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SURGERY ON THE EQUATORIAL IMMERSION I

BY

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1. Introduction

Herein, smooth immersions of closed unoriented manifolds in codimension 1 Euclidean space are studied. The geometric topology of representative immersions as it relates to stable homotopy invariants is emphasized. The author's studies [1], [2], [3], [4], [5], and [6] are continued. Please see [2] and [6] for synopses.

(1.1) For each $k = 1, 2, \ldots, m$ there is an immersion

$$e : \bigcup_{j=1}^{k} S_j^{m-2} \to S^{m-1}$$

defined by the equation

$$e\left(x_1, \ldots, \hat{x}_j, \ldots, x_m\right) = \left(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_m\right).$$

Here the domain is the disjoint union of k $(m-2)$-spheres,

$$S_j^{m-2} = \left\{\left(x_1, \ldots, \hat{x}_j, \ldots, x_m\right) : \sum x_k^2 = 1\right\};$$

$e|_{S^{m-2}}$ is an embedding, but the union is immersed. Such an immersion is called *an equatorial immersion* since each S_j is embedded as an equator of S^{m-1}. The multiple points of $(e, \cup_{j=1}^{k} S_j^{m-2})$ are spheres of lower dimensions. This immersion is null bordant since it is obtained by a piggy back sequence [6] of $(0,0), (0,1), \ldots, (0, k-1)$ surgeries on the empty immersion. Please recall a (j, r)*-surgery* attaches a hollow j-handle, $D^j \times S^{n-j}$, to an immersion $i : M \to \mathbf{R}^{n+1}$; the core disk, $D^j \times \{0\}$, lies in the r-tuple set $(0 \in D^{n+1-j})$.

The equatorial immersion is a prototype for piggy back sequences of surgeries in the following sense. If a piggy back sequence of $(j, 0), \ldots,$

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$(j, k-1)$ surgeries is performed to an immersion $i : M^n \to \mathbf{R}^{n+1}$, an equatorial immersion in $D^{m-1} = \mathrm{cl}(S^{m-1} - D^{m-1})$ appears as the intersection of the hollow handles of the surgery and an $(m-1)$-disk transverse to the initial core; here $m = n + 2 - j$.

The equatorial immersion in S^{m-1} is the boundary of a small neighborhood of a generic k-tuple point in D^m. Thus to find immersions in S^m with unusual or prescribed k-tuple behavior, one may look for null bordisms of the equatorial immersion that are not standard (such examples appear in [7]).

(1.2) A most important example of such an immersion in S^m is an immersion with one m-tuple point. Thus the equatorial immersion, $e : \bigcup_{j=1}^m S_j^{m-2} \to S^{m-1}$, of m intersecting $(m-2)$-spheres is the boundary of a small neighborhood of a generic m-tuple point. Henceforth, *the equatorial immersion* will mean this particular equatorial immersion. For example, the equatorial immersion in the 2-sphere consists of three great circles: the prime meridian, the equator, and the meridonal great circle 90 degrees from the prime meridian. The following propositions are trivial.

(1.2.1) PROPOSITION. *There is a codimension one immersion in S^m with one m-tuple point if and only if the equatorial immersion $e : \bigcup_{j=1}^m S_j^{m-2} \to S^{m-1}$ bounds an immersion in D^m with no m-tuple points.*

(1.2.2) PROPOSITION. *There is a codimension one immersion in S^m with one m-tuple point if and only if every null bordant immersion is S^{m-1} bounds two types of null bordisms: an* ***untainted*** *null bordism has no m-tuple points; a* ***tainted*** *null bordism has exactly one m-tuple point.*

A theorem of Eccles [8] relates this to the Kervaire invariant problem: *If $m = 4k + 2$, there is a codimension one immersion in S^m with one m-tuple point if and only if there is a framed m-manifold of Kervaire invariant* 1.

The organization of this paper is as follows. In Section 2 the generalized Boy's immersions are reviewed. A conceptually easier construction is given. Paragraph (2.2) contains some variations on the construction that will be used in the companion paper [7]. Also in paragraph (2.2) is a product formula for the self intersections of an immersion that is the disjoint union of various immersions that are superimposed. This formula has the flavor of a Cartan formula, but that terminology was used in [9] in a different context.

