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PRESENTATIONS OF MODULES WHEN IDEALS NEED NOT BE PRINCIPAL

BY

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Dedicated to the memory of Irving Reiner, with gratitude for his caring and encouragement

We extend elementary divisor theory by studying presentations of modules over a class of rings that includes coordinate rings of affine curves, and the orders over Dedekind domains studied in integral representation theory. As an application, we answer a question of Nakayama about the uniqueness of the diagonal form of matrices over noncommutative principal ideal domains.

As a further application of our methods, we extend the Drozd Cancellation Theorem to modules over the rings we work with.

Let f and g: $P \twoheadrightarrow U$ be *presentations* of a module over a ring Λ. In other words, P is a projective module, and f and g are surjective Λ-module homomorphisms. (Modules are always finitely generated in this paper, unless otherwise stated.) As in matrix theory, we say that f is *equivalent* to g (notation: $f \sim g$) if there exist automorphisms α and β of P and U, respectively, such that $\beta f \alpha = g$.

Using this terminology, we can restate the main result of elementary divisor theory in the following form: If Λ is a commutative PID (principal ideal domain) and there is a presentation $P \twoheadrightarrow U$, then any two presentations of U by P are equivalent to each other. We say, more briefly, U is *uniquely presentable* by P.

To see the relation between this and more conventional statements of the elementary divisor theorem, let g: $\Lambda^n \twoheadrightarrow U$ be a presentation of an arbitrary Λ-module. We want to "diagonalize" g, that is, show that g is equivalent to a direct sum of presentations of cyclic modules. Assuming that we already know that U is a direct sum of cyclic modules, it is not hard to show that U is a direct sum of n cyclic modules. Say $U = U_1 \oplus \cdots \oplus U_n$. For each i there is a

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presentation $\Lambda \twoheadrightarrow U_i$ since U_i is cyclic, so taking the direct sum of these presentations gives a presentation f: $\Lambda^n \twoheadrightarrow U$. By unique presentability we have $g \sim f$, as desired.

Very little seems to be known about the structure of presentations of modules over rings with nonprincipal ideals. (And the problem of Nakayama, discussed below, shows that the basic facts are not yet completely known for noncommutative PID's.) Is it common for modules to be uniquely presentable? Is it common for modules to have infinitely many inequivalent presentations?

The rings Λ whose module presentations we study are a class of module-finite algebras, over a commutative noetherian ring R of Krull dimension 1, such that Λ is an R-order in a semisimple artinian ring. This class includes (when R is a Dedekind domain) the orders that occur in integral representation theory, and (when $\Lambda = R$) coordinate rings of affine curves. We call them *ring-order algebras*, and give a precise definition in Section 2, together with an example, due to Heinzer and Huneke, of a ring of algebraic numbers that is a ring-order but not a classical order over a Dedekind domain.

One known result is that, if Λ is a Dedekind domain, then every (finitely generated) Λ-module U is uniquely presentable by every projective module P that can be mapped onto U. This is proved in [L '66, 1.9] but is actually implicit in Steinitz's 1911 paper [S '11]. R. B. Warfield [W '78, Theorems 9 and 10], improving on results of Fitting, showed that if Λ is a ring with 1 in its stable range and P is free, then unique presentability holds for every U that can be presented by P. For ring-order algebras without 1 in their stable range, his results show that unique presentability holds if P is free and of rank at least 2 greater than the minimal number of generators of U. On the other hand, H. Byun [By '84] has studied presentations by projective modules whose rank equals the minimal number of generators of U. She showed the existence of many non-uniquely-presentable modules of finite length over Dedekind-like rings, a class of commutative ring-orders that includes some integral group rings and some coordinate rings of affine curves. Suitable such modules can have either finitely many or infinitely many inequivalent presentations, and are the motivating examples for the theory constructed below.

Consider a presentation $P \twoheadrightarrow U$ of a module over any ring-order R-algebra Λ. For every maximal ideal $\mathfrak{m}$ of R, the localized ring $\Lambda_{\mathfrak{m}}$ has 1 in its stable range. So, by a slight modification of Warfield's results $U_{\mathfrak{m}}$ is uniquely presentable by $P_{\mathfrak{m}}$. Thus the question to be addressed is: what are the "non-local" invariants of presentations of modules?

Our first step is to observe that the collection of all equivalence classes of presentations $P \twoheadrightarrow U$, with P and U fixed, has an algebraic structure.

Given presentations f, g, h: $P \twoheadrightarrow U$ there is a presentation s: $P \twoheadrightarrow U$ such that

$$(0.1) \qquad f \oplus s \sim g \oplus h.$$

If Λ is commutative, we show that the equivalence class $[s]$ of s is uniquely determined by (0.1), and we write

$$[s] = [g] + [h] \quad (\text{sum with respect to } f). \tag{0.2}$$

It then follows easily that the set of all equivalence classes of presentations $P \twoheadrightarrow U$ becomes a group with respect to this addition; and we denote this group $\operatorname{pres}_f(P, U)$ to emphasize its dependence on the arbitrarily selected f.

If Λ is noncommutative, the equivalence class of s can fail to be uniquely determined by (0.1). So we define $[s] = [t]$, to mean that

$$f \oplus s \sim f \oplus t \tag{0.3}$$

and say that s and t are *stably equivalent* presentations of U by P. The set of stable equivalence classes of presentations of modules stably isomorphic to U by modules stably isomorphic to P forms a group with respect to the addition in (0.1) and (0.2), and we again denote this group by $\operatorname{pres}_f(P, U)$. As in K-theory, stable equivalence reduces to ordinary equivalence, except in some "low rank" situations (and, when Λ is commutative, it always reduces to ordinary equivalence).

If P is a progenerator and $\ker f$ is faithful, the group $\operatorname{pres}_f(P, U)$ turns out to be a homomorphic image of $\mathbf{K}_1(\Lambda/I\Lambda)$, where I is a conductor ideal from a maximal order to Λ (Theorem 4.11). We deduce this from a Mayer-Vietoris sequence (Theorem 4.8) that relates what we call the "genus class group" of a presentation to the locally free class group of a maximal order Γ containing Λ and the groups $\mathbf{K}_1(\Gamma)$ and $\mathbf{K}_1(\Gamma/I\Gamma)$. We now describe some consequences of these facts.

Our most detailed results occur in the commutative case. Here we show that, if U has finite length, $\operatorname{pres}_f(P, U)$ is a torsion group with exponent dividing the rank of P, even if Λ has infinite residue fields (Theorem 6.5). In less technical language, if $f, g\colon P \twoheadrightarrow U$ are presentations and rank $P = n$, then $f^n, g^n\colon P^n \twoheadrightarrow U^n$ are equivalent presentations.

If Λ is a finitely generated algebra over an algebraically closed field of characteristic zero, we prove that *every module of finite length is uniquely presentable by every projective module that presents it*. This becomes false for algebraically closed fields of characteristic $p \neq 0$.

On the other hand, if U is not of finite length and Λ has infinite residue fields, then $\operatorname{pres}_f(P, U)$ can contain elements of both finite and infinite order. In fact, we construct an example where Λ is an integral domain, finitely generated as an algebra over an arbitrary field k, and U is a torsion-free Λ-module of rank 1 such that $\operatorname{pres}_f(\Lambda^2, U) \cong k^*$ (the multiplicative group of k).

If Λ is one of the orders that occur in integral representation theory, then $\Lambda/I\Lambda$ is a finite ring, hence $\mathbf{K}_1(\Lambda/I\Lambda)$ is a finite group. We conclude that every $\operatorname{pres}_f(P, U)$ is a finite group. In fact the orders of thee presentation class groups have a uniform bound $n = n(\Lambda)$ that is independent of P and U. If Λ is noncommutative, $\operatorname{pres}_f(P, U)$ can consist of stable rather than actual presentation classes. But in the presence of a suitable Eichler condition, our uniform bound $n = n(\Lambda)$ applies to actual, rather than stable presentation classes. When this Eichler condition fails, every presentation set $P \twoheadrightarrow U$ remains finite, but a uniform bound can fail to exist.

Nakayama's question. Let Λ be a noncommutative PID (every left ideal and every right ideal is principal), and let $\mathbf{A}$ be an $m \times n$ matrix over Λ. Then, for suitable invertible matrices $\mathbf{P}$ and $\mathbf{Q}$ over Λ, the matrix $\mathbf{D} = \mathbf{PAQ}$ is a diagonal matrix, $\mathbf{D} = \operatorname{diag}(d_1, d_2, \dots)$. Three obvious invariants for the equivalence class of $\mathbf{A}$ are m, n and the isomorphism class of the left Λ-module

$$U = \Lambda^n/\Lambda^m\mathbf{A} \cong \bigoplus_i \Lambda/\Lambda d_i$$

presented by $\mathbf{A}$. If Λ is commutative, the elementary divisor theorem states that these invariants suffice to determine the equivalence class of $\mathbf{A}$. For noncommutative Λ a complete set of invariants is not known. Asano [A '38, pp. 27-28] proved the "stability" result that the invariants m, n and U determine the equivalence class of the matrix $\mathbf{A} \oplus \mathbf{I}_1$ where $\mathbf{I}_1$ is the 1×1 identity matrix. Then Nakayama wrote the paper [N '38] whose main purpose was to lament the fact that a complete set of invariants for the equivalence class of $\mathbf{A}$ was not known.

Examples of the failure of m, n and U to determine the equivalence class of $\mathbf{A}$ already occur when $\mathbf{A}$ is a 1×1 matrix, and such examples seem to exist for most noncommutative PID's Λ. For specific examples see [LR '74, 4.6] and Example 5.11 of the present paper.

In Section 2 we prove that, if Λ is any PID, module-finite over its center, and $\mathbf{A}$ has rank ≥ 2, then the equivalence class of $\mathbf{A}$ is determined by m, n, and U, just as in the commutative case. (This was proved, for Λ the ring of integral quaternions, in [K '87].)

This result also holds for noncommutative PID's that are not module-finite over their center, as we show in [GLO '88].

Outline. This paper is organized as follows. In Section 1, *Genus and presentation class groups* we prove that $\operatorname{pres}_f(P, U)$ is a group, in fact, a subgroup of the "genus class group" $\mathcal{G}(f)$, consisting of all stable equivalence classes of presentations locally equivalent to f. This resemblance to module theory is no accident since the category of presentations of Λ-modules turns out to be equivalent to a category of modules over the ring $T_2(\Lambda)$ of lower triangular 2×2 matrices over Λ. It follows easily from Bass's cancellation theorem that, for every f, $\mathscr{G}(f) \cong \mathscr{G}(f \oplus f)$; and this latter group consists of actual (rather than stable) presentation classes, a fact we use often in the rest of the paper.

In Section 2, *Ring-order algebras, regular localizations, Drozd condition*, we introduce ring-order algebras and discuss some relevant localizations with respect to sets of regular elements of R. The main result is the Drozd Cancellation Theorem, a sufficient condition for direct-sum cancellation to hold in the genus of a presentation $f\colon P \twoheadrightarrow U$, hence for $\mathscr{G}(f)$ to consist of actual presentation classes. To state this condition, let A denote the semisimple artinian ring in which Λ is an order.

