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SOME GEOMETRIC CONSEQUENCES OF THE WEITZENBÖCK FORMULA ON RIEMANNIAN ALMOST-PRODUCT MANIFOLDS; WEAK-HARMONIC DISTRIBUTIONS

BY

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0. Introduction

In this paper, we prove some geometric consequences obtained from certain linear relations among linear invariants of Riemannian almost-product manifolds. We also define and study weak-harmonic distributions.

In Section 1, we obtain a consequence of the Weitzenböck formula, (Theorem 1.2), which will be used in the next section.

Section 2 begins with general concepts on Riemannian almost-product manifolds.

A Riemannian almost-product manifold is a triplet $(\mathscr{M}, g, P)$, where $(\mathscr{M}, g)$ is a Riemannian manifold and P is a $(1,1)$-tensor field on $\mathscr{M}$ satisfying $P^2 = I$ and $g(PM, PN) = g(M, N)$, $M, N \in \mathscr{X}(\mathscr{M})$. The eigenspaces of P corresponding to the eigenvalues 1 and -1, at each point, determine two distributions $\mathscr{V}$ and $\mathscr{H}$, respectively called vertical and horizontal.

Next, we get a linear relation among linear invariants of Riemannian almost-product manifolds, (Theorem 2.8), by using Theorem 1.2, from which we deduce some geometric consequences. Among these it is necessary to note that:

THEOREM. *A Riemannian almost-product manifold $(\mathscr{M}, g, P)$ with non-negative sectional curvature in which $\mathscr{V}$ and $\mathscr{H}$ are foliations whose mean curvatures, restricted to each horizontal and vertical leaf respectively, have zero divergence, is necessarily locally a product.*

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One thus generalizes two results obtained in [1], where this conclusion is proved, when $\mathscr{V}$ and $\mathscr{H}$ are both foliations with minimal leaves, or both totally umbilical foliations with mean curvatures as in the theorem.

It is shown in [12] that one cannot find two complementary and orthogonal totally umbilical foliations on compact Riemannian manifolds with non-positive sectional curvature, unless each one of them is 1-dimensional or a totally geodesic foliation. As a consequence of Theorem 2.8, we get, in Corollary 2.11, an improvement of this result for non-integrable distributions.

In the last section we generalize the concept of harmonic foliation that appears in [7] and [8]. The distribution $\mathscr{V}$ of a Riemannian almost-product manifold ís said to be weak-harmonic if the canonical projection $\mathscr{h}: T\mathscr{M} \to \mathscr{H}$ from the tangent bundle onto horizontal bundle is an $\mathscr{H}$-valued 1-form orthogonal to $\Delta^{\mathscr{H}}\mathscr{h}$, with $\Delta^{\mathscr{H}}$ the Laplacian operator induced by the following connection on $\mathscr{H}$:

$$\nabla_A^{\mathscr{H}} X = \mathscr{h}[A, X], \quad A \in \mathscr{V}, X \in \mathscr{H},$$

$$\nabla_Y^{\mathscr{H}} X = \mathscr{h}(\nabla_Y X), \quad X, Y \in \mathscr{H},$$

where ∇ is the Levi-Civita connection of $\mathscr{M}$.

We prove that some of the main results of [8] on harmonic foliations (Corollary 2.27, Theorem 2.34) remain valid for weak-harmonic distributions. (On the other hand, these results are consequences of Theorem 2.8.) Furthermore, we show some new results about weak-harmonicity, among which are the following:

(i) A weak-harmonic distribution with the property AF (Definition 2.3) is a totally geodesic foliation.

(ii) Let $(\mathscr{M}, g, P)$ be a Riemannian almost-product manifold with non-negative sectional curvature in which the horizontal distribution is a foliation with minimal leaves. Then, if the distribution $\mathscr{V}$ is weak-harmonic, the manifold is locally a product.

All geometric objects considered throughout the paper will be of class C^∞.

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1. A consequence of the Weitzenböck formula

Let $(\mathscr{M}, g)$ be an n-dimensional Riemannian manifold and $\mathscr{E}$ a vector bundle over $\mathscr{M}$ with a covariant differentiation D.

We shall denote $\Lambda^p(\mathscr{E}, \mathscr{M})$ the vector space of all $\mathscr{E}$-valued differential p-forms on $\mathscr{M}$.

It is a well known fact that the covariant differentiation D induces the following operators on $\mathscr{E}$-valued p-forms: the covariant differential acting on forms, $\overset{*}{D}$, the exterior differential operator, d^D, the exterior codifferential, δ^D, and the Laplacian operator, Δ^D.

Furthermore, if $\mathscr{E}$ is a vector bundle over $\mathscr{M}$ with a metric $\langle\ ,\ \rangle$, we have on $\Lambda^p(\mathscr{E}, \mathscr{M})$ the metric induced by the metrics $\langle\ ,\ \rangle$ and g:

If $\theta, \eta \in \Lambda^p(\mathscr{E}, \mathscr{M})$, then $\langle\theta, \eta\rangle$ is the function on $\mathscr{M}$ given by

$$\langle\theta, \eta\rangle(x) = \frac{1}{p!} \sum_{i_1,\dots,i_p=1}^{n} \langle\theta(e_{i_1},\dots,e_{i_p}), \eta(e_{i_1},\dots,e_{i_p})\rangle$$

where $\{e_1,\dots,e_n\}$ denote an orthonormal basis of $T_x\mathscr{M}$.

Let $\mathscr{E}$ be a vector bundle over $\mathscr{M}$ with a metric $\langle\ ,\ \rangle$ and a metric covariant differentiation D. If the manifold $\mathscr{M}$ is compact and oriented, we can define the inner product

$$(\theta, \eta) = \int_{\mathscr{M}} \langle\theta, \eta\rangle * 1, \quad \theta, \eta \in \Lambda^p(\mathscr{E}, \mathscr{M}),$$

for which the operator δ^D is the adjoint operator of d^D; that is,

$$(d^D\theta, \eta) = (\theta, \delta^D\eta), \quad \forall\theta \in \Lambda^p(\mathscr{E}, \mathscr{M}), \eta \in \Lambda^{p+1}(\mathscr{E}, \mathscr{M}).$$

Consequently, for $\theta \in \Lambda^p(\mathscr{E}, \mathscr{M})$,

$$(\Delta^D\theta, \theta) = (d^D\theta, d^D\theta) + (\delta^D\theta, \delta^D\theta).$$

