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DUALITY IN SPACES OF OPERATORS AND SMOOTH NORMS ON BANACH SPACES

BY

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Introduction

This work deals with the spaces of operators between Banach spaces and their duality, from an infinite-dimensional point of view. We use isomorphic as well as isometric tools. In particular we investigate and use the fruitful interplay between metric and weak topological properties of Banach spaces. Let us summarize the content of this paper.

In Section 1 we use the technique of [6] for obtaining a general representation (1.3) of the space $K(X,Y)^{**}$, X and Y being reflexive spaces. Our method leads to an improvement (1.5) of a classical result of A. Grothendieck, and of a result (1.6) of [3].

In Section 2 we define and use the unique extension property (U.E.P.) which turns out to be the natural tool for lifting the M.A.P. from E to E^* (2.2). A geometrical lemma (2.4) enables us to find a usable condition for obtaining the U.E.P. (2.5).

In Section 3 we show that many spaces have the U.E.P. However, the class is not stable by 1-complemented subspaces (3.1). We find a surprising characterization (3.3) of the dual norms on the James space. We notice that there is a space with a Frechet-differentiable norm but no equivalent Hahn-Banach smooth norm (3.4), and we present a renorming problem.

Section 4 presents an isomorphic version (4.3) of the results of Section 2. We obtain, in particular, an extension of a result of [35].

In Section 5 we use the smoothness of the norm of $K(X,Y)$ (X and Y reflexive) for showing that such a space is "far" from being a dual space if it is not reflexive ((5.2) and its corollaries)). This improves Theorem 2 of [6]. We show also that if X is reflexive, the space $L(X)$, equipped with the operator norm, has an unique isometric predual (5.11).

This work contains many examples which show as far as possible that our results are sharp. Let us mention that, unless otherwise specified, the Banach

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spaces we are considering are *not* assumed to have any of the approximation properties.

Notations

All the normed spaces considered in this work are real. A normed space X will be considered, without special notation, as a subspace of its bidual X^{**}. The closed unit ball of X is denoted by X_1. Two linear topologies on a vector space are said to be compatible if they have the same continuous linear forms. For two Banach spaces X and Y we denote by $L(X, Y)$ (resp. $K(X, Y)$) the space of bounded (resp. compact) linear operators from X to Y. If $X = Y$ we simply write $L(X, Y) = L(X)$, $K(X, Y) = K(X)$. The topology τ defined on $L(X, Y)$ is the topology of the uniform convergence on the norm compact subsets of X. A Banach space E is said to have the approximation property (A.P.) if the identity on E, id_E, is in the τ-closure of the space of linear operators of finite rank on E, $R(E)$. For any real number $\lambda \geq 1$, one says that E has the λ-bounded approximation property (λ-B.A.P.) if $\mathrm{id}_E \in \overline{\lambda R(E)_1}^{\tau}$. The 1-B.A.P. is called the metric approximation property (M.A.P.) One says also that E has the compact approximation property (C.A.P.) (resp. the 1-compact approximation property (1-C.A.P.)) if $\mathrm{id}_E \in \overline{K(E)}^{\tau}$ (resp. $\mathrm{id}_E \in \overline{K(E)_1}^{\tau}$). We denote by π (resp. ε) the projective (resp. injective) tensor norm on $E \otimes F$. The tensor product $E \otimes F$ endowed with π (resp. ε) and completed, will be denoted $X \otimes_\pi Y$ (resp. $X \otimes_\varepsilon Y$). For basic facts on tensor products, a useful reference is [30, Chap. IV.2]. The other notations we use are classical or will be defined before use.

1. Representations of $K(X, Y)^{**}$ and the C.A.P.

Our first result is a reformulation of the main construction of [6]. For two Banach spaces X and Y we denote by w^* the weak star topology on $L(X^{**}, Y^{**})$ induced by the duality with $X^{**} \otimes_\pi Y^*$. Then, we have the following:

PROPOSITION 1.1. *Let X and Y be two Banach spaces such that X^{**} or Y^* has the Radon-Nikodym property. Then $K(X, Y)^{**}$ is isometrically isomorphic to the w^*-closure of the space $Z = \{K^{**} \mid K \in K(X, Y)\}$ in the space $(L(X^{**}, Y^{**}), w^*)$.*

Proof. Let us recall the method of [6]. We assume first that X^{**} has the R.N.P. Let Γ be a set such that there exists an isometric injection j: $Y \to l^\infty(\Gamma)$. Let J be the corresponding injection ($J(K) = jK$) from $K(X, Y)$ into $K(X, l^\infty(\Gamma)) = X^* \otimes_\varepsilon l^\infty(\Gamma)$. Since X^{**} has the R.N.P. and $l^\infty(\Gamma)$ the

M.A.P., one has

$$(X^* \otimes_\varepsilon l^\infty(\Gamma))^* = X^{**} \otimes_\pi l^\infty(\Gamma)^*.$$

We define the following operators:
i is the canonical injection from $X^{**} \otimes_\pi Y^*$ into its bidual $L(X^{**}, Y^{**})^*$;
s (from $K(X, Y)$ into $L(X^{**}, Y^{**})$) is given by $s(T) = T^{**}$;
$Q = s^* i$.
It is easy to show that the following diagram is commutative:

$$\begin{tikzcd}
L(X^{**}, Y^{**})^* \arrow[d, "s^*"'] & X^{**} \otimes_\pi Y^* \arrow[l, "i"'] \arrow[dl, "Q"'] \\
K(X, Y)^* & X^{**} \otimes_\pi l^\infty(\Gamma)^* \arrow[l, "J^*"] \arrow[u, "\mathrm{id}_{X^{**}} \otimes j^*"']
\end{tikzcd}$$

Therefore the operator Q is a quotient map. Moreover, if u denotes the canonical injection from $K(X, Y)$ into $K(X, Y)^{**}$, one has

$$(1) \qquad Q^*(u(K)) = s(K) = K^{**} \quad \text{for } K \in K(X, Y).$$

It suffices now to use (1), the fact that Q^* is an isometric $(w^* - w^*)$-continuous injection from $K(X, Y)^{**}$ into $L(X^{**}, Y^{**})$, and the w^*-density of $u(K(X, Y))$ into $K(X, Y)^{**}$ for obtaining the conclusion.

In the case where Y^* has the Radon-Nikodym property, a similar proof can be given, which this time uses a quotient map from $l^1(\Gamma)$ onto X. □

Let us emphasize a special case:

COROLLARY 1.2. *Let Y be a reflexive Banach space, and X any Banach space. Then $K(X, Y)^{**}$ is isometric to the τ-closure of the space $Z = \{K^{**} \mid K \in K(X, Y)\}$ in $L(X^{**}, Y)$.*

Proof. By a result of Grothendieck (see [24], p. 31) the dual of $(L(X^{**}, Y), \tau)$ is a quotient of $X^{**} \otimes_\pi Y^*$. Therefore, the w^* and τ-topologies on $L(X^{**}, Y)$ are compatible, and thus the closed convex sets are the same for the two topologies. This observation, together with 1.1, shows 1.2. In particular, one has:

COROLLARY 1.3. *Let X and Y be reflexive Banach spaces. Then $K(X, Y)^{**}$ is isometric to the τ-closure of $K(X, Y)$ in $L(X, Y)$. In particular, if X or Y has the C.A.P., then $K(X, Y)^{**} = L(X, Y)$.*

Remark 1.4. If $X = Y$ is reflexive and has the C.A.P., then of course $K(X)^{**} = L(X)$. Let us note that in this case, the implication C.A.P. $\Rightarrow$ 1-C.A.P. (see [3] and 1.6 below) is immediate by the w^*-density of the unit ball of $K(X)$ into the unit ball of $K(X)^{**}$, and the trivial fact that $\|\mathrm{id}_X\| = 1$. On the other hand, if $K(X)^{**}$ is "canonically" isometric to $L(X)$, that is, if $Q^*(K(X)^{**}) = L(X)$, then necessarily X has the C.A.P. This shows that Lemma 5.1 of [20] is a necessary step in the proof of Theorem 5.3 of [20].

Let us now show that the Proposition 1.1, together with a simple application of the local reflexivity principle, give an improvement of a classical result of A. Grothendieck (see [24], Theorem 1.e.15).