Section 3 contains the principal result of the paper. Let $k = n + 1 - r$. Theorem (3.1) states:

An embedded standardly framed k-sphere can be removed from the r-tuple set of an immersion if and only if one of the following conditions hold:

1. *a push-off of the sphere bounds a disk in the next lower self intersection stratum*;

2. *it is possible to remove the* $(k-u-1)$*-dimensional stratum from the equatorial immersion in* S^{r-1} *via an untainted bordism*;

3. *it is possible to engulf the linking stratum without introducing further* k*-dimensional self intersections.*

Here $n \geq 5$ and $k < (n+1)/2$. The integer $u = u(k)$ is the largest number such that every immersion of a closed manifold in $\mathbf{R}^{k+1}$ is bordant to an immersion without $(k+1-u(k))$-tuple points (equivalently without u-dimensional multiple points). It is unknown if conditions 1 and 3 alone are necessary.

Section 4 contains a summary of the sequel [7].

2. The generalized Boy's immersion revisited

(2.1) Please recall from [5]:

For each $n = 1, 2, \ldots,$ *there is an immersion* $B : (M^n, \partial M) \to (D^{n+1}, S^n)$ *with exactly one* $(n+1)$*-tuple point at* $0 \in D^{n+1}$*. Here* M *denotes the (twisted iff* n *is even)* $(n-1)$*-disk bundle over the circle. The immersion* (B, M) *has an* $(n+1)$*-fold cyclic symmetry.*

The construction of (B, M) may be obtained as the trace of surgery on the equatorial immersion in S^n as follows. Let

$$f_j(t) = (\cos t)e_j - (\sin t)e_{j+1} \quad \text{for } t \in [0, \pi/2]$$

where the index, j, is reduced modulo $m = n+1$, and e_j denotes the j-*th* unit vector of the standard basis for $\mathbf{R}^m$. The arcs f_j are arcs of $(m-2)$-tuple points of the equatorial immersion (see paragraph (1.2)). Each of the $(m-1)$-tuple points of the immersion e appears as the end point of some arc f_j. These arcs, f_j, are the cores for $(1, m-2)$-surgery.

Subsequent surgeries are parametrized by reticulate subsets of $\{1, 2, \ldots, m\}$ and certain pairs of reticulate sets. A subset, R, is *reticulate* if no two elements in R are consecutive, where m and 1 are, by definition, consecutive. In fact, a reticulate set, R, of order r uniquely determines the $(m-r)$-tuple sphere for an $(r, m-r-1)$-surgery. The set does not determine the core disk uniquely. However, once a choice of core disk has been made, the subsets R' of R such that the pair (R', R) parametrizes further "delta shaped" surgeries are determined. This construction differs from that given in [5] where every pair $(R' \subseteq R)$ parametrizes the core of some collection of intersecting handles. Explicit choices for the core disk are left to the reader.

(2.2) Some variations on the above construction are obtained below. Partition the set $\{1, \ldots, m\}$ into a collection P of subsets

$$T_1 = \{1, \ldots, j_1\}, T_2 = \{j_1+1, \ldots, j_2\}, \ldots, T_k = \{j_{k-1}+1, \ldots, m\}.$$

Let $\lambda_1 = j_1 - 1$; when α is between 2 and $k-1$, let $\lambda_\alpha = j_\alpha - j_{\alpha-1} - 1$; and let $\lambda_k = m - j_{k-1} - 1$. Thus the subsphere

$$S(T_\alpha) = \left\{(x_b, \dots, x_c) : \sum x_i^2 = 1, b = j_\alpha + 1, c = j_{\alpha+1}\right\}$$

is a sphere of dimension λ_α. The equatorial immersion in S^{m-1} restricted to $S(T_\alpha)$ is an equatorial immersion of $(\lambda_\alpha - 1)$-spheres.

