_{4p} gives $K_1(ZD) \to K_1(\mathscr{F}_2D) \to D(ZD_{4p}) \to 0$. Using [28, Cor. 10.5], we see that

$$U(\mathscr{F}_2D)/U(ZD) \xrightarrow{\approx} D(ZD_{4p})$$

which is 0 by hypothesis.

8. Negative results

The results obtained so far suggest that TFC may hold for ZG whenever $D(ZD) = 0$. However this is not the case as the following result shows.

THEOREM 8.1. *Let d^2 divide n where $d = 5$ or $d \geq 7$. Then* TFC *does not hold for ZD_{2n}.*

In particular TFC fails for D_{2^n} if $n \geq 7$ although $D(ZD_{2^n}) = 0$ for all n [10]. Similarly TFC fails for ZD_{2p^2} for prime $p \geq 5$ but $D(ZD_{2p^2}) = 0$ if p is a regular prime [15] [8].

Proof. Let $D = D_{2n} = \langle x, y \colon x^n = y^2 = 1, yxy = x^{-1}\rangle$ and let $C = C_n = \langle x \rangle \subset D$. Make ZC a ZD module by $y \cdot x^i = x^{-i}$ as in the case (u_2) of §7.

Let I be the augmentation ideal of ZC so that $I = (x - 1)$. Then (I, d) is a ZD-submodule of ZC.

LEMMA 8.2. *$(I, d)/(I^2, d) \approx Z/dZ$ with x acting as 1 and y acting as -1.*

Proof. Since $dZC \cap I = dI$, we have

$$(I, d)/(I^2, d) = I/\big(I \cap (I^2, d)\big) = I/(I^2 + dI).$$

Since $I/I^2 \approx C \approx Z/nZ$ and $d|n$, it follows that $(I, d)/(I^2, d) \approx Z/dZ$ generated by the image of $x - 1$. Note that x acts trivially since $(x - 1)^2 \in I$ and $y(x - 1) = (x^{-1} - 1)(1 - x) + (1 - x)$ so y acts as -1.

LEMMA 8.3. *$\mathrm{End}_{ZC}(I, d) = ZC + Z \cdot N/d$ where $N = \Sigma x^i$.*

Proof. Clearly the endomorphism ring lies in $\mathrm{End}_{QC}(QC) = QC$ and contains ZC. It also contains N/d since $d|n$ and $N \equiv n \bmod I$; thus $N \in (I, d)$. Suppose that $a \in QC$ and $a(I, d) \subset (I, d)$. Then

$$a(x - 1) = b(x - 1) + cd$$

with b and c in ZC. Taking augmentations shows that $\varepsilon(c) = 0$ and so $c = c_1(x - 1)$. It follows that $a - b - c_1 d$ annihilates $x - 1$ and therefore has the form qN with $q \in Q$. Since $qN(I, d) \subset ZC$ we see that $dqN \in ZC$ and hence $dq \in Z$.

Remark The same result holds for any finite group G if $d||G|$. As above we show that any endomorphism of (I, d) preserves I and use the fact that $\mathrm{End}_{ZG}(I) = ZG/(N)$. To see this note that if φ is an endomorphism of I, then $x \mapsto \varphi(x - 1)$ is a 1-cocycle which then splits in ZG; thus $\varphi(i) = ia$ for some $a \in ZG$. It is also quite easy to show that (N, d) has the same endomorphism ring as (I, d).

Note that the augmentation of QC takes N/d to n/d in Z and so defines a map ε: $\mathrm{End}_{ZC}(I, d) \to Z$.

LEMMA 8.4. *For any $\alpha \in \mathrm{End}_{ZC}(I, d)$ the diagram*

$$\begin{array}{ccc} (I, d) & \xrightarrow{\alpha} & (I, d) \\ \downarrow & & \downarrow \\ Z/dZ & \xrightarrow{\varepsilon(\alpha)} & Z/dZ \end{array}$$

commutes.

The vertical map here is that given by 8.2. The assumption that $d^2|n$ will be needed in the proof.

Proof. This is trivial for $\alpha \in ZC$ so we need only check the case $\alpha = N/d$. Here $\varepsilon(\alpha) = n/d \equiv 0 \bmod d$ since $d^2|n$. Clearly $\alpha(x-1) = 0$ and $\alpha d = N$ so we must show that N maps onto 0 in Z/dZ. Now

$$\begin{aligned} N = \sum x^i &= n + \sum (x^i - 1) \\ &= n + (x-1)\sum(1 + x + \cdots + x^{i-1}) \\ &\equiv n + (x-1)\sum i \\ &= n + (x-1)n(n-1)/2 \bmod (x-1)^2. \end{aligned}$$

Since $d^2|n$, we see that $d|n(n-1)/2$ and so $N \mapsto 0$ in Z/dZ as required.

Now let $Z' = Z$ with x acting as 1 and y acting as -1. Define M to be the pullback in the diagram

$$\begin{array}{ccc} M & \longrightarrow & (I, d) \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z/dZ. \end{array}$$

Since $QM = QC \oplus Q'$ and $\mathrm{Hom}_{QD}(Q', QC) = \mathrm{Hom}_{QD}(Q_C, Q') = 0$, the decomposition $QM = QC \oplus Q'$, and hence the diagram, is preserved by all endomorphisms of M. Using 8.3 we get a cartesian diagram

$$(\dagger) \qquad \begin{array}{ccc} \mathrm{End}_{ZD}(M) & \longrightarrow & \mathrm{End}_{ZD}(I, d) \\ \downarrow & & \downarrow{\scriptstyle \bar{\varepsilon}} \\ \mathrm{End}_{ZD}(Z') & \longrightarrow & \mathrm{End}_{ZD}(Z/dZ) \end{array}$$

The bottom line of ($\dagger$) is just $Z \to Z/dZ$ and the Mayer-Vietoris sequence of ($\dagger$) gives

$$\begin{aligned} K_1(\mathrm{End}_{ZD}(I, d)) \oplus K_1(Z) \to K_1(Z/dZ) &\xrightarrow{\partial} K_0(\mathrm{End}_{ZD}(M)) \\ &\to K_0(\mathrm{End}_{ZD}(I, d)) \oplus K_0(Z). \end{aligned}$$

The image of ∂ clearly lies in $D(\mathrm{End}_{ZD}(M))$. The map $\bar{\varepsilon}$ of ($\dagger$) factors as

$$\mathrm{End}_{ZD}(I, d) \xrightarrow{\varepsilon} Z \to Z/dZ$$

so we see that $D(\mathrm{End}_{ZD}(M))$ contains the group $K_1(Z/dZ)/K_1(Z) = (Z/dZ)^*/Z^*$. This group is non-trivial for the range of d specified in 8.1; thus $D(\mathrm{End}_{ZD}(M)) \neq 0$ and TFC fails by 2.1.

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