THEOREM 1.1 (WEITZENBÖCK'S FORMULA). *Let $\mathscr{E}$ be a vector bundle over $\mathscr{M}$ with a metric $\langle\ ,\ \rangle$ and a metric covariant differentiation D. If θ is an $\mathscr{E}$-valued 1-form, then*

$$\langle\Delta^D\theta, \theta\rangle = \frac{1}{2}\Delta\langle\theta, \theta\rangle + \langle\overset{*}{D}\theta, \overset{*}{D}\theta\rangle + A$$

where Δ is the Laplacian operator of the Riemannian manifold $\mathscr{M}$ and A is a function on $\mathscr{M}$ defined by

$$A(x) = \sum_{i=1}^{n} \langle\theta(S(e_i)), \theta(e_i)\rangle - \sum_{i,j=1}^{n} R^D(e_i, e_j, \theta(e_i), \theta(e_j))$$

where $\{e_1, \dots, e_n\}$ *is an orthonormal basis for* $T_x\mathcal{M}$, S *is the endomorphism of* $T_x\mathcal{M}$ *defined by Ricci tensor of* $\mathcal{M}$, *that is,* $S(e_i) = \sum_{k=1}^n S_{ki} e_k$, *and*

$$R^D(M, N, \phi, \psi) = \langle D_{[M,N]}\phi - D_M(D_N\phi) + D_N(D_M\phi), \psi\rangle,$$
$$M, N \in \mathcal{X}(\mathcal{M}), \phi, \psi \in \Gamma(\mathcal{E}).$$

THEOREM 1.2. *Let* $\mathcal{E}$ *be a vector bundle over the Riemannian manifold* $(\mathcal{M}, g)$, *with a metric* $\langle\ ,\ \rangle$, *and a metric covariant differentiation* D. *If* θ *is an* $\mathcal{E}$*-valued* 1*-form satisfying*

$$\langle\theta(M), \theta(N)\rangle = g(M, N), \quad M, N \in \mathcal{X}(\mathcal{M}),$$

then

$$\tau - \tau^\theta = 2\delta\mu^\theta + \langle d^D\theta, d^D\theta\rangle + \langle\delta^D\theta, \delta^D\theta\rangle - \langle \overset{*}{D}\theta, \overset{*}{D}\theta\rangle,$$

where μ^θ *is the* 1*-form defined by* $\mu^\theta(M) = -\langle\delta^D\theta, \theta(M)\rangle$, τ *is the scalar curvature of* $\mathcal{M}$ *and* τ^θ *the function on* $\mathcal{M}$ *given by*

$$\tau^\theta(x) = \sum_{i,j=1}^n R^D\big(e_i, e_j, \theta(e_i), \theta(e_j)\big)$$

with $\{e_i\}_{i=1}^n$ *an orthonormal basis of* $T_x\mathcal{M}$.

Proof. First, we will prove that

$$\langle\Delta^D\theta, \theta\rangle = 2\delta\mu^\theta + \langle d^D\theta, d^D\theta\rangle + \langle\delta^D\theta, \delta^D\theta\rangle.$$

Since $\langle\theta(M), \theta(N)\rangle = g(M, N)$, $M, N \in \mathcal{X}(\mathcal{M})$, we have

$$\Big\langle\big(\overset{*}{D}_L\theta\big)(M), \theta(N)\Big\rangle = -\Big\langle\big(\overset{*}{D}_L\theta\big)(N), \theta(M)\Big\rangle, \quad L, M, N \in \mathcal{X}(\mathcal{M}).$$

Let $\{E_i\}_{i=1}^n$ be a local orthonormal frame of $T\mathcal{M}$. Then,

$$\langle\Delta^D\theta, \theta\rangle = -\sum_{i,k=1}^n \Big\langle\big(\overset{*}{D}_{E_k} d^D\theta\big)(E_k, E_i), \theta(E_i)\Big\rangle + \sum_{i=1}^n \langle D_{E_i}(\delta^D\theta), \theta(E_i)\rangle.$$

Now,

$$\sum_{i,k=1}^{n}\left\langle\left(\overset{*}{D}_{E_k}d^D\theta\right)(E_k,E_i),\theta(E_i)\right\rangle$$

$$= \sum_{i,k=1}^{n}\left\{\left\langle D_{E_k}\left(\left(\overset{*}{D}_{E_k}\theta\right)(E_i)\right),\theta(E_i)\right\rangle\right.$$

$$-\left\langle D_{E_k}\left(\left(\overset{*}{D}_{E_i}\theta\right)(E_k)\right),\theta(E_i)\right\rangle+\left\langle\left(\overset{*}{D}_{E_i}\theta\right)\left(\nabla_{E_k}E_k\right),\theta(E_i)\right\rangle$$

$$-\left\langle\left(\overset{*}{D}_{E_k}\theta\right)\left(\nabla_{E_k}E_i\right),\theta(E_i)\right\rangle$$

$$\left.+\left\langle\left(\overset{*}{D}_{\nabla_{E_k}E_i}\theta\right)(E_k),\theta(E_i)\right\rangle\right\}$$

$$= \sum_{i,k=1}^{n}\left\{-\left\langle\left(\overset{*}{D}_{E_k}\theta\right)(E_i),\left(\overset{*}{D}_{E_k}\theta\right)(E_i)\right\rangle-\left\langle\left(\overset{*}{D}_{E_k}\theta\right)(E_i),\theta\left(\nabla_{E_k}E_i\right)\right\rangle\right.$$

$$-E_k\left\langle\left(\overset{*}{D}_{E_i}\theta\right)(E_k),\theta(E_i)\right\rangle+\left\langle\left(\overset{*}{D}_{E_i}\theta\right)(E_k),\left(\overset{*}{D}_{E_k}\theta\right)(E_i)\right\rangle$$

$$+\left\langle\left(\overset{*}{D}_{E_i}\theta\right)(E_k),\theta\left(\nabla_{E_k}E_i\right)\right\rangle-\left\langle\left(\overset{*}{D}_{E_i}\theta\right)(E_i),\theta\left(\nabla_{E_k}E_k\right)\right\rangle$$

$$\left.+\left\langle\left(\overset{*}{D}_{E_k}\theta\right)(E_i),\theta\left(\nabla_{E_k}E_i\right)\right\rangle-\left\langle\left(\overset{*}{D}_{\nabla_{E_k}E_i}\theta\right)(E_i),\theta(E_k)\right\rangle\right\}$$

$$= -\langle d^D\theta, d^D\theta\rangle+\sum_{k=1}^{n}\left\{-E_k\langle\delta^D\theta,\theta(E_k)\rangle+\langle\delta^D\theta,\theta\left(\nabla_{E_k}E_k\right)\rangle\right\}$$

since,

$$\sum_{i,k=1}^{n}\left\{\left\langle\left(\overset{*}{D}_{E_k}\theta\right)(E_i),\left(\overset{*}{D}_{E_k}\theta\right)(E_i)\right\rangle-\left\langle\left(\overset{*}{D}_{E_i}\theta\right)(E_k),\left(\overset{*}{D}_{E_k}\theta\right)(E_i)\right\rangle\right\}$$