Our theorem states that cancellation holds in $\mathscr{G}(f)$ if every composition factor of the A-modules $A \otimes_R P$ and $A \otimes_R U$ either occurs at least twice or has a commutative endomorphism ring. (We do not know whether the hypothesis on $A \otimes U$ is really needed.)

This sharpens the above-mentioned consequence of the Bass Cancellation theorem (cancellation always holds in $\mathscr{G}(f \oplus f)$) in two ways: The Drozd condition is always satisfied when Λ, hence A, is commutative; and when Λ is noncommutative, the Drozd condition is often satisfied by indecomposable presentations f.

We also prove a Drozd cancellation for modules over ring-order algebras that generalizes the version of the Drozd Cancellation Theorem for modules given in [G '87, 5.7].

In Section 3, *Maximal orders, Nakayama's question*, we let Γ be a maximal ring-order R-algebra in a simple artinian ring A, and let $f\colon P \twoheadrightarrow U$ be a presentation of a Γ-module. The purpose of Section 3 is to prove that if $\ker f$ has uniform rank ≥ 2, then U is uniquely presentable by P. When Λ is a PID, this becomes our answer to Nakayama's question. But the more general result is needed, independently of Nakayama's question, because the proofs of our main theorems, in Section 4, require unique presentability over Γ. The groups $\mathrm{pres}_f(P, U)$ are of no interest over maximal orders since, as we show, they always consist of a single element.

Section 4, *Restricted genus of a presentation*, is the guts of this paper. Let Γ be a maximal order containing Λ, and let r.gen(f) denote the "restricted genus" of f, that is, the set of all equivalence classes of presentations $g\colon S \twoheadrightarrow V$ in the genus of $f\colon P \twoheadrightarrow U$ such that $1 \otimes g\colon \Gamma \otimes S \twoheadrightarrow \Gamma \otimes V$ is equivalent to $1 \otimes f\colon \Gamma \otimes P \twoheadrightarrow \Gamma \otimes U$ as presentations of Γ-modules. Our main results are: (i) the one-to-one correspondence, when appropriate unique presentability holds over Γ, between the presentation classes in r.gen(f) and certain double cosets of units of the localization Γ_π, π the finite set of maximal ideals of R that contain the conductor ideal I; (ii) our Mayer-Vietoris sequence for the genus class group of a presentation; and (iii) our explicit formula for $\mathrm{pres}_f(P, U)$ as a homomorphic image of $\mathbf{K}_1(\Lambda/I\Lambda)$. The idea of the proofs is to adapt the description of the restricted genus of a module, given in [G '87] to presentations.

Section 5, *Global fields*, proves our finiteness results for orders over Dedekind domains in global fields.

Section 6, *Commutative case*, completes this paper.

1. Genus and presentation class groups

In this section Λ denotes a module-finite algebra over a commutative Noetherian ring R of Krull dimension ≤ 1.

The *genus* of a Λ-module M, denoted by $\operatorname{gen}(M)$, means the collection of all Λ-modules N such that $N_\mathfrak{m} \cong M_\mathfrak{m}$ for all maximal ideals of R. To see that *this definition is independent of the particular coefficient ring R* being used, let S be the center of Λ. Since ${}_RS$ is finitely generated, $S_\mathfrak{m}$ is a semilocal ring for every maximal ideal $\mathfrak{m}$ of R. Since local isomorphism implies global isomorphism, for modules over noetherian local rings [GW '76], Λ-modules in the same genus with respect to S are in the same genus with respect to R.

1.1 LEMMA. *Let M be a left Λ-module and $E = E(M)$, the endomorphism ring of M.*

(i) *The functor* $\hom_\Lambda(M, \ldots)$ *is a category equivalence between* $\operatorname{div}(M)$, *the category of all direct summands of finite direct sums of copies of M and the category of all projective left E-modules. Its inverse is $M \otimes_E (\ldots)$.*

(ii) $\operatorname{gen}(M) \subseteq \operatorname{div}(M)$.

(iii) *The functor in* (i) *provides a bijection between all isomorphism classes of Λ-modules in the genus of M and all isomorphism classes of left E-modules in the genus of E itself.*

Proof. Statement (i) is a well-known observation of Dress [D '69, p. 985], (ii) is proved in [G '84, 3.1]; and (iii) holds because of (ii) and the fact that the two functors in (i) localize properly. These statements all hold without the assumption that R has dimension ≤ 1. □

1.2 LEMMA. *Let M be a Λ-module. Then for every $X \in \operatorname{gen}(M^n)$ $(n \geq 2)$ we have $X \cong M^{n-1} \oplus M'$ with $M' \in \operatorname{gen}(M)$.*

Proof. Let $E = E(M)$, and note that E is a module-finite R-algebra. After applying the functor $\hom_\Lambda(M, \ldots)$ in Lemma 1.1, we can assume that M and X are projective E-modules, with $M = E$ and $X \in \operatorname{gen}(E^n)$. Then, for every maximal ideal of R, $X_\mathfrak{m}$ has a direct summand isomorphic to $(E_\mathfrak{m})^n$. Since R has Krull dimension ≤ 1 and $n \geq 2$, it follows from the version of Serre's theorem proved in [Sw '68, 11.2] that $X \cong E \oplus X'$ for some X'. Since direct-sum cancellation holds for Λ-modules when R is local [E '73], [GW '76], we have $X' \in \operatorname{gen}(M^{n-1})$, and induction now completes the proof. □

1.3 LEMMA. *Let M be a Λ-module, and let N, H, $H_1 \dots H_n$ be Λ-modules in* gen(M). *Then*

$$M \oplus H_1 \oplus \cdots \oplus H_n \cong N \oplus H_1 \oplus \cdots \oplus H_n \Rightarrow M \oplus H \cong N \oplus H$$

Proof. We want to cancel every H_i from the isomorphism

$$(1.3.1) \qquad M \oplus H_1 \oplus \cdots \oplus H_n \oplus H \cong N \oplus H_1 \oplus \cdots \oplus H_n \oplus H$$

By Lemma 1.1 we can suppose that each direct summand in (1.3.1) is a projective $E = E(M)$-module in the genus of E. Since R has dimension ≤ 1 and two summands remain on each side after the desired cancellation, Bass's Cancellation Theorem [B '68, 3.5] now completes the proof. □

We can now extend the well-known notion of "genus class group" so that it applies where we shall need it.

1.4 DEFINITIONS. Let M be a Λ-module. We say that modules N and N' in gen(M) are *stably isomorphic*, and write $[N] = [N']$, if $N \oplus H \cong N' \oplus H$ for some $H \in$ gen(M). This is an equivalence relation by Lemma 1.3.

Let $\mathcal{G}(M)$ be the collection of stable isomorphism classes $[N]$ with $N \in$ gen(M). For $S, G, H \in$ gen(M) we define

$$(1.4.1) \qquad [S] = [G] + [H] \quad \text{(sum with respect to } M\text{)}$$

to mean that

$$(1.4.2) \qquad M \oplus S \cong G \oplus H.$$

By Lemmas 1.2 and 1.3 this addition is well-defined and makes $\mathcal{G}(M)$ into an abelian group in which $[M] = 0$. We call $\mathcal{G}(M)$ the *genus class group* of M.

Let M and N be Λ-modules. Then there is a natural homomorphism of abelian groups

$$(1.5) \qquad \nu\colon \mathcal{G}(M) \to \mathcal{G}(M \oplus N) \text{ given by } [H] \to [H \oplus N]$$

1.6 LEMMA. *Suppose $N \in$ gen(M^n) with $n \geq 1$. Then ν is an isomorphism. Moreover, $\mathcal{G}(M \oplus N)$ consists of actual, rather than (merely) stable isomorphism classes.*

Proof. Take any $X \in$ gen($M \oplus N$). By repeated use of Lemma 1.2, we have $X \cong H \oplus N$ with $H \in$ gen(M); so ν is a surjection The proof is completed by Lemma 1.3. □

1.7 COROLLARY. *Let M be an Λ-module, and $N \in \operatorname{gen}(M)$. Then*

(i) $\mathcal{G}(M) \cong \mathcal{G}(N)$, *and*

(ii) *For every $n \geq 2$, $\mathcal{G}(M) \cong \mathcal{G}(M^n)$ via ν, and $\mathcal{G}(M^n)$ consists of actual isomorphism classes.*

Proof. By the Lemma, both $\mathcal{G}(M)$ and $\mathcal{G}(N)$ are isomorphic to $\mathcal{G}(M \oplus N)$. Statement (ii) also follows immediately from the Lemma. □

We postpone further discussion of when $\mathcal{G}(M)$ consists of actual isomorphism classes to the discussions of the Drozd and Eichler conditions in Sections 2 and 5. In particular, in Section 2, we prove that $\mathcal{G}(M)$ always consists of isomorphism classes when Λ is any commutative ring-order. We do not know whether this holds for arbitrary noetherian commutative rings Λ of dimension 1.

1.8 *Remarks on $T_2(\Lambda)$.* It is often easier to understand the category of presentations—or more generally, homomorphisms—of Λ-modules, if one realizes that this category is equivalent to a category of modules over a different ring. We learned the following device from [GR '78, p. 61].

Let $T_2(\Lambda)$ be the ring of 2×2 lower triangular matrices over Λ.

For each homomorphism $f\colon P \to U$ of left Λ-modules we define an associated left $T_2(\Lambda)$-module $M(f)$ as follows. As a left Λ-module, $M(f) = P \oplus U$. We make $M(f)$ into a $T_2(\Lambda)$-module by letting the matrix units e_{11} and e_{22} act on P and U, respectively, and by letting $e_{21} \cdot p = f(p)$, for $p \in P$. More completely, we define

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} p \\ u \end{bmatrix} = \begin{bmatrix} ap \\ b \cdot f(p) + cu \end{bmatrix}$$

Conversely, let Y be a left $T_2(\Lambda)$-module. We get a homomorphism $f\colon P \to U$ of left Λ-modules by letting $P = e_{11}Y$, $U = e_{22}Y$, and $f =$ multiplication by e_{21}.

It is easy to verify that for Λ-homomorphisms f and g, we have

$$f \sim g \Leftrightarrow M(f) \cong M(g) \quad (\text{isomorphism of } T_2(\Lambda)\text{-modules}).$$

In other words, the functors in the two preceding paragraphs are "inverses" of each other.

It is also easy to verify that, for Λ-homomorphisms f and g and a maximal ideal $\mathfrak{m}$ of R, we have $f_\mathfrak{m} \sim g_\mathfrak{m}$ (as homomorphisms of $\Lambda_\mathfrak{m}$-modules) if and only if $M(f)_\mathfrak{m} \cong M(g)_\mathfrak{m}$ (isomorphism of $T_2(\Lambda_\mathfrak{m})$-modules).

1.9 FUNDAMENTAL DEFINITIONS. Let $f\colon P \to U$ and $g\colon Q \to V$ be homomorphisms of Λ-modules. We call f *equivalent* to g, and write $f \sim g$, if there exist Λ-module isomorphisms $\alpha\colon Q \cong P$ and $\beta\colon U \cong V$ such that $\beta f \alpha = g$.