THEOREM 1.5. *Let X and Y be two Banach spaces such that X^{**} or Y^* has the Radon-Nikodym property. Let C be a convex set in $K(X, Y)$. We let $C^* = \{K^* | K \in C\}$. Let $T \in L(Y^*, X^*)$ be in the closure of C^* in $(L(Y^*, X^*), \tau)$. Then for every $\varepsilon > 0$, T belongs to the τ-closure of $\{K^* \in C^* \mid \|K^*\| < \|T\| + \varepsilon\}$.*

Proof. By a result of Grothendieck (see [24], p. 31), the dual of $(L(Y^*, X^*), \tau)$ is a quotient of $Y^* \otimes_\pi X^{**}$. Therefore, if (K_α^*) is a net in C^* such that

$$K_\alpha^* \xrightarrow{\tau} T \in L(Y^*, X^*),$$

then

$$\lim_\alpha \sum_i \langle K_\alpha^* y_i^*, x_i^{**} \rangle = \sum_i \langle T y_i^*, x_i^{**} \rangle,$$

for every family (x_i^{**}, y_i^*) in $X^{**} \times Y^*$ such that $\Sigma_i \|y_i^*\| \cdot \|x_i^{**}\| < \infty$. Now, 1.1 shows that the net (K_α^{**}) converges to T^* in $(K(X, Y)^{**}, w^*)$, and thus T^* is in the closure of C in $((K(X, Y))^{**}, w^*)$.

Now, let C_1 be a convex subset of a Banach space E and let x be in the closure of C_1 in (E^{**}, w^*). Then, for any $\varepsilon > 0$, x belongs to the closure of the set

$$\{t | t \in C_1, \|t\| < \|x\| + \varepsilon\} \quad \text{in} \quad (E^{**}, w^*).$$

Let us prove this easy result for completeness:

Let U be a convex w^*-closed neighbourhood of x and $V = U \cap C_1$. Let

$$B_\varepsilon = \{t | t \in E, \|t\| < \|x\| + \varepsilon\}.$$

We have to show that $V \cap B_\varepsilon \neq \varnothing$. Let ψ be the operator from $E \times E$ into E defined by $\psi(x_1, x_2) = x_1 - x_2$. We have $x \in \overline{V}^{w^*} \cap \overline{B}_{\varepsilon/2}^{w^*}$ and thus 0 belongs to the weak-closure of $\psi(V \times B_{\varepsilon/2})$. But this implies that there exist $x_1 \in V$, $x_2 \in B_{\varepsilon/2}$, with $\|x_1 - x_2\| < \varepsilon/2$, and thus $V \cap B_\varepsilon \neq \emptyset$.

This shows that T^* belongs to the closure of $\{K \in C \mid \|K\| < \|T\| + \varepsilon\}$ in $(K(X, Y)^{**}, w^*)$ and thus there is a net (T_β) in C with $\|T_\beta\| < \|T\| + \varepsilon$ and

$$\lim_\beta \sum_i \langle y_i^*, T_\beta^{**} x_i^{**} \rangle = \sum_i \langle y_i^*, T^* x_i^{**} \rangle,$$

for every family (x_i^{**}, y_i^*) in $X^{**} \times Y^*$ such that $\Sigma_i \|y_i^*\| \cdot \|x_i^{**}\| < \infty$. But again, this means that by Grothendieck's result [24, p. 31], T is in the closure of the set $\mathrm{co}(T_\beta^*)$ in $(L(Y^*, X^*), \tau)$. □

One says that E^* has the C.A.P. with conjugate operators if there exists a net (T_α) of elements of $K(E)$ such that $\tau: T_\alpha^* \to \mathrm{id}_{E^*}$. Let us deduce from 1.5 an improvement of Prop. 1 in [3].

COROLLARY 1.6. *Let E be a Banach space such that E^* or E^{**} has the Radon-Nikodym property. If E^* has the C.A.P. with conjugate operators, then E^* and E have the* 1-*C.A.P.*

Proof. By assumption, id_{E^*} is in the τ-closure of $C^* = \{K^* \mid K \in K(E)\}$. By 1.5, id_{E^*} is in the τ-closure of $\{K^* \mid K^* \in C^*, \|K^*\| < 1 + \varepsilon\}$ for every $\varepsilon > 0$ and this clearly shows that E^* and E have the 1-C.A.P. □

Remark 1.7. Corollary 1.6 is proved—under the assumption that E^* has the R.N.P.—in [3], by an adaptation of the proof of Theorem 1.e.15 in [24]. Our proof should be considered as an adaptation of Grothendieck's original proof in [19]. Let us notice that the assumption that E^* has the C.A.P. with conjugate operators is also needed in [3]. It is an interesting problem to decide whether or not this hypothesis is actually necessary. Concerning this question, we have the following claim: There exists a separable dual X^* and an operator $T \in K(X^*)$ such that T does not belong to the τ-closure of the space of conjugate compact operators. Here is the proof. There exists a reflexive separable Banach space G and a compact operator T on G such that T is not in the norm closure of $R(G)$. Using [36, Prop. 3.1] it is possible to build a reflexive separable Banach space Z, $A \in K(G, Z)$, $B \in K(Z, G)$ with $T = BA$. Then, on the reflexive separable Banach space $E = G \oplus Z$ one may define [21] two compact operators K_1 and K_2, such that $\inf\|K_2 K_1 - S\| > 0$, the infimum being taken on all operators S on E of finite rank. By [25], there exists a Banach space Y such that Y^{**} has a basis and Y^{**}/Y is isomorphic to E^*. We have $Y^{***} = Y^* \oplus Y^\perp$ and there exists an isomorphism U from $Y^\perp$ onto E. We let $X = Y^{**}$ and we denote by P the projection from Y^{***} to $Y^\perp$ with kernel Y^* and by j the canonical injection from $Y^\perp$ to Y^{***}. We define the operators

$$S_i = U^{-1}K_iU \text{ and } T_i = jS_iP \quad (i = 1, 2).$$

Let us assume that $T_2 = \lim_\alpha K_\alpha^*$ in $(K(X^*), \tau)$ with $K_\alpha \in K(X)$. Since X has the A.P., one has $T_2 = \lim_\alpha R_\alpha^*$ in $(K(X^*), \tau)$ with $R_\alpha \in K(X)$ and $\operatorname{rank}(R_\alpha) < +\infty$. But then $\lim_\alpha \|T_2T_1 - R_\alpha^* T_1\| = 0$ and thus

$$\lim_\alpha \|S_2S_1 - PR_\alpha^* jS_1\| = 0.$$

This is clearly a contradiction which shows that T_2 is not in the τ-closure of the space of conjugate compact operators on X^*.

Remark 1.8. The "James-tree space" JT (see [23]) is a separable dual space such that JT^{**}/JT is isomorphic to $l^2(\Gamma)$, with Γ uncountable. Therefore JT^{**} has the R.N.P. but JT^* has not.

Remark 1.9. It is not clear whether or not the C.A.P. for E^* always implies the C.A.P. for E.

Remark 1.10. The recent and interesting Theorem 1.1 in [27] is another extension of Grothendieck's result. □

2. The unique extension property

There is a Banach E with the M.A.P. and such that E^* is separable and does not have the A.P. (see [25]). On the other hand, if E has the M.A.P. for every equivalent norm, then E^* has the M.A.P. (see [22]). The same conclusion holds if E has the λ-B.A.P. for a fixed $\lambda \in R$ and each equivalent norm, and if E^* has the R.N.P. (see [7]).

We will show now, along the same lines, that the M.A.P. for E^* can be obtained from the M.A.P. for E equipped with *one* given norm, as soon as this norm is "smooth enough".

The smoothness condition we are considering is very simple (2.1). The interesting point is that this condition is satisfied in many natural situations (see 2.5 and Section 3.).

LEMMA 2.1. *Let E be a Banach space. The following are equivalent*:

(1) *The only operator $T \in L(E^{**})$ such that $\|T\| \le 1$ and $T|_E = \mathrm{id}_E$ is $T = \mathrm{id}_{E^{**}}$*;

(2) *For every surjective isometry U of E, the only $T \in L(E^{**})$ such that $T|_E = U$ and $\|T\| \le 1$ is $T = U^{**}$.*

A Banach space E which satisfies these conditions is said to have the unique extension property (U.E.P. in short). One says also that the norm of E has the U.E.P.

Proof. If U is a surjective isometry of E and if $T \in L(E^{**})$ is such that $\|T\| \leq 1$ and $T|_E = U$, we let $V = (U^{**})^{-1}T$. It is clear that V satisfies $\|V\| \leq 1$ and $V|_E = \mathrm{id}_E$. It follows that (1) and (2) are equivalent. □

Let X be a Banach space. We denote by w_{op} the weak operator topology on $L(X)$; that is, $(T_\alpha) \in L(X)$ converges to T in $(L(X), w_{\mathrm{op}})$ if

$$\lim_\alpha \langle T_\alpha x, x^* \rangle = \langle Tx, x^* \rangle \quad \text{for } x \in X,\ x^* \in X^*.$$

Let us compare the U.E.P. with properties of convergence of operators, and show that the U.E.P. permits lifting the M.A.P. and the 1-C.A.P. from a Banach space E to E^*.