$$= \langle d^D\theta, d^D\theta\rangle$$

and,

$$\sum_{i,k=1}^{n}\left\{\left\langle\left(\overset{*}{D}_{E_i}\theta\right)\left(\nabla_{E_k}E_i\right),\theta(E_k)\right\rangle+\left\langle\left(\overset{*}{D}_{\nabla_{E_k}E_i}\theta\right)(E_i),\theta(E_k)\right\rangle\right\}$$

$$= \sum_{i,k,j=1}^{n} g\left(\nabla_{E_k}E_i,E_j\right)\left\{\left\langle\left(\overset{*}{D}_{E_i}\theta\right)(E_j),\theta(E_k)\right\rangle+\left\langle\left(\overset{*}{D}_{E_j}\theta\right)(E_i),\theta(E_k)\right\rangle\right\}$$

$$= \sum_{i,k,j=1}^{n}\left\langle\left(\overset{*}{D}_{E_i}\theta\right)(E_j),\theta(E_k)\right\rangle\left\{g\left(\nabla_{E_k}E_i,E_j\right)+g\left(\nabla_{E_k}E_j,E_i\right)\right\}$$

$$= 0.$$

It follows that

$$\begin{aligned}\langle \Delta^D\theta, \theta\rangle &= \langle d^D\theta, d^D\theta\rangle + \sum_{i=1}^{n} \left\{ E_i\langle \delta^D\theta, \theta(E_i)\rangle - \langle \delta^D\theta, \theta(\nabla_{E_i}E_i)\rangle \right. \\ &\qquad \left. + \langle D_{E_i}(\delta^D\theta), \theta(E_i)\rangle \right\} \\ &= \langle d^D\theta, d^D\theta\rangle - \langle \delta^D\theta, \delta^D\theta\rangle + 2\sum_{i=1}^{n} \langle D_{E_i}(\delta^D\theta), \theta(E_i)\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}\delta\mu^\theta &= -\sum_{i=1}^{n} \left(\nabla_{E_i}\mu^\theta\right)(E_i) = -\sum_{i=1}^{n} \left\{\nabla_{E_i}\left(\mu^\theta(E_i)\right) - \mu^\theta\left(\nabla_{E_i}E_i\right)\right\} \\ &= -\sum_{i=1}^{n} \left\{ -E_i\langle \delta^D\theta, \theta(E_i)\rangle + \langle \delta^D\theta, \theta(\nabla_{E_i}E_i)\rangle \right\} \\ &= \sum_{i=1}^{n} \langle D_{E_i}(\delta^D\theta), \theta(E_i)\rangle - \langle \delta^D\theta, \delta^D\theta\rangle.\end{aligned}$$

Therefore, we have

$$\langle \Delta^D\theta, \theta\rangle = 2\delta\mu^\theta + \langle d^D\theta, d^D\theta\rangle + \langle \delta^D\theta, \delta^D\theta\rangle.$$

Now, by using the Weitzenböck formula, and considering that, in this case, $A = \tau - \tau^\theta$ and $\Delta\langle\theta, \theta\rangle = 0$, we have the required result.

2. A linear relation among linear invariants of Riemannian almost-product manifolds: geometric consequences

A Riemannian almost-product manifold is a triplet $(\mathscr{M}, g, P)$, where $(\mathscr{M}, g)$ is a Riemannian manifold and P is a $(1,1)$-tensor field on $\mathscr{M}$ satisfying

$$P^2 = I \quad \text{and} \quad g(PM, PN) = g(M, N) \text{ for } M, N \in \mathfrak{X}(\mathscr{M}).$$

A Riemannian almost-product structure P, determines two distributions $\mathscr{V}$ and $\mathscr{H}$ corresponding to the eigenvalues of P, 1 and -1, respectively called vertical and horizontal. In turn, a distribution $\mathscr{D}$ determines, on a Riemannian manifold, a complementary distribution $\mathscr{D}^\perp$, and hence, a Riemannian almost-product structure whose vertical and horizontal distributions are $\mathscr{D}$ and $\mathscr{D}^\perp$ respectively; this structure will be called Riemannian almost-product structure associated to $\mathscr{D}$.

LEMMA 2.1 [11]. *In any Riemannian almost-product manifold* $(\mathcal{M}, g, P)$, *we have*

(i) $g((\nabla_L P)M, N) = g((\nabla_L P)N, M)$ *and*
(ii) $g((\nabla_L P)PM, PN) = -g((\nabla_L P)M, N)$
for $L, M, N \in \mathfrak{X}(\mathcal{M})$.

The proof is immediate.

It is shown in [11] that there are 36 different classes of Riemannian almost-product manifolds, each one of which is characterized by some algebraic condition on ∇P. This classification was obtained by decomposition of the space of covariant tensors of order 3 that have the same algebraic properties as the tensor γ, given by $\gamma(L, M, N) = g((\nabla_L P)M, N)$ (Lemma 2.1), under the action of the structural group of $(\mathcal{M}, g, P)$, $0(p) \times 0(q)$, where p and $q = n - p$ are the respective dimensions of the distributions $\mathcal{V}$ and $\mathcal{H}$. Some non-trivial examples for every one of these classes are given in [10]; and in [4] the algebraic conditions, which define the classes, are interpreted in terms of geometric properties of the vertical and horizontal distributions.

In Definition 2.3, we describe the algebraic conditions on ∇P which characterize the properties of $\mathcal{V}$ and $\mathcal{H}$ in the different classes of Riemannian almost-product manifolds.

DEFINITION 2.2. A foliation $\mathcal{D}$ on a Riemannian manifold $(\mathcal{M}, g)$ is said to be a totally geodesic or totally umbilical foliation if all the maximal integral manifolds of $\mathcal{D}$ are totally geodesic or totally umbilical submanifolds of $\mathcal{M}$ respectively.

DEFINITION 2.3 [4], [11]. Let $\mathcal{D}$ be a distribution on a Riemannian manifold and P the almost-product structure associated to $\mathcal{D}$.

(i) $\mathcal{D}$ is a foliation (property F) if and only if $(\nabla_A P)B = (\nabla_B P)A$, $A, B \in \mathcal{D}$.
(ii) $\mathcal{D}$ is a distribution with the property $_{\mathrm{A}}$F if $(\nabla_A P)A = 0$, $A \in \mathcal{D}$.
(iii) A foliation with the property AF is a totally geodesic foliation (property TGF).
(iv) $\mathcal{D}$ is a totally umbilical foliation (property F_2) if and only if

$$(\nabla_A P)B = \frac{1}{p} g(A, B)\alpha^{\mathcal{D}}, \quad A, B \in \mathcal{D}$$

where $\alpha^{\mathcal{D}} = \sum_{a=1}^{p}(\nabla_{E_a} P)E_a$, $\{E_a\}_{a=1}^{p}$ is a local orthonormal reference of $\mathcal{D}$.