By analogy with modules, we define the *genus* of a homomorphism $f\colon P \twoheadrightarrow U$ of Λ-modules, denoted by gen(f), to be the collection of all Λ-homomorphisms g such that $g_{\mathfrak{m}} \sim f_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R. This definition is independent of the particular coefficient ring R being used, because we are really talking about the genus of the $T_2(\Lambda)$-modules $M(f)$ and $M(g)$ defined in Remarks 1.8.

Let $g \in \operatorname{gen}(f)$. We say that f and g are *stably equivalent*, and write $[f] = [g]$ if there is a homomorphism $h \in \operatorname{gen}(f)$ such that $f \oplus h \sim g \oplus h$. To show that stable equivalence is indeed an equivalence relation we prove:

(1.9.1) Let f be a Λ-module homomorphism, and let $g, h, h_1, \ldots, h_n$ be Λ-module homomorphisms in gen(f). Then

$$f \oplus h_1 \oplus \cdots \oplus h_n \sim g \oplus h_1 \oplus \cdots \oplus h_n \Rightarrow f \oplus h \sim g \oplus h.$$

Changing to $T_2(\Lambda)$-modules shows that this is a special case of Lemma 1.3.

Let $\mathcal{G}(f)$ be the collection of stable isomorphism classes $[g]$ with $g \in \operatorname{gen}(f)$. For $s, g, h \in \operatorname{gen}(f)$ we define

$$[s] = [g] + [h] \quad \text{(sum with respect to } f\text{)} \tag{1.9.2}$$

to mean that

$$f \oplus s \sim g \oplus h \tag{1.9.3}$$

This addition is well-defined and makes $\mathcal{G}(f)$ into an abelian group, the *genus class group of* f, in which $[f] = 0$. This holds because changing to $T_2(\Lambda)$-modules shows that, in effect, $\mathcal{G}(f) = \mathcal{G}(M(f))$, an identification we shall freely make from now on.

We say that gen(f) (for $f\colon P \to U$), is *fully stable* if direct-sum cancellation holds in gen(f), gen(P), and gen(U). This is equivalent to saying that $\mathcal{G}(f)$ consists of actual, rather than stable equivalence classes of homomorphisms, and $\mathcal{G}(P)$ and $\mathcal{G}(U)$ consist of isomorphism classes, rather than stable isomorphism classes, of modules.

Finally, let $f\colon P \twoheadrightarrow U$ be a presentation of a Λ-module. We define the *presentation class group* $\operatorname{pres}_f(P, U)$ to be the subgroup of $\mathcal{G}(f)$ consisting of all stable equivalence classes $[g]$ of presentations $g\colon S \twoheadrightarrow V$ where $[S] = [P]$ in $\mathcal{G}(P)$ and $[V] = [U]$ in $\mathcal{G}(U)$. This definition is justified by the following result.

1.10 PROPOSITION. *Let f: $P \twoheadrightarrow U$ be a presentation of a Λ-module. Then*:

(i) *Every presentation g: $P \twoheadrightarrow U$ is in* gen(f).

(ii) *When full stability holds in* gen(f), $\mathrm{pres}_f(P, U)$ *consists of all equivalence classes of presentations $P \twoheadrightarrow U$.*

Proof. To prove (i) we can suppose that R is local. Then Λ has 1 in its stable range [W '80, 3.4]. So g is equivalent to f by [W '78, Theorem 4]. The point of (ii) is that full stability guarantees that, if g and h are presentations $P \twoheadrightarrow U$, then so are any representatives of $[g] + [h]$ and $-[g]$. □

Now let f and g be presentations of Λ-modules, and consider the natural homomorphism

$$\nu\colon \mathscr{G}(f) \to \mathscr{G}(f \oplus g) \quad \text{given by } [h] \to [h \oplus g] \tag{1.11}$$

It is obvious that ν takes $\mathrm{pres}_f(P, U)$ into $\mathrm{pres}_{f\oplus g}(P \oplus S,\ U \oplus V)$.

1.12 COROLLARY. *Let f: $P \twoheadrightarrow U$ be a presentation of a Λ-module, and let* $(g\colon Q \twoheadrightarrow V) \in \mathrm{gen}(f)$. *Then* $\mathscr{G}(f) \cong \mathscr{G}(g)$ *and* $\mathscr{G}(f) \cong \mathscr{G}(f^n)$ $(\forall n)$. *Moreover*:

(i) $\mathrm{pres}_f(P, U) \cong \mathrm{pres}_g(Q, V)$.

(ii) $\mathrm{pres}_f(P, U) \cong \mathrm{pres}_{f^n}(P^n, U^n)$ *via* ν.

(iii) *Full stability holds in* gen(f^n) *for all* $n \geq 2$.

Proof. The isomorphisms involving $\mathscr{G}(f)$ are special cases of Lemma 1.6. Statement (iii) is a consequence of Lemma 1.6, applied to both Λ-modules and presentations of Λ-modules. Similarly, full stability holds in gen($f \oplus g$). Statement (i) is now easily proved by showing that $\mathrm{pres}_f(\cdots)$ and $\mathrm{pres}_g(\cdots)$ are both isomorphic to $\mathrm{pres}_{f\oplus g}(\cdots)$. The proof of (ii) is similar. □

If full stability does not hold in gen(f), we do not know whether $\mathrm{pres}_f(P, U)$ consists of, or is generated by, presentations $P \twoheadrightarrow U$.

We close this section with a variation of a result of Warfield, which we discuss after proving it. Recall that a *progenerator* is a projective Λ-module S such that there is a surjection $S^d \twoheadrightarrow \Lambda$ for some d.

1.13 LEMMA. *Let f, g: $P \twoheadrightarrow U$ be presentations of a Λ-module, and let S be any progenerator. Then $(f, 0)$ and $(g, 0)$ are stably equivalent presentations $P \oplus S \twoheadrightarrow U$.*

Proof. By [W '78, Lemma 1], $(f, 0)$ and $(g, 0)$ are equivalent presentations $P \oplus P \twoheadrightarrow U$. Since S is a progenerator, P is a direct summand of S^n, for some n. So we have equivalent presentations

$$(f, 0), (g, 0)\colon P \oplus S^n \twoheadrightarrow U.$$

So $(f,0)^n$ and $(f,0)^{n-1} \oplus (g,0)$ are equivalent presentations $(P \oplus S)^n \twoheadrightarrow U$. By (1.9.1) we have $[(f,0)] = [(g,0)]$ as desired. □

1.14 THEOREM (Triviality of inefficient presentations). *Let f: $P \twoheadrightarrow U$ be a presentation of a Λ-module, and suppose that $P = P' \oplus S$, where P' can be mapped onto U, and S is a progenerator. Then* $\operatorname{pres}_f(P, U) = \{0\}$.

Proof. After replacing f by f^2 we can assume that full stability holds in gen(f). Let g': $P' \twoheadrightarrow U$ be as in the hypotheses. Then

$$(g',0): P' \oplus S = P \twoheadrightarrow U$$

is a presentation of U. Let h be an arbitrary presentation $P \twoheadrightarrow U$. By Lemma 1.13, the presentations

$$(g',0,0) \text{ and } (h,0): P' \oplus S \oplus S = P \oplus S \twoheadrightarrow U$$

are stably equivalent, hence (by full stability) equivalent. Adding g' to this equivalence and regrouping terms gives

$$(g',0)^2 \sim (g',0) \oplus h \quad (\text{equivalent presentations } P^2 \twoheadrightarrow U^2)$$

so, by full stability in gen(f), we have $(g',0) \sim h$. In other words, every presentation h: $P \twoheadrightarrow U$ is equivalent to the fixed presentation $(g',0)$, as desired. □

1.15 *Example.* It is easy to see that the hypothesis of Theorem 1.14 does not imply that U is uniquely presented by P. There exist maximal orders Λ in simple algebras whose projective modules do not satisfy direct-sum cancellation [Sw '62]. Say $P' \oplus S \cong P'' \oplus S$ with P', P'', S projective and $P' \ncong P''$ (and S is a progenerator since every projective module over a maximal order in a simple algebra is a progenerator). The coordinate projections $P' \oplus S \twoheadrightarrow S$ and $P'' \oplus S \twoheadrightarrow S$ are inequivalent presentations of S because they do not have isomorphic kernels. □

Our ring Λ has 2 in its stable range [W '80, 3.4]. So, by [W '78, Theorem 9], if there is a presentation $\Lambda^n \twoheadrightarrow U$, then U is uniquely presented by Λ^{n+2}. Warfield actually states a stronger result: any two presentations f, g: $\Lambda^{n+2} \twoheadrightarrow U$ are *right equivalent* in the sense that $g = f\alpha$ for some automorphism α of Λ^{n+2}. But the type of direct-sum cancellation used in his proof only establishes ordinary equivalence. However, he comments in a later paper that using the stronger form of cancellation called *n-substitution* in [W '80] yields the stronger result.

Our methods do not seem to yield any information about this stronger form of equivalence.

2. Ring-order algebras, regular localizations, Drozd condition

In this section we introduce ring-order algebras, extend some familiar tools to work with them, and give an example of a subring of a quadratic number field that is a ring-order but not an order in the sense of integral representation theory. The principal result of this section is a sufficient condition, called the Drozd Condition, for full stability to hold in the genus of a presentation $f\colon P \twoheadrightarrow U$ of a Λ-module, where Λ is a ring-order algebra.

We begin with some *notation that will remain fixed through the rest of this paper*, unless otherwise stated.

2.1 *Notation* (Λ, Γ, R, A). Let Λ be an *R-order* in a semisimple artinian ring A, that is, Λ is a module-finite algebra over a central subring R, and $A = Q(R) \cdot \Lambda$, where $Q(R)$ denotes the classical quotient ring of R.

We call Λ a *ring-order R-algebra* if R is noetherian of Krull dimension ≤ 1 and Λ is contained in some maximal R-order Γ in A.

Note that the central subring R cannot have nilpotent elements $\neq 0$, because such elements would generate nilpotent ideals of A. Moreover:

(2.1.1) The integral closure $\tilde{R}$ of R in $Q(R)$ is contained in Γ.

To prove this, let $x \in \tilde{R}$. Then $R[x]$, and hence $R[x] \cdot \Gamma$, is a module-finite R-algebra. Since the second of these contains the maximal R-order Γ, we have $R[x] \cdot \Gamma = \Gamma$, hence $x \in \Gamma$.

Since R, hence ${}_R\Gamma$ is noetherian, (2.1.1) shows that $\tilde{R}$ is a finitely generated R-module. So R is itself a ring-order R-algebra. As in [HL], we call a commutative ring R a *ring-order* if it is a ring-order algebra over itself; in other words, if R is a noetherian commutative reduced ring of dimension ≤ 1 with finite normalization.

The reason for the name "ring-order algebra" is that the most familiar examples are the orders over Dedekind domains studied in integral representation theory, and the (commutative) coordinate rings of affine curves studied in algebraic geometry [ZS '58, p. 267].