We consider the following properties for a Banach space E:

($*$) For every net (T_α) in the unit ball of $L(E)$, such that $\lim_\alpha T_\alpha = U$ in $(L(E), w_{\mathrm{op}})$ where U is a surjective isometry, $\lim_\alpha T_\alpha^* = U^*$ in $(L(E^*), w_{\mathrm{op}})$;

($**$) The same property holds with (T_α) in the unit ball of $K(E)$ and $U = \mathrm{id}_E$.

With this terminology, one has

THEOREM 2.2. *Let E be a Banach space.*

(i) *U.E.P. $\Rightarrow$ ($*$) $\Rightarrow$ ($**$), for E.*

(ii) *If ($**$) holds and E has the M.A.P. (resp. 1-C.A.P.) then E^* has the M.A.P. (resp. 1-C.A.P.).*

(iii) *If E^* or E^{**} has the R.N.P. and if E has the M.A.P. then U.E.P., ($*$) and ($**$) are equivalent for E.*

Proof. (i) ($*$) $\Rightarrow$ ($**$) is obvious. Let E be a Banach space which has the U.E.P. and (T_α) a net in the unit ball of $L(E)$, such that $\lim_\alpha T_\alpha = U$ in $(L(E), w_{\mathrm{op}})$, U being a surjective isometry of E. To show that U.E.P. $\Rightarrow$ ($*$) we have to show that

$$\lim_\alpha \langle x^*, T_\alpha^{**} x^{**} \rangle = \langle x^*, U^{**} x^{**} \rangle \quad \text{for } (x^*, x^{**}) \in E^* \times E^{**},$$

which means that (T_α^{**}) converges to U^{**} in $(L(E^{**}), w^*)$. By w^*-compactness, it is enough to show that U^{**} is the only w^*-cluster point of the net (T_α^{**}). But if V is such a cluster point, then clearly $\|V\| \leq 1$ and $V|_E = U$. Thus $V = U^{**}$.

(ii) We show it only for the 1-C.A.P. Let (K_α) be a net in $K(E)$ with $\|K_\alpha\| \leq 1$ and $\lim K_\alpha = \mathrm{id}_E$ in $(L(E), \tau)$. Since τ is finer than w_{op}, by $(**)$ we have $\lim_\alpha K_\alpha^* = \mathrm{id}_{E^*}$ in $(L(E^*), w_{\mathrm{op}})$. Since the weak and strong operator topologies are compatible, id_{E^*} belongs to the closure of $\mathrm{co}((K_\alpha^*))$ for the strong operator topology. The result follows.

(iii) We must show that $(**) \Rightarrow$ U.E.P. By (ii), if $(**)$ holds and E has the M.A.P., then E^* has the M.A.P. If, moreover, E^* or E^{**} has the R.N.P., we have the representations

$$K(E) = E^* \otimes_\varepsilon E, \quad K(E)^* = E^{**} \otimes_\pi E^*, \quad K(E)^{**} = L(E^{**}).$$

Let $T \in L(E^{**})$ be such that $\|T\| \leq 1$ and $T|_E = \mathrm{id}_E$. Since $K(E)^{**} = L(E^{**})$, we may write $T = \lim_\alpha K_\alpha$ in $(L(E^{**}), w^*)$, with (K_α) in the unit ball of $K(E)$. This means that

$$\lim_\alpha K_\alpha^{**} x^{**} = T x^{**} \tag{2}$$

in (E^{**}, w^*) for every $x^{**} \in E^{**}$. Since $T|_E = \mathrm{id}_E$, (2) shows that $\lim_\alpha K_\alpha = \mathrm{id}_E$ in $(L(E), w_{\mathrm{op}})$. Now, by $(**)$, one has $\lim_\alpha K_\alpha^* = \mathrm{id}_{E^*}$ in $(L(E^*), w_{\mathrm{op}})$, which means exactly that

$$\lim_\alpha K_\alpha^{**} x^{**} = x^{**} \tag{3}$$

in (E^{**}, w^*) for every $x^{**} \in E^{**}$. Now (2) and (3) show that $T = \mathrm{id}_{E^{**}}$. □

Let us point out an important special case.

COROLLARY 2.3. *Let E be a separable Banach space.*
(i) *If E has the M.A.P., then U.E.P., $(*)$ and $(**)$ are equivalent.*
(ii) *If E has the 1-C.A.P. and $(**)$ holds, then E has the 1-C.A.P. for every equivalent norm.*

Proof. (i) *By 2.2 it suffices to show that if E has the M.A.P., $(**)$ implies that E^* is separable. But if $id_E = lim_n R_n$ in $(L(E), \tau)$ with $\|R_n\| \leq 1$ and $rank(R_n) < \infty$ (note that the unit ball of $L(E)$ is τ-metrizable), by $(**)$ we have $id_{E^*} = lim_n R_n^*$ in $(L(E^*), w_{op})$, and thus the norm-closed linear span of $\bigcup_{n \geq 1} R_n^*(E^*)$ is E^*.*

(ii) Suppose that E has the C.A.P. and $(**)$ holds. Then, we can prove as above that E^* is separable. By $(**)$ it is clear that E^* has the C.A.P. with conjugate operators. Then, 1.6 shows that E has the 1-C.A.P. for every equivalent norm. □

We now need a device for showing that a Banach space enjoys the U.E.P. This device is our next lemma, 2.4, which seems to have independent interest.

Let us recall that a subspace X of E^* is *norming* if

$$\|x\| = \sup\{t(x) | t \in X, \|t\| \leq 1\} \quad \text{for } x \in E.$$

Moreover, for any Banach space E, we denote by $B(t, a)$ the closed ball of E^{**} with center $t \in E$ and radius $a \geq 0$.

LEMMA 2.4. *Let E be a Banach space. The following are equivalent.*
(i) *For all* $x \in E^{**}$, $\bigcap_{u \in E} B(u, \|x - u\|) = \{x\}$.
(ii) *For all* $x \in E^{**} \setminus E$, $\bigcap_{u \in E} B(u, \|x - u\|) \bigcap E = \varnothing$.
(iii) E^* *contains no proper norming subspace.*

Proof. (i) $\Rightarrow$ (ii). Obvious.

(ii) $\Leftrightarrow$ (iii). Let Y be a closed subspace of E^*. Then Y^* is isometric to $E^{**}/Y^{\perp}$. One checks easily that the norm of the restriction of $t \in E$ to Y satisfies $\|t | Y\| = d(t, Y^{\perp})$. But Y is norming if and only if, for each $t \in E$, $\|t|_Y\| = \|t\|$, and thus if and only if for all $t \in E$, $\|t\| = d(t, Y^{\perp})$. Therefore, there exists a proper norming subspace in E^* if and only if there exists $x \in E^{**} \setminus \{0\}$ such that for all $t \in E$, $\|x - t\| \geq \|t\|$, and this proves (ii) $\Leftrightarrow$ (iii).

(ii) $\Rightarrow$ (i). Let x be in E^{**}, and $\hat{x}$ be the u.s.c. hull of x, considered as a function on (E_1^*, w^*). $\hat{x}$ is easily seen to be concave, and an application of the Hahn-Banach theorem [11, Lemma 1] shows that for $t \in E_1^*$,

$$\hat{x}(t) = \inf_{u \in E} \{t(u) + \|x - u\|\}.$$

Therefore, by [11, Lemma 2], we have

$$\begin{aligned} y \in \bigcap_{u \in E} B(u, \|x - u\|) &\Leftrightarrow \|y - u\| \leq \|x - u\|, \quad \forall u \in E \\ &\Leftrightarrow \hat{y} \leq \hat{x} \end{aligned}$$

Let $y \in E^{**}$ be such that $\hat{y} \leq \hat{x}$. The function $(x - y)\hat{} + \hat{y}$ is u.s.c. and bigger than $(x - y) + y = x$. Thus $(x - y)\hat{} + \hat{y} \geq \hat{x}$, and this implies $(x - y)\hat{} \geq \hat{x} - \hat{y} \geq 0$. But $(x - y)\hat{} \geq 0$ means that $0 \in \bigcap_{u \in E} B(u, \|(x - y) - u\|)$. By (ii), this implies that $x - y = 0$, which is (i). $\square$

We now have:

PROPOSITION 2.5. *Let E be a Banach space such that* E^* *contains no proper norming subspace. Then E has the unique extension property.*

Proof. If $T \in L(E^{**})$ satisfies $\|T\| \leq 1$ and $T|_E = \mathrm{id}_E$, then for every $x \in E^{**}$,

$$T(x) \in \bigcap_{t \in E} B(t, \|x - t\|),$$

and thus, by Lemma 2.4, $T(x) = x$ and $T = \mathrm{id}_{E^{**}}$. □

Remark 2.6. By 2.2 and 2.5, E has the convergence property $(*)$ as soon as E^* has no proper norming subspace. Let us point out that the family (T_α) of $(*)$ is *not* assumed to consist of commuting operators. This result should be compared, for instance, to the example of [31, p. 410].