(v) $\mathcal{D}$ is a distribution with the property D_2 if

$$(\nabla_A P)B + (\nabla_B P)A = \frac{2}{p} g(A, B)\alpha^{\mathcal{D}}, \quad A, B \in \mathcal{D}.$$

If $\mathscr{D}$ is a foliation on a Riemannian manifold, it is obvious that $\alpha^{\mathscr{D}}$ is, up to a constant, its mean curvature. So:

(vi) A foliation $\mathscr{D}$ is a foliation with minimal leaves (property F_1) if and only if $\alpha^{\mathscr{D}} = 0$.

(vii) A distribution $\mathscr{D}$ which satisfies $\alpha^{\mathscr{D}} = 0$ will be said to be a distribution with the property D_1.

It is evident that a distribution has the property AF if and only if it has the properties D_1 and D_2.

A Riemannian almost-product manifold $(\mathscr{M}, g, P)$ will be said to be of type (α, β) if the vertical distribution has the property α and the horizontal one has the property β.

Observe that in a Riemannian almost-product manifold $(\mathscr{M}, g, P)$, the almost-product structure associated to $\mathscr{V}$ is P, and the one associated to $\mathscr{H}$ is $-P$.

DEFINITION 2.4 [5], [13]. We define the configuration tensors T and O of a Riemannian almost-product manifold $(\mathscr{M}, g, P)$ by

$$T_M N = \frac{1}{2}(\nabla_{vM} P) PN, \quad O_M N = \frac{1}{2}(\nabla_{hM} P) PN$$

for $M, N \in \mathscr{X}(\mathscr{M})$, where $v = 1/2(I + P)$ and $h = 1/2(I - P)$ are the projectors onto $\mathscr{V}$ and $\mathscr{H}$ respectively.

It is obvious that T (resp. O) vanishes if and only if $\mathscr{V}$ (resp. $\mathscr{H}$) is a totally geodesic foliation.

DEFINITION 2.5. On a Riemannian almost-product manifold we can define

$$S_1(M, N) = h[vM, vN], \quad S_2(M, N) = v[hM, hN]$$

for $M, N \in \mathscr{X}(\mathscr{M})$.

Evidently, S_1 (resp. S_2) vanishes if and only if $\mathscr{V}$ (resp. $\mathscr{H}$) is a foliation.

LEMMA 2.6. *In any Riemannian almost-product manifold we have*:

(i)
$$\|T\|^2 = \frac{1}{2} \sum_{a,b=1}^{p} g\big((\nabla_{E_a} P) E_b, (\nabla_{E_a} P) E_b\big),$$

$$\|O\|^2 = \frac{1}{2} \sum_{u,v=p+1}^{n} g\big((\nabla_{E_u} P) E_v, (\nabla_{E_u} P) E_v\big);$$

(ii)
$$\|\nabla P\|^2 = 4(\|T\|^2 + \|O\|^2);$$

(iii)
$$4\|S_1\|^2 = 2\|T\|^2 - A_1, \quad 4\|S_2\|^2 = 2\|O\|^2 - A_2$$

where A_1 *and* A_2 *are the linear invariants* [1] *given by*

$$A_1 = \sum_{a,b=1}^{p} g\big((\nabla_{E_a}P)E_b, (\nabla_{E_b}P)E_a\big), A_2 = \sum_{u,v=p+1}^{n} g\big((\nabla_{E_u}P)E_v, (\nabla_{E_v}P)E_u\big);$$

(iv) $$\|\nabla P\|^2 - \|dP\|^2 = A_1 + A_2;$$

(v) $$\|dP\|^2 = \frac{1}{2}\|\nabla P\|^2 + 4\big(\|S_1\|^2 + \|S_2\|^2\big);$$

(vi) $$\|\delta P\|^2 = \|\alpha^{\mathscr{V}}\|^2 + \|\alpha^{\mathscr{H}}\|^2;$$

where $\{E_a\}_{a=1}^{p}$ *and* $\{E_u\}_{u=p+1}^{n}$ *are local orthonormal frames of* $\mathscr{V}$ *and* $\mathscr{H}$ *respectively.*

The proof is immediate.

DEFINITION 2.7. On a Riemannian almost-product manifold $(\mathscr{M}, g, P)$, we can define

$$\tau^{\mathscr{V}} = \sum_{a,b=1}^{p} R(E_a, E_b, E_a, E_b),$$

$$\tau^{\mathscr{H}} = \sum_{u,v=p+1}^{n} R(E_u, E_v, E_u, E_v),$$

$$\tau^{\mathscr{V}\mathscr{H}} = \sum_{a=1}^{p} \sum_{u=p+1}^{n} R(E_a, E_u, E_a, E_u)$$

where R is the Riemannian curvature operator of the manifold, and $\{E_a\}_{a=1}^{p}$ and $\{E_u\}_{u=p+1}^{n}$ are local orthonormal frames of $\mathscr{V}$ and $\mathscr{H}$ respectively.

It is obvious that the scalar curvature of $(\mathscr{M}, g, P)$, τ, can be written as

$$\tau = \tau^{\mathscr{V}} + 2\tau^{\mathscr{V}\mathscr{H}} + \tau^{\mathscr{H}}.$$

THEOREM 2.8. *Let* $(\mathscr{M}, g, P)$ *be a Riemannian almost-product manifold. Then*

$$4\tau^{\mathscr{V}\mathscr{H}} = \|dP\|^2 - \|\nabla P\|^2 + 2\,\mathrm{div}_{\mathscr{V}}\,\alpha^{\mathscr{H}} + 2\,\mathrm{div}_{\mathscr{H}}\,\alpha^{\mathscr{V}}$$

where $\mathrm{div}_{\mathscr{V}}\,\alpha^{\mathscr{H}} = \sum_{a=1}^{p} g(\nabla_{E_a}\alpha^{\mathscr{H}}, E_a)$, $\mathrm{div}_{\mathscr{H}}\,\alpha^{\mathscr{V}} = \sum_{u=p+1}^{n} g(\nabla_{E_u}\alpha^{\mathscr{V}}, E_u)$, *and* $\{E_a\}_{a=1}^{p}$ *and* $\{E_u\}_{u=p+1}^{n}$ *are local orthonormal frames of* $\mathscr{V}$ *and* $\mathscr{H}$ *respectively.*