Define a subset, $R \subseteq \{1, 2, \dots, m\}$, to be *reticulate with respect to the partition P* if and only if $R \cap T_\alpha$ is reticulate when the set T_α is identified with the set $\{1, \dots, \lambda_\alpha + 1\}$ in an order preserving fashion. Perform surgeries to $(e, \bigcup_{j=1}^m S_j^{m-2})$ where the attaching spheres are parametrized by the sets that are reticulate with respect to P. Further surgeries, parametrized by certain reticulate pairs, should also be performed. As above these are determined by the previous choices of core disks. Then for each $\alpha = 1, 2, \dots, k$, the equatorial immersion in $S(T_\alpha)$ has been surgered to be the boundary of Boy's immersion B: $(\partial M)^{\lambda_\alpha - 1} \to S(T_\alpha)$. (The disk bundle, M_α, is of dimension λ_α.)

Such surgeries eliminate all of the 0-dimensional multiple points of the equatorial immersion in S^{m-1} unless, of course, some set T_α has exactly one element. If two of the T_α's each have one element, they may be juxtaposed and further surgeries may be performed. Thus assume that at most one of λ_α is equal to 0, for $\alpha = 1, 2, \dots, k$. In the sequel [7] it will be advantageous to use a partition in which some T_α's are singletons.

Finally, any haphazard choice of arcs similar to the f_j's that is employed to remove the 0-dimensional multiple points of the equatorial immersion will yield some partition of $\{1, 2, \dots, m\}$. This partition is equivalent to P upon permuting the coordinates of S^{m-1}. The property that each element, T_α, of P contains consecutive integers is a bookkeeping convenience.

Let $B(P) : M(P) \to D^m$ denote the result of gluing the trace of the above surgeries to the immersion of the unit coordinate $(m-1)$-disks in D^m. That is $B(P)$ is the trace of the sequence of $(0,0), \dots, (0, m-1)$ surgeries to the empty immersion together with the trace of the above surgeries performed to the equatorial immersion.

Let $C_\alpha = \{1, \dots, m\} - T_\alpha$. Let $D(C_\alpha)$ denote the unit $(m - \lambda_\alpha - 1)$-disk in D^m with coordinates x_j where $j \in C_\alpha$. Let $S(C_\alpha) = \partial D(C_\alpha)$. Therefore, $S(C_\alpha)$ and $S(T_\alpha)$ form a generalized Hopf link in S^{m-1}. Similarly, let $D(T_\alpha)$ denote the disk that $S(T_\alpha)$ bounds. Consider the immersion

$$\bigsqcup_{\alpha=1}^{k} B_\alpha \times (\mathrm{id}) : \bigsqcup_{\alpha=1}^{k} M_\alpha \times D(C_\alpha) \to D^m$$

where $B_\alpha : M_\alpha \to D(T_\alpha)$ is a generalized Boy's immersion, and the dimension of M_α is λ_α. Furthermore, each immersion B_α is adjusted so that its boundary is contained in a sphere concentric with and interior to the sphere $S(T_\alpha)$. This rescaling is made to ensure that the immersions of $\partial M_\alpha \times S(C_\alpha)$ do not intersect in the tropical torus—the torus $S(T_\alpha) \times S(C_\alpha)$.

(2.2.1) PROPOSITION.

$$\bigsqcup_{\alpha=1}^{k} (B_\alpha \times (\mathrm{id})) = B(P).$$

More generally let $i_\alpha : A_\alpha \to D(T_\alpha)$ denote a codimension one immersion of an (λ_α)-manifold, A_α, for $\alpha = 1, 2, \ldots, k$. If A_α has non empty boundary, assume the immersion i_α when restricted to the boundary takes this to a sphere concentric with and interior to $S(T_\alpha)$ as above. Identify D^m with the Cartesian product $D(T_\alpha) \times D(C_\alpha)$. Define an immersion $f_\alpha : A_\alpha \times D(C_\alpha) \to D^m$ by the equation

$$f_\alpha(x, y) = (i_\alpha(x), y).$$

Then inductively define

$$f = \bigsqcup_{\alpha=1}^{k} f_\alpha : \bigsqcup_{\alpha=1}^{k} A_\alpha \times D(C_\alpha) \to D^m$$

where the image of the first $(k-1)$-A_α's is identified with $D(C_k)$, and then $D(T_k) \times D(C_k)$ and $D(C_k) \times D(T_k)$ are identified with D^m. Let the r_α-tuple manifold of A_α be denoted by $A_\alpha(r_\alpha)$. Please recall from [8] this manifold of dimension $(\lambda_\alpha + 1 - r_\alpha)$ is immersed in $D(T_\alpha)$. The image of $A_\alpha(r_\alpha)$ is the collection of points of $f_\alpha(A_\alpha)$ of multiplicity greater than or equal to r_α.