Moreover, being a ring-order algebra is a ring-theoretic property of Λ, in the sense that Λ is a ring-order algebra over its center and the semisimple artinian ring $A = Q(R)\Lambda$ is the Goldie quotient ring of Λ.

We often make the following nontriviality assumption.

(2.1.2) Λ is an indecomposable ring and is not artinian. In particular, no nonzero R-submodule of Γ has finite length, and no minimal prime ideal of R is maximal.

To prove the assertions in the second sentence, note that if Γ has an R-submodule of finite length, so does Λ (since ${}_R\Gamma$ is finitely generated and $\Lambda \subseteq Q(R)\Lambda = Q(R)\Gamma$). Moreover, if Rx ($x \in \Lambda$) has finite length, then so does the R-module Λx, which is a sum of homomorphic images of Rx since ${}_R\Lambda$ is finitely generated. In particular, Λx has finite length as a Λ-module. So the socle soc(Λ) is nonzero. Since $A = Q(R)\Lambda$ is semisimple artinian, Λ is a semiprime ring. Since Λ is also noetherian, its socle is generated by a central idempotent [CR '80], contrary to our indecomposability hypothesis in (2.1.2).

Finally, let $\mathfrak{p}$ be any minimal prime of R, and let $\mathfrak{q}$ be the product of all other minimal primes of R. Then $\mathfrak{q} \neq 0$ (else $\mathfrak{p}$ contains a minimal prime other than itself); and $\mathfrak{p}\mathfrak{q} = 0$ since R has no nonzero nilpotent elements. If $\mathfrak{p}$ were a maximal ideal, then the $R/\mathfrak{p}$-module $\mathfrak{q}$ would have finite length. But we have just shown that no nonzero R-submodule of Γ (in particular of R) has finite length.

2.2 *Notation* (I, π). Since ${}_R\Gamma$ is finitely generated and contained in $Q(R)\Lambda$, there is a *conductor* ideal I from Γ to Λ, that is, an ideal I of R such that $I\Gamma \subseteq \Lambda$, and such that I contains *regular elements* (non-zero-divisors) of R. Note that we do not require I to be the largest possible conductor ideal from Γ to Λ. We fix such an ideal I for the remainder of this paper, and let π denote the set of all maximal ideals of R that contain I.

Since every minimal prime ideal of a noetherian commutative ring consists of zero-divisors, R/I is noetherian of Krull dimension zero, that is, *R/I is artinian*. Hence *π is a finite set.*

2.3 *Regular localizations.* A *regular localization* of R means a localization of the form $S^{-1}R$ where S consists of regular elements (non-zero-divisors) of R. Let $\mathfrak{m}$ be a maximal ideal of R.

We use the notation $R_{\mathfrak{m}\rho}$ for the *regular* $\mathfrak{m}$*-localization* $S^{-1}R$ where

(2.3.1)
$$S = \{\text{regular elements of } R - \mathfrak{m}\} = R - \mathfrak{m} - \bigcup\{\mathfrak{n} \in \operatorname{minspec} R\}$$

The second equality above results from the fact that, since R is noetherian and has no nilpotent elements $\neq 0$, its set of zero-divisors is the union of all minimal prime ideals of R.

We often find it more convenient to use the regular localization $R_{\mathfrak{m}\rho}$ rather than the ordinary localization $R_\mathfrak{m}$ because the natural map $R \to R_{\mathfrak{m}\rho}$ is always one-to-one, and because $R_{\mathfrak{m}\rho}$ is a subring, with the same identity element, of the classical quotient ring $Q(R)$. To display the precise relation between the regular and ordinary $\mathfrak{m}$-localizations of R, let $Q(R) = \bigoplus_k Q_k$ where each Q_k is a field (because R has no nilpotent elements $\neq 0$). Then

$$(2.3.2)\quad R_{\mathfrak{m}\rho} = R_\mathfrak{m} \oplus \left(\oplus\{Q_k | (Q_k)_\mathfrak{m} = 0\}\right) \quad\text{and}\quad \left(R_{\mathfrak{m}\rho}\right)_\mathfrak{m} = R_\mathfrak{m}.$$

Of course, it can happen that an ordinary localization is a regular localization. For example, let R_π denote the localization $S^{-1}R$, where $S = R - \bigcup\pi$, and let $R_{\pi\rho}$ denote the *regular π-localization* $T^{-1}R$, where T is the set of regular elements in π. Then we have:

2.4 LEMMA. *$R_\pi = R_{\pi\rho}$ when* (2.1.2) *holds and $I \neq R$.*

Proof. Since R is a ring-order, we have $Q(R) = \oplus_k Q_k$ with each Q_k a field. So the set of zero-divisors equals of R equals the union of the minimal prime ideals $\mathfrak{p}_k = \ker(R \to Q_k)$ of R. Thus it suffices to show that, for each k, there is a maximal ideal $\mathfrak{m} \supseteq \mathfrak{p}_k$ such that $\mathfrak{m} \in \pi$. By symmetry it suffices to work with $\mathfrak{p} = \mathfrak{p}_1$; and such an $\mathfrak{m}$ exists if $I + \mathfrak{p} \neq R$. So we suppose that $I + \mathfrak{p} = R$. Then there is an expression $i + p = 1$ with $i \in I$ and $p \in \mathfrak{p}$.

Since the Q_1-coordinate of $\mathfrak{p}$ equals 0, we see that the Q_1-coordinate of i is the identity element e of Q_1. Since e is integral over R ($e^2 = e$), e is an element of the integral closure $\tilde{R}$ of R in $Q(R)$. By (2.1.1), $\tilde{R} \subseteq \Gamma$. Moreover, $Ie \subseteq I\Gamma \subseteq \Lambda$. Therefore $e = ie$ is a nontrivial central idempotent element of Λ, contrary to our indecomposability hypothesis (2.2.1). □

2.5 LEMMA. *Let L and X be submodules of a Λ-module V (not necessarily finitely generated). Suppose that $L_\pi = X_\pi$ (as submodules of V_π) and $L_{\mathfrak{m}\rho} = X_{\mathfrak{m}\rho}$ (as submodules of $V_{\mathfrak{m}\rho}$) for every maximal ideal $\mathfrak{m} \notin \pi$. Then $L = X$.*

Proof. $(L_{\mathfrak{m}\rho})_{\mathfrak{m}} = L_{\mathfrak{m}}$ by (2.3.2). So $L_{\mathfrak{m}} = X_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R, hence $L = X$. □

By a *Λ-lattice* we mean a (finitely generated) submodule of a free Λ-module; equivalently, a Λ-submodule of an A-module. A *full* Λ-lattice in an A-module V means a Λ-submodule M of V such that $AM = V$.

We need to know the precise amount of freedom one has in prescribing the regular localizations of a Λ-lattice. One obviously necessary condition is the "consistency condition" that if L and M are full Λ-lattices in V, then $L_{\mathfrak{m}\rho} = M_{\mathfrak{m}\rho}$ for almost all maximal ideals $\mathfrak{m}$ of R (because there exist regular elements d and e in R such that $dL \subseteq M$ and $eM \subseteq L$). The following theorem shows that, as is known for orders over a Dedekind domain, this necessary condition is also sufficient. The word "strong" refers to the fact that we are prescribing the localizations up to *equality*, not merely up to isomorphism.

2.6 STRONG CONSISTENCY THEOREM. *Let M be a full Λ-lattice in an A-module V, and let F be a finite set of maximal ideals of R. For each maximal ideal $\mathfrak{m} \in F$, let $X(\mathfrak{m})$ be a full $\Lambda_{\mathfrak{m}\rho}$-lattice in V. Then there is a unique full*

Λ-lattice L in V such that

$$(2.6.1) \qquad L_{\mathfrak{m}\rho} = \begin{cases} X(\mathfrak{m}) \text{ if } \mathfrak{m} \in F, \\ M_{\mathfrak{m}\rho} \text{ otherwise.} \end{cases}$$

Proof. The uniqueness assertion follows from Lemma 2.5.

In proving the existence of L we can suppose that F consists of a single maximal ideal $\mathfrak{n}$ and Λ satisfies nontriviality condition (2.1.2). Let X be the Λ-module generated by some finite set of $\Lambda_{\mathfrak{n}\rho}$-generators of $X(\mathfrak{n})$. Then X is a full Λ-lattice in V, and $X_{\mathfrak{n}\rho} = X(\mathfrak{n})$. We find elements d and e in R such that $L = dM + eX$ has the desired properties.

Let D be the finite set of maximal ideals $\mathfrak{m} \neq \mathfrak{n}$ such that $M_{\mathfrak{m}\rho} \neq X_{\mathfrak{m}\rho}$. It suffices to choose d and e satisfying the following conditions:

(2.6.2) $dM_{\mathfrak{n}\rho} \subseteq X_{\mathfrak{n}\rho}$ and $e \equiv 1 \pmod{\mathfrak{n}}$.

(2.6.3) $d \equiv 1 \pmod{\mathfrak{m}}$ and $eX_{\mathfrak{m}\rho} \subseteq M_{\mathfrak{m}\rho}$ when $\mathfrak{m} \in D$.

(2.6.4) d is regular in R (so d becomes a unit in $R_{\mathfrak{m}\rho}$ except for a finite set E of maximal ideals $\mathfrak{m}$) and $e \equiv 1 \pmod{\mathfrak{m}}$ when $\mathfrak{m} \in E - D - \{\mathfrak{n}\}$.

The element d exists, by the Chinese Remainder theorem, once we show that requirements (2.6.2) and (2.6.4) follow from a finite number of congruence relations at distinct maximal ideals. Since $M_{\mathfrak{n}\rho}$ and $X_{\mathfrak{n}\rho}$ are both full $\Lambda_{\mathfrak{n}\rho}$-lattices in the same A-module, there is a regular element y of $\Lambda_{\mathfrak{n}\rho}$ such that $yM_{\mathfrak{n}\rho} \subseteq X_{\mathfrak{n}\rho}$. Since $R_{\mathfrak{n}\rho}/\langle y \rangle$ is an artinian ring, its radical $\mathfrak{n}_{\mathfrak{n}\rho}/\langle y \rangle$ is nilpotent. So $\mathfrak{n}^s_{\mathfrak{n}\rho} \subseteq \langle y \rangle$ for some s. Choose any such s, and then take $d \in \mathfrak{n}^s - \mathfrak{n}^{s+1}$. (Note that $\mathfrak{n}^s \neq \mathfrak{n}^{s+1}$ since, by (2.1.2), the maximal ideal $\mathfrak{n}_{\mathfrak{n}}$ of $R_{\mathfrak{n}}$ is nonzero.) Any such d has the property required in (2.6.2). In order for d to be as in (2.6.4), we must keep d out of all minimal prime ideals. This can be accomplished by choosing a finite number of maximal ideals, and keeping d out of some power of each of them. We already have $d \notin \mathfrak{n}^{s+1}$. This, together with (2.6.3) and conditions of the form $d \equiv 1$ modulo a finite number of additional maximal ideals, constitute the needed Chinese Remainder conditions for the existence of d.