Proof. By applying Theorem 1.2 to the $T\mathcal{M}$-valued 1-form P, we obtain

$$\tau - \tau^P = 2\delta\mu^P + \|dP\|^2 + \|\delta P\|^2 - \|\nabla P\|^2.$$

Now, $\tau - \tau^P = 4\tau^{\mathscr{V}\mathscr{H}}$, and

$$\begin{aligned}
\delta\mu^P + \|\delta P\|^2 &= \sum_{i=1}^{n} g\big(\nabla_{E_i}(\delta P), PE_i\big) \\
&= -\sum_{a=1}^{p} g\big(\nabla_{E_a}\alpha^{\mathscr{V}}, E_a\big) + \sum_{a=1}^{p} g\big(\nabla_{E_a}\alpha^{\mathscr{H}}, E_a\big) \\
&\quad + \sum_{u=p+1}^{n} g\big(\nabla_{E_u}\alpha^{\mathscr{V}}, E_u\big) - \sum_{u=p+1}^{n} g\big(\nabla_{E_u}\alpha^{\mathscr{H}}, E_u\big) \\
&= \frac{1}{2}\sum_{a=1}^{p} g\big((\nabla_{E_a}P)E_a, \alpha^{\mathscr{V}}\big) + \operatorname{div}_{\mathscr{V}}\alpha^{\mathscr{H}} \\
&\quad + \operatorname{div}_{\mathscr{H}}\alpha^{\mathscr{V}} - \frac{1}{2}\sum_{u=p+1}^{n} g\big((\nabla_{E_u}P)E_u, \alpha^{\mathscr{H}}\big) \\
&= \frac{1}{2}\|\alpha^{\mathscr{V}}\|^2 + \operatorname{div}_{\mathscr{V}}\alpha^{\mathscr{H}} + \operatorname{div}_{\mathscr{H}}\alpha^{\mathscr{V}} + \frac{1}{2}\|\alpha^{\mathscr{H}}\|^2
\end{aligned}$$

which implies the result.

COROLLARY 2.9. *Let $(\mathcal{M}, g, P)$ be a Riemannian almost-product manifold.*
(i) *If $(\mathcal{M}, g, P)$ is of type* (AF, AF), *then*

$$\tau^{\mathscr{V}\mathscr{H}} = \frac{1}{8}\|\nabla P\|^2.$$

(ii) *If $(\mathcal{M}, g, P)$ is of type* (F, F), *then*

$$4\tau^{\mathscr{V}\mathscr{H}} = -\frac{1}{2}\|\nabla P\|^2 + 2\operatorname{div}_{\mathscr{V}}\alpha^{\mathscr{H}} + 2\operatorname{div}_{\mathscr{H}}\alpha^{\mathscr{V}}.$$

(iii) *If $(\mathcal{M}, g, P)$ is of type* $(\mathrm{F}_1, \mathrm{F}_1)$, *then*

$$\tau^{\mathscr{V}\mathscr{H}} = -\frac{1}{8}\|\nabla P\|^2.$$

(iv) *If $(\mathcal{M}, g, P)$ is of type* (F, AF), *then*

$$2\tau^{\mathscr{V}\mathscr{H}} = -\|T\|^2 + \|O\|^2 + \operatorname{div}_{\mathscr{H}}\alpha^{\mathscr{V}}.$$

(v) *If* $(\mathscr{M}, g, P)$ *is of type* $(\mathrm{D}_2, \mathrm{D}_2)$, *then*

$$2\tau^{\mathscr{V}\mathscr{H}} = \frac{1}{4}\|\nabla P\|^2 - \frac{1}{p}\|\alpha^{\mathscr{V}}\|^2 - \frac{1}{q}\|\alpha^{\mathscr{H}}\|^2 + \mathrm{div}_{\mathscr{V}}\,\alpha^{\mathscr{H}} + \mathrm{div}_{\mathscr{H}}\,\alpha^{\mathscr{V}}.$$

(vi) *If* $(\mathscr{M}, g, P)$ *is of type* $(\mathrm{AF}, \mathrm{D}_2)$, *then*

$$2\tau^{\mathscr{V}\mathscr{H}} = \frac{1}{4}\|\nabla P\|^2 - \frac{1}{q}\|\alpha^{\mathscr{H}}\|^2 + \mathrm{div}_{\mathscr{V}}\,\alpha^{\mathscr{H}}.$$

(vii) *If* $(\mathscr{M}, g, P)$ *is of type* $(\mathrm{F}, \mathrm{D}_2)$, *then*

$$2\tau^{\mathscr{V}\mathscr{H}} = -\frac{1}{q}\|\alpha^{\mathscr{H}}\|^2 - \|T\|^2 + \|O\|^2 + \mathrm{div}_{\mathscr{V}}\,\alpha^{\mathscr{H}} + \mathrm{div}_{\mathscr{H}}\,\alpha^{\mathscr{V}}.$$

Proof. Results (i) through (iv) follow immediately from Theorem 2.8 and Lemma 2.6.

For the remaining results, it is sufficient to consider that if $\mathscr{V}$ (resp. $\mathscr{H}$) is a distribution with the property D_2, then

$$A_1 = \frac{2}{p}\|\alpha^{\mathscr{V}}\|^2 - 2\|T\|^2 \quad \left(\text{resp. } A_2 = \frac{2}{q}\|\alpha^{\mathscr{H}}\|^2 - 2\|O\|^2\right).$$

Corollary 2.10. *Let* $(\mathscr{M}, g, P)$ *be a compact, oriented Riemannian almost-product manifold. Then*

$$4\int_{\mathscr{M}} \tau^{\mathscr{V}\mathscr{H}} * 1 = \int_{\mathscr{M}} \|dP\|^2 * 1 + \int_{\mathscr{M}} \|\delta P\|^2 * 1 - \int_{\mathscr{M}} \|\nabla P\|^2 * 1.$$

The proof follows from Theorem 2.8 by considering that

$$\int_{\mathscr{M}} \mathrm{div}_{\mathscr{V}}\,\alpha^{\mathscr{H}} * 1 = \frac{1}{2}\int_{\mathscr{M}} \|\alpha^{\mathscr{H}}\|^2 * 1 \text{ and } \int_{\mathscr{M}} \mathrm{div}_{\mathscr{H}}\,\alpha^{\mathscr{V}} * 1 = \frac{1}{2}\int_{\mathscr{M}} \|\alpha^{\mathscr{V}}\|^2 * 1.$$

Of course, the formulas of Corollary 2.10 which contain $\mathrm{div}_{\mathscr{V}}\,\alpha^{\mathscr{H}}$ or $\mathrm{div}_{\mathscr{H}}\,\alpha^{\mathscr{V}}$, can be reformulated in compact manifolds.