Remark 2.7. It is shown in [12] that E^* contains no proper norming subspace if and only if E satisfies the following geometric condition: *For every closed bounded convex set C, and every $x \notin C$, there exists a finite family of balls $\{B_1, B_2, \ldots, B_n\}$ such that $x \notin B_i$ for every i and $C \subset \bigcup_{i=1} B_i$.* □

3. Examples and counterexamples

Examples of spaces with the U.E.P. We will make use of 2.5 and for this we need a practical tool for showing that E^* contains no proper norming subspace. The tool is as follows: *Let $C(w^*, w)$ be the set of points of continuity of* id: $(E_1^*, w^*) \to (E_1^*, w)$. It is easy to check that any weak* strongly exposed point of E_1^* is an element of $C(w^*, w)$ and also that $C(w^*, w)$ is a subset of any norming subspace of E^*. It follows that if $C(w^*, w)$ generates a subspace which is norm dense in E^*, then E^* contains no proper norming subspace and E has the U.E.P. In particular, the following spaces have the U.E.P.:

(a) Hahn-Banach smooth spaces (see [32]), by [13, Lemme-def 5];
(b) in particular, Banach spaces which are M-ideals in their bidual (see Section 4 below);
(c) Banach spaces with a Frechet-differentiable norm on $E \setminus \{0\}$ (by Smulyan's lemma, see [4]);
(d) more generally, Banach spaces with the Mazur intersection property (see [9], or Remark 2.7);
(e) Banach spaces $K(X, Y)$, where X and Y are reflexive Banach spaces equipped with the operator norm (see Section 5);
(f) separable polyhedral Lindenstrauss spaces (see [10] and [26]).

Let us notice that it is formal to prove that a space E has the U.E.P. if and only if it is "uniquely decomposed" in the sense of [17], that is, if and only if there exists an unique projection on E^{***} of norm one and of kernel $E^\perp$. This is the link between the present work and [17].

Examples of spaces without the U.E.P. It is clear that if there exists a norm-one projection from E^{**} to E, then E does not have the U.E.P. Thus, for instance, a non-reflexive dual space does not have the U.E.P.

Several classes of spaces with the U.E.P. are stable by subspaces and quotients, for example (a) and (b), or just by subspaces, for example (c). The following result shows that the class of spaces with the U.E.P. has no good properties of stability. It can also furnish a counterexample for several properties of smoothness.

PROPOSITION 3.1. *There exists a Banach space E with the U.E.P. which contains a norm-one complemented hyperplane Y without the U.E.P.*

Proof. Let J be the quasi-reflexive Banach space defined by James (see [24, p. 25]). Let $E = J^* \times R$ be equipped with the supremum norm; then the dual norm of $E^* = J^{**} \times R$ will be $\|(x^{**}, \alpha)\| = \|x^{**}\| + |\alpha|$. We define in E, $e_1 = (0,1)$, and in E^*, $e_1^* = (0,1)$. Without special notations we consider J^* (resp. J^{**}) as a subspace of E (resp. E^*). By the Bishop-Phelps theorem, there exist $x_0^{**} \in J^{**} \setminus J$ and $x_0^* \in J^*$ such that $\|x_0^{**}\| = \|x_0^*\| = \langle x_0^*, x_0^{**}\rangle = 1$. We define $y_1^*, y_2^* \in J^*$ by

$$y_1^* = x_0^{**} + \frac{e_1^*}{2}, \quad y_2^* = x_0^{**} - \frac{e_1^*}{2}$$

and C by

$$C = \mathrm{co}\{ E_1^* \cup \{\pm y_1^*\} \cup \{\pm y_2^*\}\}.$$

C is clearly balanced w^*-closed bounded and convex and thus it is the unit ball of an equivalent dual norm on E^* which is noted $\|\ \ \|_1$, as well as its predual norm.

For every $x^* \in J^*$ and $x^{**} \in J^{**}$ one has $\|x^*\| = \|x^*\|_1$ and $\|x^{**}\| = \|x^{**}\|_1$. If we let

$$y_1 = \frac{2}{3}(x_0^* + e_1), \quad y_2 = \frac{2}{3}(x_0^* - e_1),$$

it is easy to check that y_i^* $(i = 1,2)$ is weak* strongly exposed in C by y_i $(i = 1,2)$. By w^*-density of J_1 in J_1^{**}, the set $SE_*(J_1^{**})$ of the weak* strongly exposed points of J_1^{**} is contained in J. Since J has the R.N.P. one has $J_1 = \overline{\mathrm{co}}^{\|\ \|}(SE_*(J_1^{**}))$ (see [34]).

Now let us consider an element u^{**} of J_1^{**} which is weak* strongly exposed in J_1^{**} by $u^* \in J^*$ of norm 1. We will prove that u^{**} is weak* strongly exposed in C by u^*. First, one has $\|u^*\|_1 = \|u^{**}\|_1 = \langle u^*, u^{**}\rangle = 1$.

Every sequence $z_n^* \in C$ may be written as

$$z_n^* = \lambda_0^n x_n^{**} \pm \lambda_1^n e_1^* + (\lambda_2^n - \lambda_3^n) y_1^* + (\lambda_4^n - \lambda_5^n) y_2^*$$

with $x_n^{**} \in J_1^{**}$, $\lambda_i^n \geq 0$ and $\Sigma_{i=0}^5 \lambda_i^n = 1$. One has $\langle u^*, e_1^* \rangle = 0$ and

$$\langle u^*, y_i^* \rangle = \langle u^*, x_0^{**} \rangle \neq \langle u^*, \pm u^{**} \rangle = \pm 1 \quad (i = 1,2).$$

This implies that if $\langle u^*, z_n^* \rangle$ converges to 1, we must have $\lambda_0^n \to 1$ and thus

$$\|z^* - x_n^{**}\|_1 \to 0 \quad \text{and} \quad \langle u^*, x_n^{**} \rangle \to 1.$$

Since u^{**} is weak* strongly exposed in J_1^{**} by u^*, this implies that

$$\|x_n^{**} - u^{**}\| \to 0$$

and thus

$$\|z_n^* - u^{**}\|_1 \to 0.$$

This means that u^{**} is weak* strongly exposed in C by u^*. A similar—and easier—proof shows that e_1^* is weak* strongly exposed in C by e_1. It follows that the norm-closed linear span of the weak* strongly exposed points of C, which contains J, $x_0^{**} = \frac{1}{2}(y_1^* + y_2^*)$ and e_1^* is equal to E^*. Therefore $(E, \| \quad \|_1)$ has the U.E.P.

Note that if P_0 is the projection on E^* of image J^{**} and such that $P_0(e_1^*) = 0$, one has $P_0(C) \subset C$. Thus $\|P_0\|_1 = 1$. If P is the projection on E of image J^* and $P(e_1) = 0$, one has $P^* = P_0$ and thus $\|P\|_1 = 1$. Finally, the subspace $Y = J^*$ of $(E, \| \quad \|_1)$ is a non-reflexive dual space and thus it does not have the U.E.P. □

Remark 3.2. There is one situation where a nonreflexive Banach space E is a dual space if and only if it does not have the U.E.P.; namely, if $\dim E^{**}/E = 1$. Indeed, if such an E does not have the U.E.P., then by 2.5, E^* contains a proper norming subspace X. Since $\dim E^{**}/E = 1$, one has $E^{**} = E \oplus X^\perp$. The projection π: $E^{**} \to E$ of kernel $X^\perp$ has norm one since X is norming. Thus $X^* = E^{**}/X^\perp = E$.

Applying this remark (and 2.3) to the space J of James (see [24, p. 25]) leads to the following surprising result:

PROPOSITION 3.3. *Let $\| \quad \|$ be an equivalent norm on the space J of James. Then $(J, \| \quad \|)$ is isometric to a dual space if and only if there exists a sequence $(T_n)_{n \geq 1}$ in $L(J)$ with $\|T_n\| \leq 1$ and*

(a) *T_n converges to id_J in $(L(J), w_{\mathrm{op}})$,*

(b) *T_n^* does not converge to id_{J^*} in $(L(J^*), w_{\mathrm{op}})$.*

Let us mention that the above method fails in general, even if dim $E^{**}/E = 2$. For instance if $E = J \times J$ is equipped with the norm $\sup(\| \ \|_1, \| \ \|_2)$ where $\| \ \|_1$ is a dual norm and $\| \ \|_2$ is not, it is easily seen that E is not a dual space, even though it does not have the U.E.P.