A similar, but easier argument then yields e. $\square$

One application of the preceding theorem is a very simple proof that being a maximal order localizes.

2.7 COROLLARY. *Γ_π is a maximal R_π-order in A.*

Proof. Suppose $\Gamma_\pi \subseteq X$ for some R_π-order X in A. By the theorem, there is a full Λ-lattice L in A such that $L_\pi = X$ and such that $L_{\mathfrak{m}\rho} = \Gamma_{\mathfrak{m}\rho}$ when $\mathfrak{m} \notin \pi$. It is easily checked, locally, that $L^2 = L$, hence L is an R-order in A and $L \supseteq \Gamma$. So, by maximality of Γ we have $L = \Gamma$, hence $X = L_\pi = \Gamma_\pi$ as desired. □

2.8 *Notation* (Structure of Γ_π). Since Γ is a maximal order in A, there is a decomposition $\Gamma = \bigoplus_k \Gamma_k$ where each Γ_k is a maximal order, over its center, in one of the simple ring-direct summands of A. By Corollary 2.7, each $(\Gamma_k)_\pi$ is then a maximal order over its semilocal center. Hence, as shown below, *$(\Gamma_k)_\pi$ is a full matrix ring over a (non-commutative) principal ideal domain $\Delta_k(\pi)$.*

For the proof of the last assertion, and other related facts we need later, we show that *each Γ_k is a maximal order over a Dedekind domain*, namely its center, which we call R_k. By maximality of Γ_k and (2.1.1) the integral closure $\tilde{R}_k$ of R_k in $Q(R_k)$ is contained in the center of Γ_k. Hence $R_k = \tilde{R}_k$ which is a Dedekind domain.

Therefore the assertion about $(\Gamma_k)_\pi$ being a full matrix ring follows from the proof of [R '75, (17.3) and (18.2)], together with the fact that, because of the semilocal center, local module isomorphism is the same as actual isomorphism.

2.9 *Drozd condition.* Let $Q = Q(R)$ so that $A = Q\Lambda$. We say that a Λ-module U satisfies the *Drozd condition* if, whenever a simple A-module S occurs exactly once as a composition factor of $Q \otimes_R U$, the endomorphism ring $E(S)$ is commutative.

The Drozd condition places no restriction on composition factors of $Q \otimes U$ that appear more than once.

Our main stability result so far (Cor. 1.7) has been the consequence of the Bass cancellation theorem: direct-sum cancellation always holds in gen$(U \oplus U)$. We now prove a sequence of cancellation results which show that, over ring-order algebras, stability occurs much more often than this. For example, the Drozd condition is satisfied by $U \oplus \Lambda$ for every faithful Λ-module U. It is also satisfied for many indecomposable Λ-modules U; and if Λ is commutative (see Cor. 2.19), it is satisfied by every Λ-module U.

2.10 DROZD CANCELLATION LEMMA (maximal orders). *Suppose $M \oplus X \cong N \oplus X$ for Γ-lattices M, N, X and suppose that M satisfies the Drozd condition. Then $M \cong N$.*

Proof. We can suppose the maximal order Γ is an indecomposable ring and its center is R. Then, as observed in 2.8, Γ is a maximal order over the Dedekind domain R, in the sense of integral representation theory.

We can now suppose that $A = Q \cdot \Gamma$ is a full matrix ring over a division ring. By [R '75, 21.5], every Γ-lattice is projective and, for every indecomposable Γ-lattice U, the A-module $Q \cdot U$ is a simple module. Moreover, since every localization of Γ at a maximal ideal of R is a full matrix ring over a (noncommutative) PID, it follows that all indecomposable Γ-lattices are in the same genus.

Write M, N and X as direct sums of indecomposable lattices. If M is the direct sum of at least two indecomposables, the desired conclusion follows by Lemma 1.3.

The remaining case is that M is indecomposable. Then, by the Drozd condition, the simple A-module $Q \cdot M$ has a commutative endomorphism ring. But then the endomorphism ring of M is our Dedekind domain R, and R is Morita equivalent to Γ. The desired cancellation now follows by passage from Γ-modules to R-modules, where it is well known. □

2.11 Drozd Cancellation Theorem (Lattices). *Let M be a Λ-lattice that satisfies the Drozd condition. Then direct-sum cancellation holds in the genus of M.*

Proof. Let $f\colon M \to N$ be a homomorphism of Λ-lattices and $Q = Q(R)$. As in integral representation theory, f has a unique natural extension to a homomorphism $QM \to QN$ of $A = Q\Lambda$-modules, since $QM = Q \otimes_R M$. Again calling this extended map f we get a natural embedding $E(M) \subseteq E(QM)$. Consequently we have $E(M) \subseteq E(\Gamma M) \subseteq E(QM)$ (even though $\Gamma M \not\cong \Gamma \otimes_\Lambda M$) and $E(M) \subseteq E(S^{-1}M)E(QM)$ for every multiplicatively closed set S of regular elements of R. We use these identifications without explicit mention.

Let f be as in the preceding paragraph, and suppose that M and N are also modules over some larger subring Λ' of $Q\Lambda$. Then f is also a Λ'-homomorphism. This follows from the discussion in the preceding paragraph. However, it becomes false for modules that are not lattices.

By the *restricted genus* of the Λ-lattice M we mean the collection of all Λ-lattices $N \in \operatorname{gen}(M)$ such that $\Gamma N \cong \Gamma M$ (as Γ-modules or Λ-modules; it does not matter which since M and N are lattices).

For $\alpha \in E^*(\Gamma M_\pi)$ [the invertible elements of $E(\Gamma M_\pi)$] let $M_\alpha = M_\pi \alpha \cap \Gamma M$. Then M_α is in the restricted genus of M, and every Λ-lattice in the restricted genus of M is isomorphic to some M_α. The proof is identical to that given in [G '87, 3.4] for orders over Dedekind domains, if in that proof we interpret M_P ($P \in \pi'$) to be the *regular* P-localization of M. It follows from this same proof that

$$(2.11.1) \quad M_\alpha \cong M \Leftrightarrow \alpha \in \lambda(M) \quad \text{where } \lambda(M) = E^*(M_\pi)E^*(\Gamma M).$$

Note that we are not claiming that $\lambda(M)$ is a group.

Recall, from Notation 2.8, that Γ_π is a direct sum of full matrix rings over the PID's $\Delta_k(\pi)$. We can therefore consider $E(\Gamma M_\pi)$ to be a direct sum of full matrix rings over the rings $\Delta_k(\pi)$. Now we make use of the Drozd condition. For each k, either $\Delta_k(\pi)$ is commutative or the size of the matrices over $\Delta_k(\pi)$ appearing in $E(\Gamma M_\pi)$ is at least 2×2.

Let $\mathscr{E}$ be the subgroup of $E^*(\Gamma M_\pi)$ generated by all coordinatewise elementary transvection matrices. By the Drozd condition and the fact that Γ_π has 1 in its stable range, $\mathscr{E}$ is a normal subgroup of $E^*(\Gamma M_\pi)$. See [B '68, Chap. V, 4.1], noting that SR_2 in the terminology there is what we are calling stable range 1. We claim that

$$(2.11.2) \quad \lambda(M) \cdot \mathscr{E} \subseteq \lambda(M) \quad (\text{in particular, } \mathscr{E} \subseteq \lambda(M)).$$

Let $1 + d$ $(d^2 = 0)$ be a tuple of elementary transvection matrices in $E^*(\Gamma M_\pi)$. For some $t \in R - \bigcup \pi$, we have $td \in E(\Gamma M)$. Since π is the set of all maximal ideals containing I, there exists $s \in R$ such that $st \equiv 1 \pmod{I}$. Then $1 - st \in I$, so

$$(2.11.3) \quad 1 + d = [1 + (1 - st)d](1 + std) \in [1 + IE(\Gamma M_\pi)]E^*(\Gamma M).$$

By normality of $\mathscr{E}$, (2.11.2) is equivalent to

$$(2.11.4) \qquad E^*(M_\pi) \cdot \mathscr{E} \cdot E^*(\Gamma M) \in \lambda(M).$$

Take $\varepsilon = \varepsilon_1 \varepsilon_2 \dots \varepsilon_n \in \mathscr{E}$, where each ε_i has the form $1 + d$ with $d^2 = 0$. Factor each ε_i as in (2.11.3). Since $1 + IE(\Gamma M_\pi)$ is a normal subgroup of $E^*(\Gamma M_\pi)$—the kernel of the homomorphism that reduces all entries in the matrices of $E^*(\Gamma M_\pi)$ modulo I_π times the appropriate coordinate ring $\Delta_k(\pi)$—we see that

$$(2.11.5) \qquad \varepsilon \in [1 + IE(\Gamma M_\pi)]E^*(\Gamma M) \subseteq E^*(M_\pi)E^*(\Gamma M)$$

so (2.11.4) holds, and (2.11.2) is proved.

Now suppose that $M \oplus X \cong N \oplus X$ with N and X in the genus of M. Then

$$\Gamma M \oplus \Gamma X \cong \Gamma N \oplus \Gamma X.$$

So by the Drozd Cancellation Lemma for maximal orders we have $\Gamma M \cong \Gamma N$. So N is in the restricted genus of M. Therefore we have $N \cong M_\alpha$ for some α. Moreover by Lemma 1.3, the relation $M_\alpha \oplus X \cong M \oplus X$ implies that $M_\alpha \oplus M \cong M \oplus M$.

Applying (2.11.1) to $M \oplus M$ in place of M we get

$$(2.11.6) \qquad M_\alpha \oplus M \cong M \oplus M \Leftrightarrow \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \in GL_2(E(M_\pi))GL_2(E(\Gamma M))$$

By (2.11.1) it now suffices to prove that $\alpha \in \lambda(M)$, and by (2.11.2) it suffices to do this modulo $\mathscr{E}$. Let ν denote "natural image in $\mathbf{K}_1(E(\Gamma M_\pi))$". Since

$$\nu(\alpha) = \nu(\operatorname{diag}(\alpha, 1)),$$

it suffices to show

$$(2.11.7) \qquad \nu GL_2(E(M_\pi))\nu GL_2(E(\Gamma M)) \subseteq \lambda(M) = \nu E^*(M_\pi)\nu E^*(\Gamma M).$$

Since 1 is in the stable range of $E(M_\pi)$, every element of $\nu GL_2(E(M_\pi))$ has the form $\nu \operatorname{diag}(\delta, 1) = \nu(\delta) \in \nu E^*(M_\pi)$. It therefore suffices to show that $\nu GL_2(E(\Gamma M))$ belongs to the right-hand side of (2.11.7). Let $F = E(\Gamma M)$. It now suffices to prove:

$$(2.11.8) \qquad GL_2(F) \subseteq \operatorname{diag}(F^*, 1) \cdot \ker(\nu\colon GL_2(F) \to (\mathbf{K}_1(F_\pi))$$

The properties of F that we use are that it is a maximal order whose center, which we now call R, is a direct sum of Dedekind domains and such that, by the Drozd condition, $B = Q(R)F$ is a direct sum of fields and matrix rings of size at least 2×2 over division rings. The localization in (2.11.8) is at a finite set π of maximal ideals of the center R of F. We can assume, by working one coordinate ring at a time, that B is a simple ring and R is a Dedekind domain.