Corollary 2.11. *Let* $(\mathscr{M}, g, P)$ *be a Riemannian almost-product manifold.*

(i) *If* $(\mathscr{M}, g, P)$ *is of type* $(\mathrm{AF}, \mathrm{AF})$, *then* $\tau^{\mathscr{V}\mathscr{H}} \geq 0$, *with equality holding only if the manifold is locally a product.*

(ii) *If* $(\mathscr{M}, g, P)$ *is of type* (F, F) *and the mean curvatures of the vertical and horizontal foliations, restricted to each horizontal and vertical leaf respectively, have zero divergence, then* $\tau^{\mathscr{V}\mathscr{H}} \leq 0$, *with equality holding only if* $(\mathscr{M}, g, P)$ *is a locally-product manifold.*

(iii) *If* $(\mathcal{M}, g, P)$ *is of type* (D_2, D_2), *compact and oriented, then*

$$\int_{\mathcal{M}} \tau^{\mathcal{V}\mathcal{H}} * 1 \geq 0,$$

and the equality is satisfied if and only if each distribution, $\mathcal{V}$ *and* $\mathcal{H}$, *is of dimension one or a totally geodesic foliation.*

(iv) *If* $(\mathcal{M}, g, P)$ *is compact and oriented,* $\dim \mathcal{H} = 1$ *and* $\mathcal{V}$ *is a foliation with minimal leaves, then* $\int_{\mathcal{M}} \tau^{\mathcal{V}\mathcal{H}} * 1 \leq 0$, *with equality holding if and only if* $\mathcal{V}$ *is a totally geodesic foliation.*

Proof. Results (i) and (ii) are deduced immediately from results (i) and (ii) of Corollary 2.10 respectively.

(iii) Considering that

$$\int_{\mathcal{M}} \operatorname{div}_{\mathcal{V}} \alpha^{\mathcal{H}} * 1 = \frac{1}{2} \int_{\mathcal{M}} \|\alpha^{\mathcal{H}}\|^2 * 1,$$

$$\int_{\mathcal{M}} \operatorname{div}_{\mathcal{H}} \alpha^{\mathcal{V}} * 1 = \frac{1}{2} \int_{\mathcal{M}} \|\alpha^{\mathcal{V}}\|^2 * 1$$

and by using (v) of Corollary 2.10, we deduce

$$2\int_{\mathcal{M}} \tau^{\mathcal{V}\mathcal{H}} * 1 = \frac{1}{4} \int_{\mathcal{M}} \|\nabla P\|^2 * 1 + \frac{p-2}{2p} \int_{\mathcal{M}} \|\alpha^{\mathcal{V}}\|^2 * 1 + \frac{q-2}{2q} \int_{\mathcal{M}} \|\alpha^{\mathcal{H}}\|^2 * 1$$

and so, if $\dim \mathcal{V} \geq 2$ and $\dim \mathcal{H} \geq 2$, we have $\int_{\mathcal{M}} \tau^{\mathcal{V}\mathcal{H}} * 1 \geq 0$, equality holding only if the manifold is locally a product.

If $\dim \mathcal{V} = 1$, $\mathcal{V}$ is a totally umbilical foliation. Therefore

$$2\|T\|^2 = A_1 = \frac{2}{p}\|\alpha^{\mathcal{V}}\|^2 - 2\|T\|^2$$

and the last formula can be written in the following form:

$$2\int_{\mathcal{M}} \tau^{\mathcal{V}\mathcal{H}} * 1 = \int_{\mathcal{M}} \|O\|^2 * 1 + \frac{q-2}{2q} \int_{\mathcal{M}} \|\alpha^{\mathcal{H}}\|^2 * 1.$$

So, if $q \geq 2$, then $\int_{\mathcal{M}} \tau^{\mathcal{V}\mathcal{H}} * 1 \geq 0$, equality holding if and only if $\mathcal{H}$ is a totally geodesic foliation. And if $q = 1$, the integral vanishes.

For $\dim \mathcal{H} = 1$, the argument is analogous.

(iv) If $(\mathcal{M}, g, P)$ is of type (F_1, F_2), compact and oriented, we have

$$2\int_{\mathcal{M}} \tau^{\mathcal{V}\mathcal{H}} * 1 = \frac{q-1}{2q} \int_{\mathcal{M}} \|\alpha^{\mathcal{H}}\|^2 * 1 - \int_{\mathcal{M}} \|T\|^2 * 1$$

which implies the result.

Comments. Result (i) in Corollary 2.9 (and consequently the result (i) in Corollary 2.11) was obtained in [1] by using a different method.

Result (ii) in Corollary 2.11 generalizes two results obtained in [1]. There, it was shown for a manifold of type (F_2, F_2) instead of (F, F). Furthermore, in [1], it was also shown that, on manifolds of type (F_1, F_1), we have $\tau^{\mathscr{V}\mathscr{H}} \leq 0$, the equality holding only if the manifold is locally a product.

Result (iii) in Corollary 2.11 generalizes a result obtained in [12]. There, the same conclusion is obtained for a manifold of type (F_2, F_2).

3. Weak-harmonic distributions

In [7], F.W. Kamber and Ph. Tondeur analyzed some properties of harmonic foliations and in [8] the same authors examined the relation between the harmonicity property of a foliation with bundle-like metric and the sectional curvature of the manifold, obtaining the following result: Let $(\mathscr{M}, g, P)$ be a Riemannian almost-product manifold of type (F, TGF) with non-negative sectional curvature. If $\mathscr{V}$ is a harmonic foliation, then it is a totally geodesic foliation [8, Corollary 2.27].

We shall begin this section by extending the concept of harmonicity which appears in [7] and [8], obtaining afterwards a generalization of the above result. Furthermore, we shall obtain, among other results, some generalizations of several other conclusions found in [8].

DEFINITION 3.1. Let $(\mathscr{M}, g, P)$ be a Riemannian almost-product manifold. We define the following connection on the vector bundle $\mathscr{H}$:

$$\nabla_A^{\mathscr{H}} X = h[A, X], \quad A \in \mathscr{V},\ X \in \mathscr{H}$$
$$\nabla_Y^{\mathscr{H}} X = h(\nabla_Y X), \ X, Y \in \mathscr{H}.$$

Its torsion, $T^{\mathscr{H}}$, is the $\mathscr{H}$-valued 2-form on $\mathscr{M}$ defined by

$$T^{\mathscr{H}}(M, N) = \nabla_M^{\mathscr{H}}(hN) - \nabla_N^{\mathscr{H}}(hM) - h[M, N], \quad M, N \in \mathfrak{X}(\mathscr{M}).$$

Writing this expression for vertical and horizontal vector fields, we have

$$T^{\mathscr{H}}(A, B) = -h[A, B], \quad T^{\mathscr{H}}(A, X) = 0, \quad T^{\mathscr{H}}(X, Y) = 0$$

with $A, B \in \mathscr{V}$, $X, Y \in \mathscr{H}$.