A space with the U.E.P. which is "far" from being Hahn-Banach smooth. We have already noticed that the Hahn-Banach smooth spaces have the U.E.P. In other words, if every $f \in E^* \setminus \{0\}$ has an unique extension of the same norm to $\tilde{f} \in E^{***}$, then every surjective isometry of E has an unique extension to a contraction of E^{**}. The next result shows that, despite the formal analogy between the two properties, the U.E.P. is much more general than Hahn-Banach smoothness.

PROPOSITION 3.4. *Let ω_1 be the first uncountable ordinal. Define $E = \mathscr{C}(\langle 0, \omega_1 \rangle)$. We have the following result*:

(i) *There exists an equivalent norm on E which is Frechet differentiable on $E \setminus \{0\}$* [33]. *In particular this norm has the U.E.P.*

(ii) *If K is a compact space which contains a subset which is not Borel, there is no equivalent norm on $\mathscr{C}(K)$ which is Hahn-Banach smooth. In particular, there is no Hahn-Banach smooth equivalent norm on E.*

Proof. Only (ii) has to be proved. Suppose that we have found an equivalent norm on $\mathscr{C}(K)$ which is Hahn-Banach smooth. It follows that for this norm the w^* and the w topologies coincide on the unit sphere of $\mathscr{C}(K)^*$. By [5], this implies that the weak* and weak Borel σ-fields coincide on $\mathscr{C}(K)^*$, and in particular on $\{\varepsilon_x | x \in K\}$. Since this set is discrete, every subset of it is weak Borel and thus weak* Borel. This means that every subset of K is Borel.

Finally, it is well known that a subset A of $\langle 0, \omega_1 \rangle$ which is cofinal, as well as $\langle 0, \omega_1 \rangle \setminus A$, is not Borel in $\langle 0, \omega_1 \rangle$. □

A renorming problem. Let E be a non-reflexive Banach space. We define the Godun index $\gamma(E)$ of E by

$$\gamma(E) = \sup\left[\inf\{\|P\| \,|\, P \text{ is a continuous projection from } E^{**} \text{ to } E\}\right]$$

where the supremum is taken over the set of equivalent norms on E. It is proved in [18] that $\gamma(E) \geq 2$ for any non-reflexive Banach space E. This is clearly the best possible result in full generality (take E so that dim $E^{**}/E = 1$). On the other hand, it follows from [8, Lemma 2.9] that if dim $E^{**}/E < \infty$, then $\gamma(E) \leq 3$. The following question is therefore natural.

Question 3.5. Let E be a Banach space such that E^{**}/E is infinite dimensional. Is it true that $\gamma(E) = \infty$?

Let us notice that at least we have the following result.

Proposition 3.6. *There exists a separable dual Banach space E such that $\gamma(E) = \infty$.*

Proof. Let F be a separable reflexive space without the A.P. By [25] there exists a Banach space Y such that Y^{**} has a basis and Y^{**}/Y is isomorphic to F. One has $Y^{***} = Y^* \oplus Y^\perp$ and thus Y^{***} does not have the A.P., since $Y^\perp$ is isomorphic to F^*. By [7] this implies that for any $\lambda \geq 1$, there exists an equivalent norm N_λ on Y^{**} such that (Y^{**}, N_λ) does not have the λ-B.A.P. Since Y^{**} has the R.N.P. and the B.A.P., by Theorem 1.1 of [27] this implies that $\gamma((Y^{**}, N_\lambda)) \geq \lambda$. Therefore $E = Y^{**}$ works. □

Remark 3.7. If we denote by N_n the norm which corresponds to $\lambda = n$, and if we define the l_2-sum $Z = (\Sigma_n \oplus (E, N_n))_2$, then Z is not complemented in Z^{**}, although it is the l^2-sum of spaces which are all isomorphic to dual spaces. The space Z, of course, is not isomorphic to a dual space.

Let us conclude this section by a question which is—at least formally—weaker than Question 3.5.

Question 3.8. Let E be a Banach space such that E^{**}/E is infinite dimensional, and let $\lambda \geq 1$. Does there exist a norm N_λ on E such that the Banach-Mazur distance from (E, N_λ) to any isometric dual space is $\geq \lambda$? □

4. An isomorphic version of the results of Section 2

In this section we will establish some extensions of the convergence results of Section 2. It turns out, unfortunately, that we need to replace the metric condition $\|T_\alpha\| \leq 1$ by a (very strong) algebraic condition: the commutativity of the T_α's (see 4.7).

Let us recall that the characteristic $r(X)$ of a subspace X of E^* is defined by

$$r(X) = \inf_{x \in E \setminus \{0\}} \sup_{x^* \in X \setminus \{0\}} \left[\frac{|x^*(x)|}{\|x^*\| \cdot \|x\|} \right].$$

For the definition and some of the properties of the Banach spaces which are M-ideals in their bidual, see [20]. Let us prove:

Lemma 4.1. *Let E be a Banach space which is an M-ideal in its bidual. Then for any proper subspace X of E^* one has $r(X) \leq \frac{1}{2}$.*

Proof. It is enough to show that for every $f \in E^{**}$ of norm 1, we have $r(\operatorname{Ker} f) \leq \frac{1}{2}$. By the Hahn-Banach theorem, it means that for every $\varepsilon > 0$ we have to find $x^* \in E^*$ with $\|x^*\| \leq \frac{1}{2} + \varepsilon$ and $x^* \notin \overline{(\operatorname{Ker} f \cap E^*)}^{w^*}$.

Actually, any $x^* \in E^*$ such that $\|x^*\| = \frac{1}{2} + \varepsilon$ and $f(x^*) > 1/2 - \varepsilon/2$ satisfies this condition. Indeed, if not, there is a net (x_α^*) in $E_1^* \cap \operatorname{Ker} f$ such that $\lim x_\alpha^* = x^*$ in (E_1^*, w^*). If x_0^{***} is a cluster point of (x_α^*) in (E_1^{***}, w^*), it is clear that $x_0^{***} = x^* + t$, with $t \in E^\perp$, and that $x_0^{***}(f) = 0$. Moreover,

$$1 \geq \|x_0^{***}\| = \|x^*\| + \|t\| = \left(\tfrac{1}{2} + \varepsilon\right) + \|t\|$$

and thus $\|t\| \leq \frac{1}{2} - \varepsilon$. Since $t = x_0^{***} - x^*$, we have

$$|t(f)| = |f(x^*)| > \tfrac{1}{2} - \varepsilon/2 \geq \|t\|$$

and this is impossible since $\|f\| = 1$. □

Remark 4.2. Let Y be a Banach space with an unconditional shrinking basis, such that $\|P_n\| = \|\mathrm{id}_Y - P_n\| = 1$, if the (P_n)'s are the projections associated with the basis. It is shown in [35] that any subspace or quotient space of Y satisfies the conclusion of 4.1. Therefore, the following results will apply also under this assumption. We shall not repeat it in the statements.

PROPOSITION 4.3. *Let E be a separable Banach space which is an M-ideal in its bidual. Let $(T_n)_{n\geq 1}$ be a sequence of finite rank operators on E such that*:

(i) $\sup\|T_n\| < 2$;

(ii) $T_nT_k = T_kT_n$, *for each n and k*;

(iii) $\lim\|T_nx - x\| = 0$, *for each $x \in E$.*

Then we have $\lim\|T_n^*x^* - x^*\| = 0$ *for each* $x^* \in E^*$.

Proof. By [31, p. 775], we have $\lim\|T_n^*x^* - x^*\| = 0$ for every x^* in the norm-closed subspace Γ of E^* generated by $\bigcup_{n=1}^{\infty} T_n^*(E^*)$. But it is easily checked that $r(\Gamma) \geq (\sup\|T_n\|)^{-1}$ and thus $r(\Gamma) > \frac{1}{2}$. It follows that $\Gamma = E^*$. □

A first consequence is an extension of Corollary 1 in [35].

COROLLARY 4.4. *Let E be a Banach space which is an M-ideal in its bidual and let (e_n) be a basic sequence. If the basis constant of (e_n) is strictly less than 2, then (e_n) is shrinking.*

Proof. The space X generated by (e_n) is an M-ideal in its bidual since the class is hereditary. It suffices now to apply 4.3 to the sequence (P_n) of projections associated with (e_n). □

What can be done for bettering an approximating sequence in an M-ideal? Here is an answer.