First, let B be commutative, so $F = R$. If $\alpha \in GL_2(R)$ has determinant d, then $\alpha = \operatorname{diag}(d, 1)\beta$ where $\det \beta = 1$. Since 1 is in the stable range of R_π the matrix β is a product of elementary matrices in $GL_2(R_\pi)$ whence (2.11.8) follows.

Now suppose B is not commutative. Then, by the Drozd condition, B is a full matrix ring of size at least 2×2 over a division ring D. We have $F = E({}_\Delta P)$ where P is a locally free module over a maximal order Δ in D, and R is the center of Δ [R '75, 21.6]. In particular, F_π is a full matrix ring over Δ_π.

Let $N = P_1 \oplus P_2$ with each $P_i \cong P$, so $GL_2(F) = E^*({}_\Delta N)$.

We call an automorphism $\alpha \in E(N)$ *elementary with respect to a decomposition* ${}_\Delta N = P' \oplus P''$ (P', P'' locally free) if α equals right multiplication by a matrix of the form

$$\alpha = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix} \tag{2.11.9}$$

where γ maps P' to P'' or vice versa.

Let $\mathscr{E}(N)$ denote the subgroup of $E^*(N)$ generated by all elementary automorphisms with respect to all such decompositions of N.

Every α of the form (2.11.9) becomes a product of actual elementary transvection matrices, with respect to a suitable basis of the free Δ_π-module N_π. Hence $\nu\mathscr{E}(N) = 1$ in $\mathbf{K}_1(F_\pi)$.

Now let β be any element of $GL_2(F) = E^*(N)$. By the following lemma, there exists $\alpha \in \mathscr{E}(N)$ such that $P_1\alpha = P_1\beta$. Then we have

$$\beta\alpha^{-1} = \begin{bmatrix} \gamma_1 & 0 \\ \gamma_2 & \gamma_3 \end{bmatrix} \in E^*(N) = GL_2(F)$$

so $\nu(\beta) = \nu(\gamma_1)\nu(\gamma_3) \in \nu(F^*)$. □

2.12 Lemma. *Let Δ be a module-finite R-algebra, where R is noetherian of Krull dimension* 1, *and let N be a locally free Δ-module. Suppose that $L \cong L'$ for locally-free direct summands L and L' of N, and suppose* $\operatorname{rank}(N) - \operatorname{rank}(L) \geq 2$. *Then $L' = L\alpha$ for some $\alpha \in \mathscr{E}(N)$ ($\mathscr{E}(N)$ as defined above, following* (2.11.9)).

Proof. If L is free of rank 1, then this lemma becomes a special case of [B '68, 3.4, p. 183]. Next let L be an arbitrary locally free module of rank 1. Changing categories by the category equivalence $\hom(L, \dots)$ in Lemma 1.1 makes L free of rank 1, so the previously considered case yields the desired α.

We complete the proof by induction on the rank of L. We can suppose that $\operatorname{rank}(L) \geq 2$. Write $L = L_1 \oplus L_2$ and $L' = L_1' \oplus L_2'$ with each $L_i \cong L_i'$ and L_1 locally free of rank 1. Then $L_1' = L_1\alpha$ for some $\alpha \in \mathscr{E}(N)$. Since $\mathscr{E}(N)$ is a group, we can now suppose that $L_1 = L_1'$. Let

$$N = L_1 \oplus L_2 \oplus X = L_1 \oplus L_2' \oplus Y,$$

and let $\theta\colon N \to L_2 \oplus X$ be the projection map. Then $L_2 \oplus X = L_2'\theta \oplus Y\theta$ so, by induction, there exists $\beta \in \mathscr{E}(L_2 \oplus X)$ such that $L_2'\theta = L_2\beta$. Since $\ker\theta = L_1$ and $L_2\beta = L_2\beta\theta$ we have $L_1 \oplus L_2\beta = L_1 \oplus L_2'$. Extending β to the element of $\mathscr{E}(N)$ that equals 1 on L_1 we get $(L_1 \oplus L_2)\beta = L_1 \oplus L_2'$ as desired. □

2.13 Lemma. (*Let Λ be a ring-order R-algebra in the semisimple artinian ring A.*) *Let E be a module-finite R-algebra and $S = R \cdot 1_E$. Suppose that E is* *an S-order in a semisimple artinian ring B, and every division algebra associated with B is, up to R-isomorphism, one of the division algebras associated with A. Then E is a ring-order S-algebra.*

Proof. What must be proved is that E is contained in a maximal S-order in B. The direct sum E' of the projections of E in the simple components of B is an R-order containing E. So it suffices to prove that E' is contained in a maximal order. Thus we can suppose that B is a simple artinian ring.

Let Δ be the division algebra associated with B. We have $S = R \cdot 1_\Delta$. Since Δ is R-isomorphic to one of the division algebras associated with A (2.1.1) shows that $\tilde{S}$, the integral closure of S in the center $Z(\Delta)$, is a finitely generated S-module. Hence $\tilde{S}E$ is an S-order containing E, so we can suppose $E = \tilde{S}E$. Now $Z(E)$ is the Dedekind domain $\tilde{S}$ whose field of fractions equals $Z(\Delta) = Z(B)$. Since central simple algebras are trivially separable algebras over their center, E is contained in a maximal $\tilde{S}$-order G in B [R '75], and G is also the needed maximal S-order in B. □

2.14 *Notation.* Let R be a commutative ring-order. For an R-module U, let $t(U)$ denote the *torsion submodule* of U, that is, the kernel of the natural map $U \to Q(R) \otimes_R U$. Then let $\overline{U} = U/t(U)$. If U is also a Λ-module, then $\overline{U}$ is a Λ-lattice (a submodule of a free Λ-module). The product $Q(R)\overline{U}$ then makes sense and is canonically isomorphic to $Q(R) \otimes_R U$.

For a surjection $f\colon P \to U$ of Λ-modules, we let $E(f)$, the *endomorphism ring* of f, denote the set of elements α of $E(P)$ such that for some $\beta \in E(U)$ we have $f\alpha = \beta f$. It is easily verified that $E(f) = E(M(f))$ where $M(f)$ is the $T_2(\Lambda)$-module defined in Remarks 1.8. Finally, we let $\overline{E}(f) = E(f)/t(E(f))$, a module-finite R-algebra.

We often write canonical isomorphisms as equality.

2.15 Lemma. *Let*

$$K \rightarrowtail P \xrightarrow{f} U$$

be a presentation of a Λ-module, and let $Q = Q(R)$. Then we have the following canonical R-algebra surjection γ and isomorphisms (denoted by equality):

$$Q\overline{E}(f) = Q \otimes E(f) \xrightarrow{\gamma} E(QK) \oplus E(Q \otimes U) = E(QK) \oplus E(Q\overline{U})$$
$$(\otimes = \otimes_R).$$

Moreover, $(\ker \gamma)^2 = 0$, *and $E(QK)$ and $E(Q\overline{U})$ are semisimple artinian rings whose associated division algebras are (up to R-algebra isomorphism) among those of $A = Q\Lambda$.*

Proof. The induced sequence $QK \rightarrowtail QP \twoheadrightarrow Q \otimes U = Q\bar{U}$ is exact, since Q is a flat R-module, and is split since $Q\Lambda$ is a semisimple artinian ring. Hence $E(1 \otimes f)$ can be viewed as an upper triangular matrix ring whose elements φ, acting on the left, have the form

$$\varphi = \begin{bmatrix} \varphi_1 & \varphi_2 \\ 0 & \varphi_3 \end{bmatrix} \in E(QK \oplus Q \otimes U) = E(QK \oplus Q\bar{U}).$$

Moreover, $E(1 \otimes f) = Q \otimes E(f)$, as one can see by tensoring the $T_2(\Lambda)$-module $M(f)$ with Q. So we define γ by $\gamma(\varphi) = (\varphi_1, \varphi_3)$. Since K and $\bar{U}$ are Λ-lattice, $E(QK)$ and $E(Q\bar{U})$ are semisimple artinian rings and their associated division algebras are, up to R-algebra isomorphism, among the division algebras of $Q\Lambda$. □

2.16 Drozd Cancellation Theorem (presentations). *Let*

$$K \rightarrowtail P \xrightarrow{f} U$$

be a presentation of a Λ-module, and $Q = Q(R)$. Suppose that K and U satisfy the Drozd condition 2.9. *Then direct-sum cancellation holds in the genus of f.*

Proof. We can suppose that Λ is an indecomposable, nonartinian ring-order R-algebra, where R is the center of Λ. Let $E = E(f)$, and let E' be the natural image of E in the semisimple artinian ring $E(QK) \oplus E(Q\bar{U})$ obtained by combining the natural embedding

$$E \subseteq E(1 \otimes f\colon QP \twoheadrightarrow Q\bar{U})$$

with the natural surjection γ described in Lemma 2.15. By that Lemma, E' is an $S = R \cdot 1_{E'}$-order in the semisimple artinian ring $E(QK) \oplus E(Q\bar{U})$. And, by Lemma 2.13, E' is a ring-order S-algebra.

It now suffices to define an additive functor λ on div(f), the category of direct summands of the presentations f^n ($n = 1, 2, \dots$), such that (i) for every presentation $g \in$ gen(f), $\lambda(g)$ is a (necessarily projective) E'-module in the genus of E'; and (ii) for g and h in the genus of f, $g \sim h \Leftrightarrow \lambda(g) \cong \lambda(h)$. For then λ transforms every direct-sum relation $g \oplus h \sim g \oplus k$ to a similar direct-sum relation among E'-lattices in a single genus, and the Drozd Cancellation Theorem for lattices completes the proof. (Use of this functor is a minor modification of the basic procedure described in [G '84].)

The desired functor λ is the composition of the following three functors. The first functor is $g \to M(g)$ where $M(g)$ is the $T = T_2(\Lambda)$-module described in Remarks 1.8. The second functor is

$$M(g) \to \hom_T(M(f), M(g))$$

in Lemma 1.1, which is a category equivalence between $\operatorname{div}(M(f))$ and $\operatorname{div}(E)$. The third functor is

$$X \to \beta(X) = X/\ker(E \to E') \cdot X.$$

This functor satisfies $X \cong Y \Leftrightarrow \beta(X) \cong \beta(Y)$ by uniqueness of the projective cover, because $\ker(E \to E')$ is an ideal of square zero. □

2.17 Drozd Cancellation Theorem (modules). *Let U be a Λ-module that satisfies the Drozd condition. Then direct-sum cancellation holds in the genus of U.*

This theorem was proved by Drozd [Dz '69] under the additional hypotheses that U is a Λ-lattice, Λ is an order over a Dedekind domain R and $Q(R)\Lambda$ is a separable $Q(R)$-algebra. It was extended in [G '87] by allowing U to be an arbitrary (finitely generated) module.