(2.2.2) *The r-tuple manifold of the immersion f is the disjoint union of the products of the self intersection manifolds $A_1(r_1), \ldots, A_k(r_k)$ where the disjoint union is taken over all $r_1, \ldots, r_k$ such that $r_1 + \cdots + r_k = r$, and*

$$\dim\left(\bigtimes_{\alpha=1}^{k} A_\alpha(r_\alpha) \right) = m - r;$$

by definition $A_\alpha(0) = D(T_\alpha)$.

Proof. Suppose $k = 2$. In this case the result states that the multiple points of f are as follows (in simplified notation $\dim(A) = \lambda$ and $\dim(B) = \mu =$

$m - \lambda - 2$):

1-tuples: $A \times D^{m-\lambda-1} \sqcup D^{\lambda+1} \times B$,
2-tuples: $A(2) \times D^{m-\lambda-1} \sqcup A(1) \times \mathrm{B}(1) \sqcup \mathrm{D}^{\lambda+1} \times \mathrm{B}(2)$,
⋮
$(\lambda + \mu)$-tuples: $A(\lambda + 1) \times B(\mu - 1) \sqcup A(\lambda) \times B(\mu) \sqcup A(\lambda - 1) \times B(\mu + 1)$,
$(\lambda + \mu + 1)$-tuples: $A(\lambda + 1) \times B(\mu) \sqcup A(\lambda) \times B(\mu + 1)$,
$(\lambda + \mu + 2)$-tuples: $A(\lambda + 1) \times B(\mu + 1)$.

Of course, some of the multiple points manifolds of A or B may be empty; in this case the corresponding terms vanish. So for $k = 2$, the result is trivial.

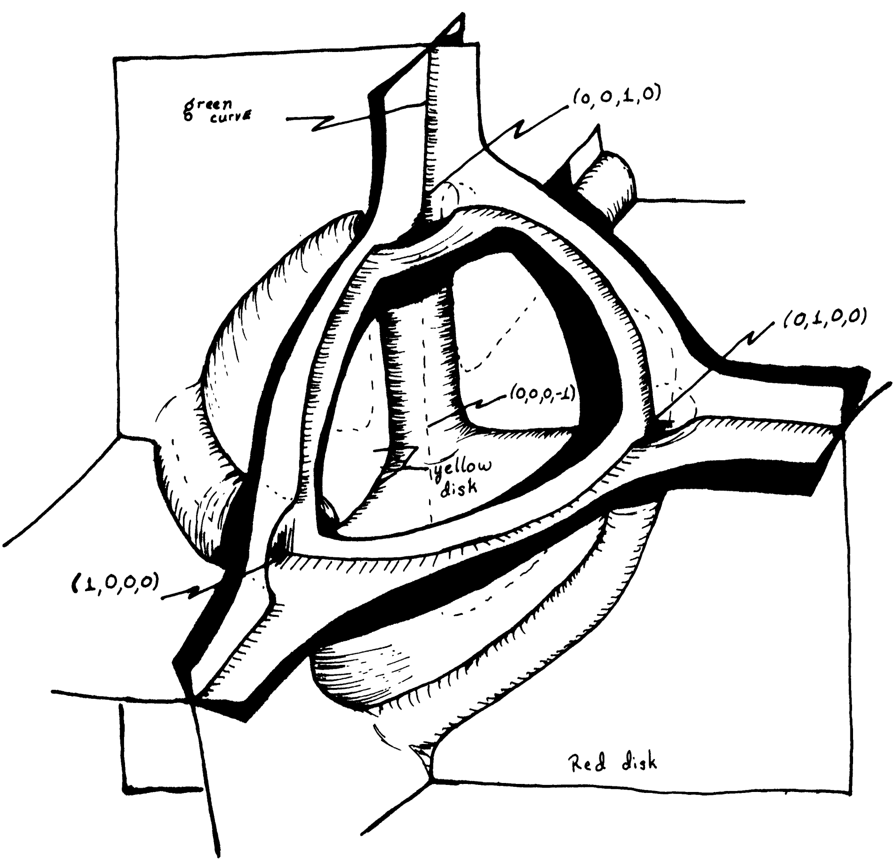

FIG. 2.4.1

The general case follows from this case and induction once the various identifications of D^m have been reconciled.