COROLLARY 4.5. *Let E be a separable Banach space which is an M-ideal in its bidual. The following are equivalent.*

(i) *There exists a sequence (T_n) of finite rank operators on E such that:*
   (a) $T_nT_k = T_kT_n$, *for each n and k;*
   (b) $\sup\|T_n\| < 2$;
   (c) $\lim\langle T_nx, y^*\rangle = \langle x, y^*\rangle$ *for each $x \in E$ and $y^* \in E^*$.*

(ii) *For every sequence of scalars (ε_n) with $0 < \varepsilon_n < 1$ and $\lim \varepsilon_n = 0$, there exists a sequence (R_n) of finite rank operators such that:*
   (a) $R_nR_k = R_{\inf(n,k)}$ *for each n and k with $n \neq k$;*
   (b) $\|R_n\| < 1 + \varepsilon_n$ *for each n;*
   (c) $\lim\|R_nx - x\| = \lim\|R_n^*y^* - y^*\| = 0$ *for each $x \in E$ and $y^* \in E^*$.*

Proof. (ii) $\Rightarrow$ (i) is obvious. (i) means that $\lim T_n = \mathrm{id}_E$ in the weak operator topology. Since the weak and strong operator topologies have the same dual, there exists a sequence (S_n) of convex combinations of the T_n's such that $\lim\|S_nx - x\| = 0$ for every $x \in E$. The family (S_n) is still commuting. Now 4.3 shows that E^* has the 2-B.A.P. and even the M.A.P., since E^* has the R.N.P. But this implies (see [31, p. 316–318]) that a sequence (R_n) satisfying (ii) can be constructed. □

Example 4.6. The example of the summing basis of c_0 shows that 4.4 is sharp, and that the assumption (i) is necessary in 4.3.

Example 4.7. Let us denote by $(e_i^*)_{i\geq 1}$ the unit basis of l^1, and let $\varepsilon > 0$ be given. We define a sequence $(P_n)_{n\geq 1}$ of projections on c_0 by

$$P_n(x) = \big(\phi_n^j(x)\big)_{j\geq 1} \quad \text{for} \quad x \in c_0,$$

where

$$\phi_n^j = \begin{cases} 0 & \text{if } j > n, \\ e_j^* + \varepsilon e_{n+1}^* & \text{if } j \leq n. \end{cases}$$

It is easily seen that (a) $\|P_n\| = 1 + \varepsilon$ for every n and (b) the sequence (P_n^*) of $L(l^1)$ does not converge in the weak operator topology. More precisely, if $B \in L(l^\infty)$ belongs to the w^*-closed convex hull of the w^*-cluster points in $L(l^\infty)$ of the sequence (P_n^{**}), then $\|B - \mathrm{id}\| = \varepsilon$.

This example shows that the assumption (ii) in 4.3 is actually necessary.

Example 4.8. If Λ is a Shapiro subset of an abelian discrete group Γ (see [15, Prop. 26] or [16]) and if $G = \hat{\Gamma}$, then the space $C(G)/C_{\Gamma\setminus\Lambda}(G)$ is an M-ideal in its bidual. A typical example is the space $C(T)/A_0(D)$ which corresponds to $\Gamma = Z$ and $\Lambda = Z^-$.

The convolution with the Fejer kernel provides us with a sequence of operators on $C(G)/C_{\Gamma\setminus\Lambda}(G)$—since $C_{\Gamma\setminus\Lambda}(G)$ is an ideal—which satisfies the assumptions of 4.3, with $\|T_n\| = 1$. The same sequence shows, of course, that the dual $L^1_{-\Lambda}(G)$ has the M.A.P. Let us notice that the existence of a basis is not clear for the spaces which belong to this class, or for their duals.

Let us conclude this section with another question.

Question 4.9. Does there exist a subspace E of c_0 with the A.P., such that E^* does not have the A.P.?

5. Projective tensor product of strongly exposed points and its applications

Several lemmas, similar to 5.1 below, have been proved in recent years by various authors (see [28], [29]). We prove it for completeness; the proof closely follows the method of [6, p. 46].

LEMMA 5.1. *Let X and Y be Banach spaces, $x \in X$ (resp. $y \in Y$) with $\|x\| = \|y\| = 1$. Assume that x (resp. y) is strongly exposed in the unit ball of X by $x^* \in X^*$ (resp. $y^* \in Y^*$). Then $x \otimes y$ is strongly exposed in the unit ball of $X \otimes_\pi Y$ by $x^* \otimes y^*$.*

Proof. We may assume that $\|x^*\| = \|y^*\| = 1$. Let ε be in $]0,1[$ and $\eta \in]0, \varepsilon/2[$ such that

$$t \in X_1,\ \langle t, x^*\rangle \geq 1 - \eta \Rightarrow \|x - t\| \leq \varepsilon/2,$$
$$s \in Y_1,\ \langle s, y^*\rangle \geq 1 - \eta \Rightarrow \|y - s\| \leq \varepsilon/2.$$

Now consider $z \in X \otimes_\pi Y$, $\|z\|_\pi \leq 1$, such that $\langle z, x^* \otimes y^*\rangle \geq 1 - \eta^2$. We have to maximize $\|z - x \otimes y\|_\pi$. We may clearly assume that $z = \sum_{j=1}^n \lambda_j t_j \otimes s_j$, with $t_j \in X_1$, $s_j \in Y_1$, $\lambda_j \geq 0$ and $\sum_{j=1}^n \lambda_j \leq 1$. Consider the set

$$I = \{\, j \mid 1 \leq j \leq n,\ \langle t_j, x^*\rangle\langle s_j, y^*\rangle < 1 - \eta \,\}$$

and denote $I' = \{1, 2, \ldots, n\} \setminus I$. We have

$$\sum_{j\in I'} \lambda_j + \sum_{j\in I} \lambda_j \langle t_j, x^*\rangle\langle s_j, y^*\rangle \geq \langle z, x^* \otimes y^*\rangle \geq 1 - \eta^2.$$

Thus,

$$\sum_{j\in I'} \lambda_j + (1 - \eta) \sum_{j\in I} \lambda_j \geq 1 - \eta^2.$$

This implies

$$\Big(1 - \sum_{j\in I} \lambda_j\Big) + (1-\eta)\sum_{j\in I}\lambda_j \geq 1 - \eta^2.$$

Finally,

$$\sum_{j\in I}\lambda_j \leq \eta < \varepsilon/2;$$

therefore

$$\Big\| z - \sum_{j\in I'} \lambda_j t_j \otimes s_j \Big\|_\pi < \varepsilon/2. \tag{4}$$

It is easy to check that one has also $\sum_{j\in I'}\lambda_j > 1 - \varepsilon/2$. If $j \in I'$, one has

$$\langle t_j, x^*\rangle\langle s_j, y^*\rangle \geq 1 - \eta$$

and since each term of the product is less than, or equal to, one,

$$\langle t_j, x^*\rangle \geq 1 - \eta \quad \text{and} \quad \langle s_j, y^*\rangle \geq 1 - \eta$$

which implies

$$\|x - t_j\| \leq \varepsilon/2 \quad \text{and} \quad \|y - s_j\| \leq \varepsilon/2.$$

Therefore

$$\|x \otimes y - t_j \otimes s_j\| \leq \varepsilon \quad \text{for} \quad j \in I'.$$

Now, by convexity, one has

$$\Big\| x \otimes y - \Big(\sum_{j\in I'}\lambda_j\Big)^{-1} \sum_{j\in I'} \lambda_j t_j \otimes s_j \Big\|_\pi \leq \varepsilon. \tag{5}$$

Finally,

$$\begin{aligned} &\Big\| \Big(\sum_{j\in I'}\lambda_j\Big)^{-1} \sum_{j\in I'} \lambda_j t_j \otimes s_j - \sum_{j\in I'} \lambda_j t_j \otimes s_j \Big\|_\pi \\ &\qquad \leq \Big(\sum_{j\in I'}\lambda_j\Big)^{-1} - 1 \leq \frac{1}{1-\varepsilon/2} - 1 < \varepsilon. \end{aligned} \tag{6}$$

The inequalities (4), (5) and (6) show that $\|z - x\otimes y\|_\pi < 5\varepsilon/2$. □

From this lemma we will deduce the next result.