Proof. Suppose $W \oplus X \cong W \oplus Y$ for modules W, X, Y in the genus of U. Present W, X, and Y inefficiently, by a huge free module F. Then the presentations

$$(F \to W) \oplus (F \to X) \quad \text{and} \quad (F \to W) \oplus (F \to Y)$$

are stably equivalent, by Theorem 1.14, hence equivalent by (1.9.1). By the Drozd Cancellation Theorem for presentations, we have $(F \to X) \sim (F \to Y)$ and hence $X \cong Y$. □

Let $f\colon P \twoheadrightarrow U$ be a presentation of a Λ-module. Recall, from 1.9, that we say that $\operatorname{gen}(f)$ is *fully stable* if direct-sum cancellation holds in $\operatorname{gen}(f)$, $\operatorname{gen}(P)$, and $\operatorname{gen}(U)$; and this is equivalent to saying that $\mathcal{G}(f)$ consists of actual, rather than stable equivalence classes of homomorphisms, and $\mathcal{G}(P)$ and $\mathcal{G}(U)$ consist of isomorphism classes, rather than stable isomorphism classes, of modules. Moreover, by Proposition 1.10, this implies that $\operatorname{pres}_f(P, U)$ consists of all equivalence classes of presentations of U by P. By combining the Drozd Cancellation Theorems for presentations and modules, we get:

2.18 Theorem (full stability). (*Let Λ be a ring-order algebra.*) *Let*

$$K \rightarrowtail P \xrightarrow{f} U$$

be a presentation of a Λ-module, and suppose that K, P, and U satisfy the Drozd condition. Then full stability holds in $\operatorname{gen}(f)$.

2.19 COROLLARY. *Let*

$$K \rightarrowtail P \xrightarrow{f} U$$

be a presentation of a Λ*-module and suppose that* Λ *is commutative. Then full stability holds in* gen(f).

Proof. What must be proved is that every Λ-module satisfies the Drozd condition. Since Λ is commutative, the semisimple artinian ring $A = Q(R)\Lambda$ is a direct sum of fields. The ring of endomorphisms of every simple A-modules is one of these fields, hence is commutative. □

2.20 *Appendix.* We give an example of a commutative ring-order R contained in a quadratic extension of the rational numbers Q, and such that R is not a module-finite algebra over any Dedekind subring. We wish to thank W. Heinzer and C. Huneke for producing this example and R. Wiegand for showing it to us.

Let $T = Z[2\sqrt{2}] = Z[x]/\langle x^2 - 8\rangle$. This ring has two maximal ideals P, S lying over $7Z$ because, modulo 7, $x^2 - 8 = x^2 - 1 = (x-1)(x+1)$. We claim that $P^e = Ty$ for some $e \geq 1$ and some $y \in T$.

The ring of all algebraic integers in $Q(T)$ is $\tilde{T} = Z[\sqrt{2}]$, and $\langle 2\rangle$ is a conductor ideal from $\tilde{T}$ to T. Moreover, $2 \notin P$ because any ideal containing 2 and 7 also contains 1. Since P does not contain the conductor ideal $\langle 2\rangle$, it is invertible. By the Jordan-Zassenhaus theorem, there exist integers $u < v$ such that $P^u \cong P^v$. Hence, by invertibility, P^{v-u} is principal, as desired.

Let $R = T[1/y]$. We claim:

(2.20.1) $R \cap Q = Z$;

(2.20.2) R is not integrally closed in $Q(R)$.

For (2.20.1), suppose that $R \cap Q \supsetneqq Z$. Then $R \cap Q$ contains $1/p$ for some prime number p. Some maximal ideal of T contains p, and since the localization $R = T[1/y]$ of T was formed by removing only one of the two maximal ideals containing 7 (and no other maximal ideal), p is in some maximal ideal H of R. But then $1 = (1/p)p \in H$, a contradiction.

For (2.20.2) note that $7T \subseteq P$ so $(7T)^e \subseteq P^e = Ty$. Since 7 becomes invertible in the localization $Z_2[2\sqrt{2}]$ of T, y is invertible there, too. Hence R has the localization $Z_2[2\sqrt{2}, 1/y] = Z_2[2\sqrt{2}]$ which is not integrally closed in its quotient field.

Now suppose R is a module-finite algebra over some Dedekind subring D, hence integral over D. We must have $Q(D) = \mathbf{Q}$ or $Q(D) = \mathbf{Q}[\sqrt{2}] = Q(R)$.

If $Q(D) = Q$, then $D \subseteq R \cap Q = Z$. So R is integral over Z, hence over T. But this is impossible because no prime ideal of R lies over the maximal ideal P of T.

If $Q(R) = Q(D)$ then integrality of R over D, which is integrally closed in $Q(D)$, shows $R = D$, contrary to (2.20.2).

3. Maximal orders, Nakayama's problem

In this section we prove a unique presentability theorem for maximal orders. Consequences of this are our answer to Nakayama's question and the fact that $\mathrm{pres}_f(P, U)$ always equals zero over maximal orders.

3.1 LEMMA. *Let Γ be a maximal ring-order R-algebra in a simple artinian ring $A = Q(R)\Gamma$. Then:*

(i) *For every essential left ideal L of Γ, ${}_\Gamma(\Gamma/L)$ has finite length.*

(ii) *For every nonzero 2-sided ideal T of Γ, Γ/Γ is an artinian* PIR *(every left ideal and every right ideal is principal).*

(iii) *Suppose Γ is not artinian. Then a (finitely generated) Γ-module has finite length if and only if it is annihilated by a regular element of R.*

Proof. For (i) and (ii) see [ER '70, 1.3 and 3.3]. For the "only if" part of (iii) it suffices to show that every cyclic module Γ/L of finite length is annihilated by a regular element of R. Since Γ/L has finite length, L is essential in Γ [ER '70, 1.3], so $AL = A$. Since $A = Q(R)\Gamma$ we have $Q(R)L = A$ so $1_A = d^{-1}x$ with d regular in R and $x \in L$. Then $d = x \in L$ so $d(\Gamma/L) = 0$. Conversely, suppose $dU = 0$ for some regular $d \in R$ and some Γ-module U. Since R is noetherian of Krull dimension 1, R/dR is an artinian ring. So U has finite length as an R-module, hence as a Γ-module. □

3.2 *Remarks.* Let Γ be a maximal ring-order R-algebra in a simple artinian ring, and suppose that Γ is not an artinian ring. We adapt some known properties of (finitely generated) Γ-modules to forms in which we shall need them.

A module U over any ring is called *uniform* if $U \neq 0$ and if the intersection of any two nonzero submodules of U is again nonzero. If $U = U_1 \oplus \cdots \oplus U_n$ with each U_i uniform, then the number n is an invariant of the isomorphism class of U (by the Krull-Schmidt theorem, applied to the injective hull of U, because the endomorphism ring of the injective hull of each U_i is a local ring). The integer n is called the *uniform rank*, u-rk(U).

Every Γ-module M has a decomposition [ER '70, p. 71]:

(3.2.1) $\quad M = P \oplus U$ with P projective and U of finite length.

Moreover, $U = \{u \in M \mid Ru$ has finite length$\}$ and is called the *torsion submodule* of M. Every projective Γ-module P has a decomposition [ER '70, 1.4]:

(3.2.2) $$P = P_1 \oplus \cdots \oplus P_n \quad \text{with every } P_i \text{ uniform.}$$

Since Γ is locally a full matrix ring over a (noncommutative) principal ideal domain [see Notation 2.8], we have:

(3.2.3) Any two projective Γ-modules of the same uniform rank are in the same genus.

By Lemma 3.1, every Γ-module U of finite length is a module over an artinian PIR; hence [J '56, pp. 78-79]

(3.2.4) $$U = U_1 \oplus \cdots \oplus U_n \quad \text{with each } U_i \text{ a uniserial module}$$

where *uniserial* means that its submodules are totally ordered by inclusion and U_i itself $\neq 0$.

Every uniserial Γ-module of finite length is a direct summand of $\Gamma/\mathfrak{M}^e$ for some unique maximal ideal $\mathfrak{M}$ of Γ and exponent e, and [J '56, Theorem 44, p. 79]:

(3.2.5) $\Gamma/\mathfrak{M}^e$ is a direct sum of mutually isomorphic uniserial modules.

Let U have finite length and be decomposed as in (3.2.4). As in abelian groups, we define the *$\mathfrak{M}$-component* $U_{\mathfrak{M}}$ of U to be the sum of all terms U_i that are annihilated by some power of $\mathfrak{M}$. This is a fully invariant submodule of U, and does not depend upon the particular decomposition (3.2.4) used to compute it. A *primary component* of U means an $\mathfrak{M}$-component for some $\mathfrak{M}$.

In what follows we have to know whether there exists a presentation $P \twoheadrightarrow U$ from a given projective P to a given U of finite length. To state the answer, we define the *capacity* of $\mathfrak{M}$ to be

(3.2.6) $$\kappa(\mathfrak{M}) = \text{u-rk}(\Gamma/\mathfrak{M})/\text{u-rk}(\Gamma).$$

Then:

(3.2.7) Let P and U be Γ-modules, with P projective and U of finite length. There is a presentation $P \twoheadrightarrow U$ if and only if

$$\text{u-rk}(U_{\mathfrak{M}}) \leq \kappa(\mathfrak{M}) \cdot \text{u-rk}(P)$$

for every maximal ideal $\mathfrak{M}$ of Γ.

Since P is projective, it can be mapped onto U if and only if it can be mapped onto $U/(\operatorname{rad} U)$. So we can suppose that U is a semisimple module. Moreover, P can be mapped onto U if and only if P can be mapped onto every nonzero $U_{\mathfrak{M}}$. So we can suppose that U is the direct sum of some number of copies of the unique simple module S over the simple artinian ring $\Gamma/\mathfrak{M}$. On the other hand, if H and J are uniform projective modules, we have $H/\mathfrak{M}H \cong J/\mathfrak{M}J$ by [LR '74, 2.6]. (For the rings we are dealing with, this can also be proved from (3.2.3).) Thus, for every uniform projective Γ-module H we have $H/\mathfrak{M}H \cong S^m$ for some m that is independent of H.

Writing $\Gamma = P_1 \oplus \cdots \oplus P_n$ with each P_i uniform, hence $n = \text{u-rk}(\Gamma)$, we get $\Gamma/\mathfrak{M} \cong S^{mn}$. This shows that $\kappa(\mathfrak{M}) = \text{u-rk}(H/\mathfrak{M}H)$ for every uniform projective Γ-module H. The desired result now follows by writing $P = P_1 \oplus \cdots \oplus P_n$ with each P_i uniform and noting that a surjection: $P \twoheadrightarrow U$ exists if and only if a surjection of semisimple modules $P/\mathfrak{M}P \twoheadrightarrow U$ exists.