Proposition (2.2.1) follows by comparing the self intersection manifolds of the two immersions. If one has developed a taste for the scaffolds of [5], then it is fun to draw the scaffolds of $\sqcup B_\alpha \times (\mathrm{id})$; these become product scaffolds. In particular it is amusing, albeit combinatorially trivial, to see how the various products of simpleces fit into a simplex of a given dimension.

(2.3) In case $m = 3$, when the equatorial immersion in S^2 is surgered according to the trivial partition $T_1 = \{1, 2, 3\}$, a Jordan curve in the plane results. This may be removed by $(2, 0)$-surgery. The trace of the surgeries that is engendered, as in the fifth paragraph of (2.2), is the standard Boy's immersion.

In case $m = 4$, when the equatorial immersion in S^3 is surgered according to the trivial partition, the result may be further surgered to an embedding. Figure 2.4.1 depicts the result of the first four $(1, 2)$-surgeries. "Windows" have been cut so the immersion may be seen easily. The indicated disks are analogous to the red and yellow disks of [4]. The long double point curve that passes through infinity corresponds to the green curve of [4]. It is not hard to see that a push off of this curve bounds a disk in the complement of the immersion. Thus a $(2, 0)$ and a $(2, 1)$ surgery will remove this double point curve. An embedding of spheres results. After these are removed, the trace of all these surgeries yields the generalized Boy's immersion of [4]. More information about this construction will be given in the sequel [7].

3. Linked self intersection sets

The self intersection sets of codimension one immersions may be linked. Thus it may be possible that the self intersection invariants vanish, and yet the self intersection sets cannot be removed. This is why the main theorem of [6] is stated in a weak form. Namely:

If (i, M) is an immersion for which the self intersections sets of dimension less than k are empty, and the self intersection invariant of degree $n + 1 - k$ vanishes, then the k-dimensional self intersection set may be made spherical. Here $n \geq 5$ and $k < (n + 1)/2$.

Necessary and sufficient conditions are given below to remove this sphere.

Suppose that S^k is an embedded sphere in the $(n + 1 - k)$-tuple set of a codimension one immersion (i, M^n). Let $r = n + 1 - k$. Suppose further that S^k has a standard framing induced by the immersion (i, M). That is if $\theta \in S^k$ and $p_1(\theta), \ldots, p_r(\theta) \in i^{-1}(\theta)$, there are normal vectors $v_1(\theta), \ldots, v_r(\theta)$ depending smoothly on θ such that $v_j(\theta)$ is normal to (i, M) at $p_j(\theta)$; this framing $(v_1, \ldots, v_r)$ of the normal bundle is standard.

Let the integer $u = u(k)$ be the largest number such that every immersion of a closed manifold in $\mathbf{R}^{k+1}$ is bordant to an immersion without $(k + 1 - u(k))$-tuple points (equivalently without u-dimensional multiple points). Let $n \geq 5$ and $k < (n + 1)/2$.

(3.1) THEOREM. *An embedded standardly framed k-sphere can be removed from the r-tuple set of an immersion if and only if one of the following conditions hold*:

1. *a push-off of the sphere bounds a disk in the next lower self intersection stratum*;
2. *it is possible to remove the* $(k - u - 1)$*-dimensional stratum from the equatorial immersion in* S^{r-1} *via an untainted bordism*;
3. *it is possible to engulf the linking stratum without introducing further k-dimensional self intersections.*

Proof. Consider the immersion

$$B \times \mathrm{id} : M^6 \times S^5 \to S^{12},$$

$$\sqcup\ \mathrm{id} \times E : S^6 \times (P^2 \times S^3) \to S^{12}$$

The immersion (B, M) is Hill-Tout's immersion with one septuple point ([7] or [8]). The immersion $(E, P^2 \times S^3)$ is Eccles's immersion [8] with one hextuple point. The spheres S^5 and S^6 form a generalized Hopf link in the 12-sphere. This linked immersion has a 5-sphere of septuple points that cannot be removed by piggy back sequences of surgeries. On the other hand, the linking intersection set can be engulfed to a ball without introducing further septuple points. This example shows the engulfing condition is necessary.