THEOREM 5.2. *Let X be a reflexive Banach space and Y a Banach space such that Y^* has the R.N.P. and contains no proper norming subspace. Then*:

(i) *The span of the weak*, strongly exposed points of the unit ball of $(X \otimes_\varepsilon Y)^*$ is norm dense in $(X \otimes_\varepsilon Y)^*$;*

(ii) *The span of the weak*, strongly exposed points of the unit ball of $K(X, Y)^*$ is norm dense in $K(X, Y)^*$.*

Proof. For any Banach space Z, we denote by $SE_*(Z_1^*)$ the set of the weak* strongly exposed points in the dual unit ball Z_1^*. Since X is reflexive, the span of $SE_*(X_1^*)$ is dense in X^*. Since Y^* has the R.N.P., one has $Y_1^* = \overline{\mathrm{co}}^{w^*}(SE_*(Y_1^*))$ and thus the space $\overline{\mathrm{span}}^{\| \ \|}(SE_*(Y_1^*))$ is a norming subspace of Y^*. It has to be Y^* itself by our assumption. Thus the space $\mathrm{span}(SE_*(Y^*))$ is norm dense in Y^*. We consider the subset $\Omega = SE_*(X_1^*) \otimes SE_*(Y_1^*)$ of the unit ball of $X^* \otimes_\pi Y^*$. It is clear that the space $\mathrm{span}(\Omega)$ is norm dense in $X^* \otimes_\pi Y^*$.

For (i), we denote by $I_1(X, Y^*)$ the space of integral operators from X to Y^* (see [19] or [30, Chap. IV, §5]). Let Q be the canonical quotient map from $X^* \otimes_\pi Y^*$ onto $(X \otimes_\varepsilon Y)^* = I_1(X, Y^*)$. Let $x^* \otimes y^* \in \Omega$ with x^* (resp. y^*) weak* strongly exposed in X_1^* (resp. Y_1^*) by $x \in X$ (resp. $y \in Y$). We claim that $Q(x^* \otimes y^*)$ is weak* strongly exposed by $x \otimes y$ in the unit ball of $(X \otimes_\varepsilon Y)^*$. Indeed, let $\| \ \|_i$ denote the norm of this space, and (T_n) be a sequence in the unit ball of $(X \otimes_\varepsilon Y)^*$ such that

$$\langle T_n - Q(x^* \otimes y^*), x \otimes y \rangle \to 0.$$

There exists $t_n \in X^* \otimes_\pi Y^*$ with $Q(t_n) = T_n$ and $\|t_n\|_\pi = \|T_n\|_i$ (see [6, Remark 1, p. 43]). Therefore, $\langle Q(t_n - x^* \otimes y^*), x \otimes y \rangle \to 0$, and it follows, by 5.1, that

$$\|t_n - x^* \otimes y^*\|_\pi \to 0.$$

This implies that $Q(\Omega)$ consists of weak* strongly exposed points of the unit ball of $(X \otimes_\varepsilon Y)^*$. Since Q is onto and $\mathrm{span}(\Omega)$ dense in $X^* \otimes_\pi Y^*$, the result follows.

The proof of (ii) is similar; X has to be replaced by X^*, and the quotient map we have to consider is Q: $X \otimes_\pi Y^* \to K(X, Y)^*$ as in [6] and in the proof of 1.1. The rest of the proof follows the same lines. □

Examples 5.3. The above result can be applied to the spaces Y which belong to the classes (a) to (f) described in 3.1, which all satisfy the assumptions of 5.2. Concerning the Mazur intersection property (d), it is not known whether it implies that Y^* has the R.N.P. if Y is not separable. However, the characterization of [9] shows that the proof of 5.2 can be completed under this assumption.

A first corollary is:

COROLLARY 5.4. *Under the hypothesis of* 5.2, *the space* $X \otimes_\varepsilon Y$ *and* $K(X, Y)$ *have the U.E.P.*

This is indeed clear by 2.5, 3 and 5.2.

Another consequence of the proof of 5.2 is the following:

THEOREM 5.5. *Let X and Y be two reflexive Banach spaces. Let $T_1, T_2 \in L(X, Y)$ be in the τ-closure of $K(X, Y)$. Then for every $\varepsilon > 0$, there exists $K \in K(X, Y)$ such that $\|T_1 - K\| > \|T_1 - T_2\| + \|T_2 - K\| - \varepsilon$.*

Proof. If Y is reflexive, the proof of 5.2 actually shows that the unit ball of $K(X, Y)^*$ is the norm-closed convex hull of the set Ω' of its weak* strongly exposed points. If T_1 and T_2 are in the τ-closure of $K(X, Y)$ into $L(X, Y)$, then by 1.3, T_1 and T_2 belong to $K(X, Y)^{**}$. Let us assume that there exists $\varepsilon_0 > 0$ such that

$$\|T_1 - K\| \leq \|T_1 - T_2\| + \|T_2 - K\| - \varepsilon_0 \quad \text{for} \quad K \in K(X, Y).$$

If we consider T_1 and T_2 as functions on the unit ball of $K(X, Y)^*$ equipped with the w^* topology and if we use the notations of the proof of 2.4, by [11, Lemma 1] this implies that

$$\hat{T}_1 \leq \hat{T}_2 + \|T_1 - T_2\| - \varepsilon_0. \tag{7}$$

But if u is weak* strongly exposed in the unit ball of $K(X, Y)^*$, then u is a point of w^*-continuity of T_1 and T_2. Thus $\hat{T}_1(u) = T_1(u)$ and $\hat{T}_2(u) = T_2(u)$. Therefore (7) implies

$$T_1(u) \leq T_2(u) + \|T_1 - T_2\| - \varepsilon_0$$

and thus

$$\sup_{\Omega'} (T_1 - T_2) \leq \|T_1 - T_2\| - \varepsilon_0.$$

But since $\overline{\text{co}}^{\|\ \|}(\Omega')$ is the unit ball of $K(X, Y)^*$, $\sup_{\Omega'}(T_1 - T_2) = \|T_1 - T_2\|$ and this is a contradiction. □

It is easier to understand the meaning of Theorem 5.5, through its corollaries.

COROLLARY 5.6. *Let X and Y be reflexive Banach spaces. The following are equivalent.*

(i) *$K(X, Y)$ is reflexive.*

(ii) *Every family $(B_\alpha)_{\alpha \in I}$ of closed balls in $K(X, Y)$ such that $\bigcap_{\alpha \in F} B_\alpha \neq \emptyset$ for every finite subset F of I, satisfies $\bigcap_{\alpha \in I} B_\alpha \neq \emptyset$.*

Proof. (i) $\Rightarrow$ (ii). Clear by w-compactness of the balls.

(ii) $\Rightarrow$ (i). If $K(X, Y)$ is not reflexive, there exists an operator

$$T \in \overline{K(X, Y)}^{\tau} \setminus K(X, Y).$$

We consider the family $\mathscr{B}$ of balls, in $K(X, Y)$,

$$\mathscr{B} = B_1(K, a) = \{K' \in K(X, Y) | \, \|K - K'\| \leq a\},$$

where $K \in K(X, Y)$ and $a > \|K - T\|$. By 1.3 and the local reflexivity principle, the intersection of any finite subfamily of $\mathscr{B}$ is non-empty.

If (ii) holds, this implies that $\cap \mathscr{B} \neq \emptyset$. If $K_0 \in \cap \mathscr{B}$, one has

$$\|K - K_0\| \leq \|K - T\| \quad \text{for} \quad \forall K \in K(X, Y).$$

But, by 5.5, there exists $K_1 \in K(X, Y)$ such that

$$|K_0 - K_1| > |T - K_1| + |T - K_0|/2$$

and this is a contradiction. □

In particular, we have the following improvement of Theorem 2 in [6].

COROLLARY 5.7. *Let X and Y be reflexive Banach spaces. Then, if $K(X, Y)$ is not reflexive, it is not norm-one complemented in its bidual. A fortiori, it is not norm-one complemented in $L(X, Y)$.*

Proof. Indeed, let $(B_\alpha)_{\alpha \in I}$ be a family of closed balls of $K(X, Y)$ with the finite intersection property and $\tilde{B}_\alpha$ the closure of B_α in $(K(X, Y)^{**}, w^*)$. By w^* compactness, $\bigcap_{\alpha \in I} \tilde{B}_\alpha \neq \emptyset$. If P is a norm one projection defined on $K(X, Y)^{**}$ with image $K(X, Y)$, it is clear that $P(\bigcap_{\alpha \in I} \tilde{B}_\alpha) = \bigcap_{\alpha \in I} B_\alpha$. By 5.6, this implies that $K(X, Y)$ is reflexive. The last assertion is clear, since $K(X, Y)^{**}$ is isometric to a subspace of $L(X, Y)$. □

Remark 5.8. In particular, if X is an infinite, dimensional, reflexive Banach space, such that $K(X)$ is not reflexive—e.g., X with the C.A.P.—then $K(X)$ is not 1-complemented in $L(X)$. This leads to:

Question 5.9. Does there exist an infinite dimensional Banach space X such that $K(X)$ is reflexive?