It is evident that $\nabla^{\mathscr{H}}$ is torsion free if and only if $\mathscr{V}$ is integrable, and in this case, $\nabla^{\mathscr{H}}$ is the basic connection which is used in [7] to define the concept of harmonic foliation.

PROPOSITION 3.2. *$\nabla^{\mathscr{H}}$ is a metric connection (with respect to the metric induced by g in $\mathscr{H}$) if and only if $\mathscr{H}$ is a distribution with the property* AF.

The proof is immediate.

The connection $\nabla^{\mathscr{H}}$ determines the operators $\overset{*}{\nabla}{}^{\mathscr{H}}$, $d^{\mathscr{H}}$, $\delta^{\mathscr{H}}$ and $\Delta^{\mathscr{H}}$ on $\mathscr{H}$-valued forms, which, in this section, will be applied to the $\mathscr{H}$-valued 1-form h.

LEMMA 3.3

(i)
$$\left(\overset{*}{\nabla}{}^{\mathscr{H}}_{A}h\right)B = -\frac{1}{2}(\nabla_A P)B, \quad A, B \in \mathscr{V};$$
$$\left(\overset{*}{\nabla}{}^{\mathscr{H}}_{A}h\right)X = \left(\overset{*}{\nabla}{}^{\mathscr{H}}_{X}h\right)A = -\frac{1}{2}(\nabla_X P)A, \quad A \in \mathscr{V},\ X \in \mathscr{H};$$
$$\left(\overset{*}{\nabla}{}^{\mathscr{H}}_{X}h\right)Y = 0, \quad X, Y \in \mathscr{H}.$$

(ii) $\delta^{\mathscr{H}}h = \frac{1}{2}\alpha^{\mathscr{V}}$.

(iii) $d^{\mathscr{H}}h(M, N) = T^{\mathscr{H}}(M, N)$, $M, N \in \mathfrak{X}(\mathscr{M})$.

The proof is immediate.

DEFINITION 3.4. (i) We will say that the distribution $\mathscr{V}$ is harmonic if the $\mathscr{H}$-valued 1-form h is $\nabla^{\mathscr{H}}$-closed and $\nabla^{\mathscr{H}}$-coclosed, that is, $d^{\mathscr{H}}h = \delta^{\mathscr{H}}h = 0$.

(ii) We will say that $\mathscr{V}$ is a weak-harmonic distribution if the $\mathscr{H}$-valued 1-form h satisfies $g(\Delta^{\mathscr{H}}h, h) = 0$.

It is evident that if $\mathscr{V}$ is a harmonic distribution, then it is a weak-harmonic distribution.

THEOREM 3.5. (i) *h is $\nabla^{\mathscr{H}}$-coclosed if and only if $\mathscr{V}$ is a distribution with the property* D_1.

(ii) *h is $\nabla^{\mathscr{H}}$-closed if and only if $\mathscr{V}$ is a foliation.*

(iii) *$\mathscr{V}$ is a harmonic distribution if and only if it is a foliation with minimal leaves.*

The proof follows immediately from Lemma 3.3.

THEOREM 3.6. *Let $(\mathscr{M}, g, P)$ be a Riemannian almost-product manifold. Then:*

(i) $g(\Delta^{\mathscr{H}}h, h) = \frac{1}{2}\operatorname{div}_{\mathscr{H}}\alpha^{\mathscr{V}} + \|S_1\|^2$.

(ii) $g(\Delta^{\mathscr{H}}h, h) = \frac{1}{8}\|\nabla P\|^2 - \|S_2\|^2 - \frac{1}{2}\operatorname{div}_{\mathscr{V}}\alpha^{\mathscr{H}} + \tau^{\mathscr{V}\mathscr{H}}$.

(iii) *If $(\mathscr{M}, g, P)$ is of type* $(-, \mathrm{AF})$, *then*
$$g(\Delta^{\mathscr{H}}h, h) = \frac{1}{2}\|T\|^2 - \frac{1}{2}\|O\|^2 + \tau^{\mathscr{V}\mathscr{H}}.$$

Proof. (i) Let $\{E_a\}_{a=1}^{p}$ and $\{E_u\}_{u=p+1}^{n}$ be local orthonormal frames of $\mathscr{V}$ and $\mathscr{H}$ respectively.

$$\begin{aligned}
g(\Delta^{\mathscr{H}}\hbar, \hbar) &= \sum_{u=p+1}^{n} g\big((\Delta^{\mathscr{H}}\hbar)E_u, E_u\big) \\
&= -\sum_{i=1}^{n}\sum_{u=p+1}^{n} g\Big(\Big(\overset{*}{\nabla}{}^{\mathscr{H}}_{E_i} d^{\mathscr{H}}\hbar\Big)(E_i, E_u), E_u\Big) + \frac{1}{2}\sum_{u=p+1}^{n} g\big(\nabla^{\mathscr{H}}_{E_u}\alpha^{\mathscr{V}}, E_u\big) \\
&= \sum_{a=1}^{p}\sum_{u=p+1}^{n} g\big(d^{\mathscr{H}}\hbar(E_a, v\nabla_{E_a}E_u), E_u\big) + \frac{1}{2}\operatorname{div}_{\mathscr{H}}\alpha^{\mathscr{V}} \\
&= \sum_{a=1}^{p}\sum_{u=p+1}^{n}\Big\{-\frac{1}{2}g\big((\nabla_{E_a}P)\nabla_{E_a}E_u, E_u\big) + \frac{1}{2}g\Big(\big(\nabla_{v\nabla_{E_a}E_u}P\big)E_a, E_u\Big)\Big\} \\
&\quad + \frac{1}{2}\operatorname{div}_{\mathscr{H}}\alpha^{\mathscr{V}} \\
&= \frac{1}{4}\sum_{a=1}^{p}\sum_{u=p+1}^{n} g\big((\nabla_{E_a}P)E_u, (\nabla_{E_a}P)E_u\big) \\
&\quad + \frac{1}{2}\sum_{a,b=1}^{p}\sum_{u=p+1}^{n} g\big((\nabla_{E_b}P)E_a, E_u\big)g\big(\nabla_{E_a}E_u, E_b\big) + \frac{1}{2}\operatorname{div}_{\mathscr{H}}\alpha^{\mathscr{V}} \\
&= \frac{1}{2}\|T\|^2 - \frac{1}{4}A_1 + \frac{1}{2}\operatorname{div}_{\mathscr{H}}\alpha^{\mathscr{V}} \\
&= \|S_1\|^2 + \frac{1}{2}\operatorname{div}_{\mathscr{H}}\alpha^{\mathscr{V}}.
\end{aligned}$$