In general, the high dimensional self intersection set of an immersion (i, M) may intersect itself along the $(n + 1 - k)$-tuple set of (i, M). Thus the condition (2) *may* be necessary. In case the linking phenonenon does not occur, condition 1 holds, and a piggy back sequence of surgeries exists. Therefore, for a particular immersion, one of these conditions is necessary.

Rescale the immersion so that the map $\theta \mapsto \theta + \Sigma \pm v_j(\theta)$ (where the sum is over any q-element subset of $\{1, \ldots, r\}$) is a push off of S^k into the $(r - q)$-tuple set. (Thus $1 \leq q \leq \mathrm{r} - 1$.) Suppose no such push off bounds a disk that is embedded in the $(r - q - 1)$-tuple set. Then in particular, the various push offs into the non-singular image of (i, M) do not bound disks in the complement of the immersion. However, if $k < (n + 1)/2$, then every push off bounds some embedded disk. In this case the r-tuple set is said to be *linked* with the self intersection sets of lower multiplicity; a piggy back sequence of strong surgeries employed to remove the k-sphere from the self intersection set does not exist.

Let $\tilde{h} : S^k \overset{\subset}{\to} \mathbf{R}^{n+1}$ denote a push off of the r-tuple sphere into the nonsingular image of (i, M). Let $h : D^{k+1}z \overset{\subset}{\to} \mathbf{R}^{n+1}$ denote an embedded disk bounded by $\tilde{h}(S^k)$. Choose these so that the inward pointing normal at $h(\theta)$ is $v_r(\theta)$ for $\theta \in S^k$.

The disk $h(D^{k+1})$ intersects the immersion $i(M)$ transversely in the former's interior. The intersection will be an immersion

$$f : N^k \to \text{int } D^{k+1}.$$

In a sufficiently small neighborhood of $h(\text{int } D^{k+1})$, $i(M)$ looks like $f(N) \times D^{r-1}$.

The immersion (f, N) can be engulfed to an $(n + 1)$-ball, by standard results in engulfing theory. However, since (f, N) is part of the larger immersion (i, M), the image of the engulfed (i, M) may have further k-dimensional self intersections. There is always a way to make the intersections be of the form

$$(f, N) \times \left(e, \bigcup_{j=1}^{r} S_j^{r-2}\right).$$

The radius of the engulfing ball sometimes can be increased to accommodate the linking set and to remove further introduced self intersections. This occurs in the example depicted in Figure 3.1.0. In this figure an immersed surface in 3-space is shown. The immersion "8"$\times$ S^1 can be engulfed to a small ball away from the horizontal double point curve on the right. It is likely that high dimensional analogs of this immersion exists.

In case conditions (1) and (3) do not hold, perform a piggy back sequence of weak (a.k.a transverse) $(k + 1, 0)$-$, \ldots, (k + 1, r - 1)$-surgeries to remove S^k from the r-tuple set. (Please recall that a weak (j, q)-surgery requires only that the core disk transversely intersects the multiple points of dimension greater than q.) These surgeries will introduce self intersections of the form $f(N^k) \times (e, \cup_{j=1}^{r} S_j^{r-2})$ to the resulting immersion which is denoted again by (i, M). The equatorial immersion, $(e, \cup S_j^{r-2})$ in D^{r-1} occurs as the intersection of the cocore of the piggy back sequence with the "vertical" disk D^{r-1}. The situation is schematized Figure (3.1.1).