Of course, such an X would have to be reflexive. Let us notice that if there exists a reflexive Banach space X such that $L(X) = K(X) \oplus \text{span}(\text{id}_X)$ (another open question), and without the C.A.P., then the corresponding $K(X)$ would be reflexive.

We conclude this article with another application of 5.1.

Let us recall that E is said to be *unique predual* if there is an unique projection $P\colon E^{***} \to E^*$ with $\|P\| = 1$ and $\operatorname{Ker} P$ w^*-closed. This implies that every Banach X, where X^* is isometric to E^*, is isometric to E. One now has:

PROPOSITION 5.10. *Let X and Y be two Banach spaces with the* R.N.P. *Then $X \otimes_\pi Y$ is the unique predual of $L(X, Y^*)$.*

Proof. Since X and Y have the R.N.P., X_1 and Y_1 are the norm-closed convex hulls of their strongly exposed points. By 5.1 this property is shared by $X \otimes_\pi Y$, and one concludes the proof by [14, Lemme 9]. □

Remark 5.11. There exist (see [2]) Banach spaces X_0 and Y_0 with the R.N.P. such that $X_0 \otimes_\pi Y_0$ contains c_0. We have the easy claim: On every separable space Z containing c_0, there exists an equivalent norm, $|\ \ |$, such that $(Z, |\ \ |)$ is not a unique predual. Here is a proof. By Sobczyk's theorem, we may write $Z \approx V \oplus c_0$. We let $|\ \ | = \sup(\|\ \ \|, \|\ \ \|_\infty)$, where $\|\ \ \|$ is some norm on V. One now has $(Z, |\ \ |)^* = V^* \oplus_1 l^1$. If α is a countable ordinal and $Z_\alpha = V \oplus_\infty \mathscr{C}(\langle 1, \alpha \rangle)$, then Z_α^* is isometric to Z^*. But there exists no Banach space with a separable dual which contains all the $\mathscr{C}(\langle 1, \alpha \rangle)$ for $\alpha < \omega_1$, (see [1]), and thus there is an α_0 such that Z_{α_0} is not even isomorphic to Z.

This claim shows in particular that there will be a norm on $X_0 \otimes_\pi Y_0$ such that this space is not unique predual, and therefore 5.10 is sharp.

Remark 5.12. Corollary 5.11 furnishes a large sample of Banach algebras with an unique isometric predual. However, it can be proved that the algebra $L(l^\infty)$, equipped with its canonical norm, has two isometric preduals which are not isomorphic. The answer to our last question is probably positive.

Question 5.13. Does there exist an algebra of operators—that is, a w^*-closed subalgebra of $L(l^2)$—with more than one isometric predual? □

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REFERENCES

1. J. BOURGAIN, *On separable Banach spaces, universal for all separable reflexive spaces*, Proc. Amer. Math. Soc., vol. 79 (1980), pp. 241–246.
2. J. BOURGAIN and G. PISIER, *Construction of new $\mathscr{L}^{\infty}_{\lambda}$-spaces*, Bol. Soc. Brasil. Mat., vol. 14 (1983), pp. 109–123.
3. CHONG-MAN CHO and W.B. JOHNSON, *A characterization of subspaces X of l^p for which $K(X)$ is an M-ideal in $L(X)$*, Proc. Amer. Math. Soc., vol. 93 (1985), pp. 466–470.
4. J. DIESTEL, *Geometry of Banach spaces, selected topics*, Lecture Notes in Math., No. 485, Springer Verlag, New York, 1975.
5. G.A. EDGAR, *Measurability in a Banach space*, Indiana Univ. Math. J., vol. 26 (1977), pp. 663–677.
6. M. FEDER and P.D. SAPHAR, *Spaces of compact operators and their dual spaces*, Israel J. Math., vol. 21 (1975), pp. 38–49.
7. T. FIGIEL and W.B. JOHNSON, *The approximation property does not imply the bounded approximation property*, Proc. Amer. Math. Soc., vol. 41 (1973), pp. 197–200.
8. T. FIGIEL, W.J. JOHNSON and L. TZAFRIRI, *On Banach lattices and spaces having local unconditional structure*, J. Approx. Theory, vol. 13 (1975), pp. 395–412.
9. J.R. GILES, D.A. GREGORY and B. SIMS, *Characterization of normed linear spaces with Mazur's intersection property*, Bull. Austral. Math. Soc., vol. 18 (1978), pp. 105–123.
10. A. GLEIT and R. MACGUIGAN, *A note on polyhedral Banach spaces*, Proc. Amer. Math. Soc., vol. 33 (1972), pp. 398–404.
11. G. GODEFROY, *Nicely smooth Banach spaces*, Longhorn Notes, The University of Texas at Austin, Functional Analysis Seminar, 1984-1985, pp. 117–124.
12. G. Godefroy and N. KALTON, *The ball topology and its applications*, to appear.
13. G. GODEFROY, *Points de Namioka, espaces normant, applications à la theorie isométrique de la dualité*, Israel J. Math., vol. 38 (1981), pp. 209–220.
14. ______, *Espaces de Banach existence et unicite de certains preduaux*, Annales Inst. Fourier, vol. 28 (1978), pp. 87–105.
15. ______, *Sous-espaces bien disposés de L^1, applications*, Trans. Amer. Math. Soc., vol. 286 (1984), pp. 227–249.
16. ______, *On Riesz subsets of abelian discrete groups*, to appear.
17. G. GODEFROY and P.D. SAPHAR, *Normes lisses et propriété d'approximation métrique*, C.R. Acad. Sci. Paris, vol. 299 Serie I (1984), pp. 753–756.
18. B.V. GODUN, *Equivalent norms on non-reflexive Banach spaces*, Soviet Math. Doklady, vol. 26 (1982), pp. 12–15.
19. A. GROTHENDIECK, *Produits tensóriels topologiques et espaces nucleaires*, Mem. Amer. Math. Soc., no. 16, 1955.
20. P. HARMAND and A. LIMA, *On Banach spaces which are M-ideals in their biduals*, Trans. Amer. Math. Soc., vol. 283 (1984), pp. 253–264.
21. W.B. JOHNSON, Personal communication, August 1985.
22. ______, *A complementably universal conjugate Banach space and its relation to the approximation problem*, Israel J. Math., vol. 13 (1972), pp. 301–310.
23. R.C. JAMES, *A separable somewhat reflexive Banach space with non-separable dual*, Bull. Amer. Math. Soc., vol. 80 (1974), pp. 738–743.
24. J. LINDENSTRAUSS and L. TZAFRIRI, *Classical Banach spaces, vol. I, Sequence spaces*, Springer-Verlag, Berlin, 1977.
25. J. LINDENSTRAUSS, *On James's paper "Separable conjugate spaces"*, Israel J. Math., vol. 9 (1971), pp. 279–284.

26. ______, *Extension of compact operators*, Mem. Amer. Math. Soc., no. 48, 1964.
27. O. REINOV, *A survey on the approximation property*, Math. Nachrichten, vol. 119 (1984), pp. 257–264.
28. W. RUESS and C. STEGALL, *Weak*-denting points in duals of operator spaces*, *Banach Spaces*, Proc. of the Missouri Conference, Colombia, Missouri, 1984; Lecture Notes in Math., no. 1166, Springer Verlag, Berlin, 1985, pp. 158–168.
29. ______, *Exposed and denting points in duals of operator spaces*, Israel J. Math., to appear.
30. H.H. SCHAEFER, *Banach lattices and positive operators*, Springer Verlag, New York, 1974.
31. I. SINGER, *Bases in Banach spaces*, vol. II, Springer Verlag, New York, 1981.
32. F. SULLIVAN, *Geometrical properties determined by the higher duals of a Banach space*, Illinois J. Math., vol. 21 (1977), pp. 315–331.
33. M. TALAGRAND, *Renormages de quelques* $C(K)$, Israel J. Math., to appear.
34. R.D. BOURGIN, *Geometric aspects of convex sets with the Radon Nikodym property*, Lecture Notes in Math., no. 993, Springer Verlag, New York, 1983.
35. B.V. GODUN, *Unconditional bases and spanning basic sequences*, Izv. Vyssh. Uchebn., Zaved Mat., vol. 24 (1980), pp. 69–72.
36. T. FIGIEL, *Factorizations of compact operators and applications to the approximation problem*, Studia Math., vol. 45 (1973), pp. 191–210.

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