(ii) By using (i), Lemma 2.6 and Theorem 2.8, we have

$$\begin{aligned}
4g(\Delta^{\mathscr{H}}\hbar, \hbar) &= 4\|S_1\|^2 + 2\operatorname{div}_{\mathscr{H}}\alpha^{\mathscr{V}} \\
&= 2\|T\|^2 + \|dP\|^2 - \|\nabla P\|^2 + A_2 + 2\operatorname{div}_{\mathscr{H}}\alpha^{\mathscr{V}} \\
&= 2\|T\|^2 + 4\tau^{\mathscr{V}\mathscr{H}} - 2\operatorname{div}_{\mathscr{V}}\alpha^{\mathscr{H}} + A_2 \\
&= \frac{1}{2}\|\nabla P\|^2 - 4\|S_2\|^2 - 2\operatorname{div}_{\mathscr{V}}\alpha^{\mathscr{H}} + 4\tau^{\mathscr{V}\mathscr{H}}.
\end{aligned}$$

Evidently, if $\mathscr{H}$ is a distribution with the property AF, this formula is that given in (iii).

The formula given in (iii) of the last Theorem was obtained in [8] in the case that $\mathscr{V}$ is a foliation.

COROLLARY 3.7. *Let* $(\mathcal{M}, g, P)$ *be a Riemannian almost-product manifold.*

(i) *If* $(\mathcal{M}, g, P)$ *is of type* $(-, \mathrm{TGF})$ *and* $\mathcal{V}$ *is a weak-harmonic distribution, then* $\tau^{\mathcal{V}\mathcal{H}} \leq 0$, *with equality holding only if the manifold is locally a product.*

(ii) *If* $(\mathcal{M}, g, P)$ *is of type* $(\mathrm{F}, \mathrm{TGF})$ *and* $\mathrm{div}_{\mathcal{H}}\, \alpha^{\mathcal{V}} = 0$, *then* $\tau^{\mathcal{V}\mathcal{H}} \leq 0$, *where the equality holds only if the manifold is locally a product.*

(iii) *If* $(\mathcal{M}, g, P)$ *is of type* $(\mathrm{AF}, -)$, *then* $\mathcal{V}$ *is not a weak-harmonic distribution, unless it is a totally geodesic foliation.*

(iv) *A distribution with the property* D_1 *is weak-harmonic if and only if it is harmonic.*

(v) *If* $(\mathcal{M}, g, P)$ *is compact and oriented, then* $\mathcal{V}$ *is a weak-harmonic distribution if and only if it is a harmonic distribution.*

(vi) *If* $\mathcal{V}$ *is a weak-harmonic distribution and* $\mathcal{H}$ *is a foliation satisfying* $\mathrm{div}_{\mathcal{V}}\, \alpha^{\mathcal{H}} = 0$ *(in particular if* $\mathcal{H}$ *is a foliation with minimal leaves), then* $\tau^{\mathcal{V}\mathcal{H}} \leq 0$, *with equality holding only if the manifold is locally a product.*

(vii) *If* $\mathcal{V}$ *is a weak-harmonic distribution,* $\dim \mathcal{H} = 1$ *and* $\alpha^{\mathcal{H}}$ *has zero divergence, then* $\tau^{\mathcal{V}\mathcal{H}} \leq 0$, *the equality holding if and only if* $\mathcal{V}$ *is a totally geodesic foliation.*

Proof. Result (i) is an immediate consequence of part (iii) of the theorem above.

(ii) If $\mathcal{V}$ is a foliation and $\mathrm{div}_{\mathcal{H}}\, \alpha^{\mathcal{V}} = 0$, then, from part (i) of the last theorem, $g(\Delta^{\mathcal{H}} h, h) = 0$, and the result follows from (i).

(iii) If $\mathcal{V}$ is a distribution with the property AF, then, by using (i) of Theorem 3.6, we deduce $g(\Delta^{\mathcal{H}} h, h) = \|T\|^2$ and the result follows.

(iv) This follows immediately from Theorem 3.6(i).

(v) By integrating formula (i) of Theorem 3.6, we have

$$\int_{\mathcal{M}} g(\Delta^{\mathcal{H}} h, h) * 1 = \int_{\mathcal{M}} \|S_1\|^2 * 1 + \frac{1}{4} \int_{\mathcal{M}} \|\alpha^{\mathcal{V}}\|^2 * 1$$

which implies the result.

(vi) This is a direct consequence of Theorem 3.6(ii).

(vii) If $\dim \mathcal{H} = 1$, then $\|S_2\|^2 = 0$ and $\|O\|^2 = 1/2 \|\alpha^{\mathcal{H}}\|^2$. So we deduce from Theorem 3.6(ii) that

$$g(\Delta^{\mathcal{H}} h, h) = \frac{1}{2}\|T\|^2 - \frac{1}{2}\mathrm{div}\, \alpha^{\mathcal{H}} + \tau^{\mathcal{V}\mathcal{H}}$$

and the result follows.

We observe that the result (ii) in the last corollary is clearly more general than Corollary 2.27 in [8]. In any case, this result is an immediate consequence of Corollary 2.11(ii).

Furthermore, we must note that if $\mathscr{V}$ is a foliation satisfying $\operatorname{div}_{\mathscr{H}} \alpha^{\mathscr{V}} = 0$, then it is a weak-harmonic distribution, but it is not necessarily a harmonic distribution.

The harmonic foliations of codimension one are also analyzed in [8], where the following result is obtained.

If $\mathscr{V}$ is a transversally orientable foliation of codimension one on a compact and oriented Riemannian manifold $\mathscr{M}$ with non-negative Ricci-curvature then:

(i) If the Ricci operator is positive for at least one point in $\mathscr{M}$, the foliation $\mathscr{V}$ is not harmonic.

(ii) If $\mathscr{V}$ is harmonic, then $\mathscr{V}$ is totally geodesic.

Since a harmonic distribution is a foliation with minimal leaves, this result is a consequence of the formula

$$\int_{\mathscr{M}} \tau^{\mathscr{V}\mathscr{H}} * 1 = -\frac{1}{2} \int_{\mathscr{M}} \|T\|^2 * 1$$

which is true if $\dim \mathscr{H} = 1$ and $\mathscr{V}$ has the property F_1 (Corollary 2.11(iv)). Furthermore, the result can be stated without the hypothesis of integrability of $\mathscr{V}$ (nevertheless, we must note that harmonicity implies integrability), and considering the assumption on $\tau^{\mathscr{V}\mathscr{H}}$, instead of that on the Ricci-curvature.

On the other hand, part (vii) in Corollary 3.7 can be considered as a version of this result for non-compact manifolds.

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