According to the product formula, (2.2.2), the k-dimensional intersections in $f(N^k) \times (e, \cup S_j^{r-2})$ are as follows: The immersion $f(N^k)$ crossed with the 0-dimensional intersections of $(e, \cup S_j^{r-2})$, the double points of (f, N) crossed with the 1-dimensional set of $(e, \cup S_j^{r-2})$, and so forth, through the 0-dimensional intersections of (f, N) crossed with the k-dimensional intersections of

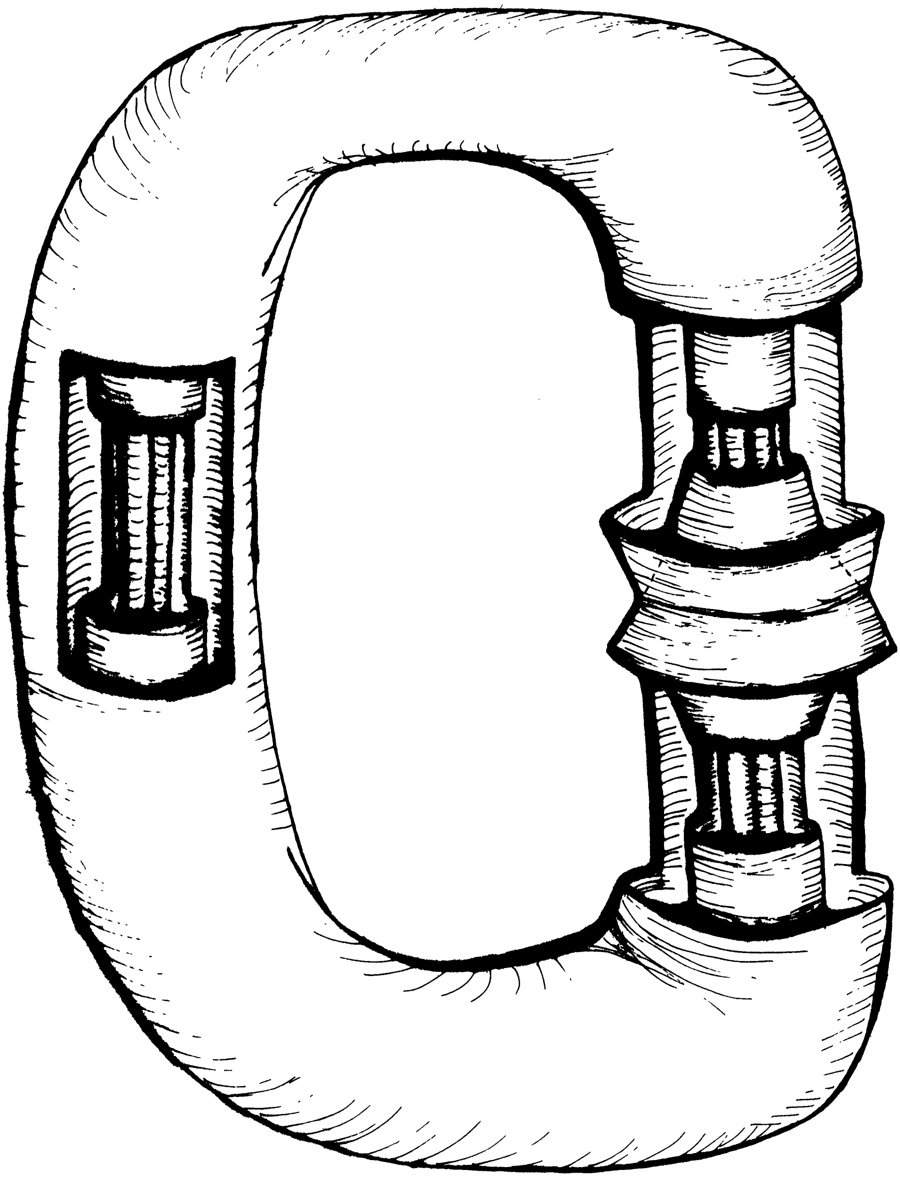

Fig. 3.1.0

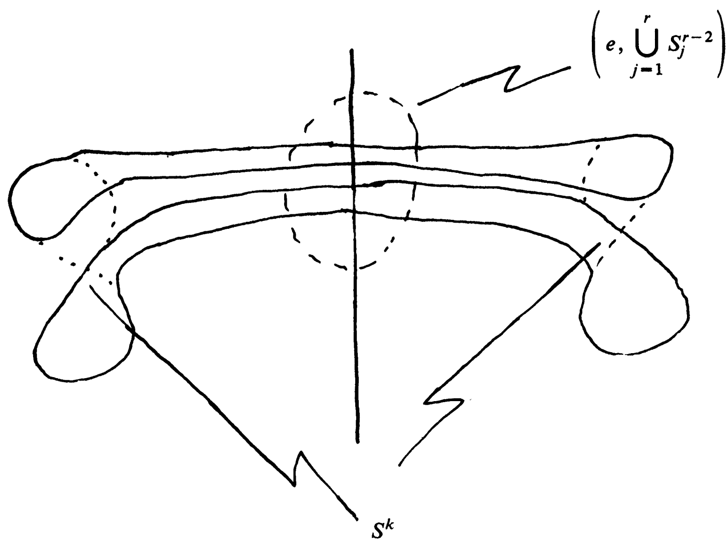

FIG. 3.1.1

$(e, \bigcup S_j^{r-2})$. Since $f(N)$ is assumed non-empty, at least the 0-dimensional multiple points of the equatorial immersion must be removed.

That the removal of the $(k - u)$-dimensional stratum is sufficient is seen as follows. A (j, q)-surgery on the equatorial immersion extends to a $(j, q + s)$-surgery on (i, M) where s is the multiplicity of the lowest dimensional self intersection set of (f, N). (Upon simplifying f, we have $s = u - 1$.) The attaching region for such a surgery—literally the attaching region for the j-handle of the bordism between (i, M) and the result of surgery—is a solid torus $S^{j-1} \times D^{n-j+1}$. The disk factor lies horizontally; each such disk is a sub-disk of the cocores of the previous surgeries. The sphere factor can be written as a union of pushed off sectors of the cores of these same surgeries. The disk factor for the first surgery is parallel to the initial core of the piggy back sequence that was used to remove S^k. The radius of the disk factor can be chosen to be larger than the radius of $f(N)$. If the (j, q)-surgery on the equatorial immersion removes a $(j - 1)$-sphere of multiple points, then the multiplicity of $f(N) \times S^{j-1}$ is reduced.

If a given $(j - 1)$-sphere of multiple points of the equatorial immersion does not bound a disk in the next lower self intersection set, then such a disk may be introduced by a piggy back sequence of surgeries. In general, no linking is introduced provided there are no m-tuple points (i.e., 0-dimensional multiple points) within the bordism of the equatorial immersion to the immersion without $(k - u - 1)$-dimensional self intersections.

If (f, N) has multiple points of dimension less than u, then there is a bordism from (f, N) to an immersion without u-dimensional multiple points. Apply a corresponding bordism to (i, M) in a neighborhood of (f, N). Hence the immersion (i, M) is surgered using handles of the bordism of (f, N) crossed with disks. In this way the linking immersion (f, N) can be assumed to be without u-dimensional multiple points. Thus only the $(k - u - 1)$-dimensional sets of the equatorial immersion need to be removed. This completes the proof of (3.1).

4. Closing remarks

Proposition (1.2.1) and Proposition (1.2.2) indicate that for "most" m there are not null bordisms for the equatorial immersion in S^{m-1} without $(m + 1)$-tuple points. The degree to which the equatorial immersion can be simplified may determine if an embedded sphere can be removed from the codimension self intersection set of an immersion. The bordism eliminating this sphere is likely to have zero dimensional multiple points. Thus the question of removing an embedded sphere is linked to the question of removing self intersections via untainted bordisms. Whether or not the engulfing condition (3) is the only condition necessary to remove this self intersection stratum is an open question.

Propositions (1.2.1) and (1.2.2) also suggest that any attempts to remove the intersections of the equatorial immersion must be ad hoc. However there is some hope that the methods herein will succeed in finding the desired null bordism.

In the sequel [7] a geometrically defined obstruction to removing the 2-dimensional multiple points of the equatorial immersion in S^4 is given. All other multiple points for this immersion can be removed in an untainted fashion. The ideas in dimension 4 are generalized to higher dimensions to address the issue: given an immersion in S^m with one m-tuple point, why (geometrically why) is there not an immersion in S^{m+1} with one $(m + 1)$-tuple point.

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