We will need to know the precise structure of P/SP where P is a uniform projective Γ-module and S is a nonzero 2-sided ideal of Γ. Every such ideal of Γ is a product of maximal ideals; and $\mathfrak{M}\mathfrak{N} = \mathfrak{N}\mathfrak{M}$ for all maximal ideals $\mathfrak{M}$ and $\mathfrak{N}$ of Γ [R '68, 2.1]. These remarks show that we get the answer to our question by repeated use of the following:

(3.2.8) Let T be a 2-sided ideal of Γ, and $\mathfrak{M}$ a maximal 2-sided ideal not containing T. Then, for every uniform projective Γ-module P, we have $P/\mathfrak{M}^eTP \cong P/\mathfrak{M}^eP \oplus P/TP$, and $P/\mathfrak{M}^eP$ is the direct sum of $\kappa(\mathfrak{M})$ mutually isomorphic uniserial modules, each of length e.

The splitting follows from the Chinese remainder Theorem. The proof that $P/\mathfrak{M}^eP$ is as claimed is similar to that of (3.2.7), so will be omitted.

Next, we prove that for projective Γ-modules $P \supseteq P'$:

(3.2.9.) $_\Gamma(P/P')$ has finite length $\Leftrightarrow$ $\text{u-rk}(P) = \text{u-rk}(P')$

If P/P' has finite length, it is annihilated by a regular element d of R, by Lemma 3.1. Hence $dP \subseteq P' \subseteq P$, so $AP = AP'$, where $A = Q(R)\Gamma$; and the stated equality of uniform rank follows. Conversely, equality of uniform rank shows that $AP = AP'$, hence $Q(R)P = Q(R)P'$, so $d(P/P') = 0$ for some regular element d of R. By Lemma 3.1, P/P' has finite length.

Finally, we recall, from [LR '74, 1.9]:

(3.2.10) If H is a noetherian module over *any* ring Γ, and T is a 2-sided deal of Γ, then all surjections $H \twoheadrightarrow H/TH$ are equivalent to each other.

3.3 "STABLE" INVARIANT FACTOR THEOREM. *Let Γ be a maximal (hence hereditary), non-artinian ring-order algebra in a simple artinian ring, and let K*

be a submodule of a projective Γ*-module* P. *Then, for some* d, *there exist compatible decompositions*

(3.3.1) $P^d = P_1 \oplus \cdots \oplus P_n$

and

(3.3.2) $K^d = T_1P_1 \oplus \cdots \oplus T_nP_n \quad (T_1 \subseteq \cdots \subseteq T_n)$

where

(3.3.3) *each* T_i *is a* 2*-sided ideal of* Γ *and each* $P_i \in \text{gen}(\Gamma)$.

If Γ is commutative, we can take d = 1.

Proof. For commutative Γ, this is Steinitz's theorem [S '11; L '66, 1.10]. Now consider the hypotheses as stated, and let f be the natural homomorphism of P onto $U = P/K$. By [LR '74, 3.1, 3.2] there exist compatible decompositions

$$(3.3.4) \qquad P = X_1 \oplus \cdots \oplus X_m \quad (\text{each } X_i \text{ uniform})$$

and

$$(3.3.5) \qquad U = U_1 \oplus \cdots \oplus U_m$$

with each $f(X_i) = U_i$. This yields a decomposition $K = \bigoplus_i (X_i \cap K)$.

Our first objective is to find d such that (3.3.1)–(3.3.3) are satisfied with each P_i *uniform*, rather than in gen(Γ). If $X_i \cap K = 0$ we can take $T_i = 0$, and then forget about X_i. So we suppose, from now on, that, for each i, $X_i \cap K \neq 0$. Then, by (3.2.9), each $X_i/(X_i \cap K)$ has finite length, so U has finite length, too.

Let $S(U)$ be the set of maximal ideals $\mathfrak{M}$ of Γ such that the $\mathfrak{M}$-component $U_{\mathfrak{M}}$ is nonzero. $S(U)$ is a finite set because U has finite length.

Let d be a common multiple of the finite set of numbers $\{\kappa(\mathfrak{M}) \mid \mathfrak{M} \in U\}$ defined in (3.2.6). The decompositions (3.3.4) and (3.3.5) yield decompositions of P^d and U^d which, after a change of notation, we again call (3.3.4) and (3.3.5). Now let each $U_i = \oplus_h V_{ih}$ with each V_{ih} uniserial. Because of our choice of d, decompositions (3.3.4) and (3.3.5) now satisfy:

(3.3.6) Suppose $V_{ih} \subseteq U_{\mathfrak{M}}$. Then the number of times, up to isomorphism, that V_{ih} occurs as a direct summand of U is a multiple of $\kappa(\mathfrak{M})$. Moreover, each U_i is a homomorphic image of every X_j.

The final assertion in (3.3.6) holds by (3.2.7), since $U_i = f(X_i)$ is a homomorphic image of X_i.

We claim that U has a new decomposition (3.3.5) in which each $U_i \cong X_i/T_iX_i$ for some 2-sided ideal $T_i \neq 0$.

Each of the following two types of reshuffling of the summands V_{ih} results in a new decomposition of U that, by (3.2.7), still satisfies (3.3.6). Type (i):

Choose two summands U_i and U_j and a maximal ideal $\mathfrak{M}$, and interchange some V_{ih} in the $\mathfrak{M}$-component of U_i with some V_{jk} in the $\mathfrak{M}$-component of U_j. Type (ii): If, for some i and $\mathfrak{M}$, the $\mathfrak{M}$-component of U_i consists of fewer than $\kappa(\mathfrak{M})$ summands V_{ih}, move a summand V_{jk} from the $\mathfrak{M}$-component of some U_j to that of U_i.

By means of a sufficient number of reshuffles of these types, and by making use of the first assertion in (3.3.6), we get a new decomposition (3.3.5) with the following properties: For each $\mathfrak{M} \in S(U)$ and each i, the $\mathfrak{M}$-component of U_i either consists of precisely $\kappa(\mathfrak{M})$ mutually isomorphic summands V_{ih} or equals zero; moreover, the length $e(i, \mathfrak{M})$ of every uniserial summand V_{ih} in the $\mathfrak{M}$-component of U_i is $\leq$ the length of every uniserial summand in the $\mathfrak{M}$-component of U_{i+1}.

Hence repeated use of (3.2.8) and the paragraph above it show that, for each i,

$$(3.3.7) \quad U_i \cong X_i/T_iX_i \quad \text{where } T_i = \Pi\{\mathfrak{M}^{e(i,\mathfrak{M})} | \mathfrak{M} \in S(U)\} \subseteq T_{i+1}.$$

However, we have lost the property that $f(X_i) = U_i$. We recover this by means of the lifting theorem of [LR '74, 1.5]: Since each X_i is projective, U_i has finite length and, by (3.3.6), every U_i is a homomorphic of every X_j, the decomposition (3.3.5) of U can be lifted to a new decomposition (3.3.4) in which each $f(X_i) = U_i$ and each new X_i is isomorphic to the old X_i. By (3.3.7) and (3.2.10), U_i is uniquely presented by X_i, so we now have $\ker(X_i \twoheadrightarrow U_i) = T_iX_i$, and therefore $K = \oplus_i T_iX_i$ as demanded in (3.3.2). Moreover, we have $T_i \subseteq T_{i+1}$. Thus the theorem is proved, but with each P_i uniform instead of in $\mathrm{gen}(\Gamma)$.

The proof is now easily finished. Any two projective Γ-modules of the same uniform rank are in the same genus, by (3.2.3). Replace (3.3.1) and (3.3.2) by direct sums of u-rk(Γ) copies of themselves, and then change notation so that each new P_i and U_i is the direct sum of the u-rk(Γ) old ones. Since each new P_i is now in $\mathrm{gen}(\Gamma)$, the proof is complete. □

3.4 Lemma. *Let K be a submodule of a projective module P over a maximal, nonartinian ring-order algebra Γ, and suppose that P/K has finite length. Then, for some d, P^d/K^d is uniquely presentable by P^d.*

Proof. Choose d as in the Stable Invariant Factor Theorem, let f be the natural homomorphism $P^d \twoheadrightarrow U = P^d/K^d$ and, in the notation of (3.3.1), let $f(P_i) = U_i$. Then we have decompositions

$$(3.4.1) \qquad P^d = P_1 \oplus \cdots \oplus P_n \quad \text{and} \quad U = U_1 \oplus \cdots \oplus U_n$$

where each $f(P_i) = U_i$. We claim that, in fact, *every P_i can be mapped onto every U_j.*

By (3.2.7) it suffices to show that all of the summands P_i have the same uniform rank. This holds since, by (3.3.3), every $P_i \in \operatorname{gen}(\Gamma)$.

Consider a presentation $f\colon P \twoheadrightarrow U$ of a module over any ring, and a pair of decompositions of the form (3.4.1). Under the following conditions, it is proved in [LR '74, 1.6] that if U_n is uniquely presentable by P_n, then U is uniquely presentable by P. Each P_i is projective, each U_i has finite length, and each P_i can be mapped onto each U_j. Moreover, if P_n is noetherian and $U_n \cong P_n/T_nP_n$ for some 2-sided ideal T_n of Γ, then U_n *is* uniquely presentable by P_n (3.2.10). Since our Stable Invariant Factor Theorem provides the needed 2-sided ideal, the proof is complete. □

3.5 Unique Presentability Theorem (maximal orders). *Let $f\colon P \twoheadrightarrow U$ be a presentation of a module over a maximal ring-order R-algebra Γ in a simple artinian ring A. Suppose that* u-rk(ker f) ≥ 2 *or* Γ *is commutative. Then U is uniquely presentable by P.*

Proof. We can suppose that Γ is not artinian. For if Γ is artinian, U is uniquely presentable by P by the proof of 1.10(i).

If Γ is commutative, then it is a Dedekind domain, and the theorem is known. See [L '66, 1.9] or the historical remarks in [LR '74, 3.6]. So we can assume, from now on, that u-rk(ker f) ≥ 2.

Suppose, first, that U has finite length (the crux of our proof). Then, by the preceding lemma, there is a $d \geq 0$ such that U^{d+1} is uniquely presentable by P^{d+1}.

Now let g and h be arbitrary presentations $P \twoheadrightarrow U$. Then $g \oplus f^d$ and $h \oplus f^d$ are presentations $P^{d+1} \twoheadrightarrow U^{d+1}$. Hence $g \oplus f^d \sim h \oplus f^d$.

Thus, all that remains is to cancel f^d. By (1.9.1) (effectively, the Bass Cancellation Theorem for presentations) we get $g \oplus f \sim h \oplus f$. To cancel the final f we verify the hypotheses of the Drozd Cancellation Theorem for Presentations 2.16. Let $Q = Q(R)$. Since u-rk(ker f) ≥ 2, $E(Q \cdot (\ker f))$ is a full matrix ring of size at least 2×2 over a division ring. And since U has finite length and $\Gamma \neq Q\Gamma$, $Q \otimes U = 0$ by Lemma 3.1(iii). Therefore the Drozd conditions are satisfied, and the theorem is proved when U has finite length.

Now we drop the additional hypothesis that U has finite length. But we can still suppose that Γ is not artinian. The critical observation here is that, since Γ is maximal, every presentation of every Γ-module is equivalent to a presentation of the form

$$(P' \twoheadrightarrow U') \oplus (P'' \rightarrowtail\!\!\!\!\twoheadrightarrow P'') \tag{3.5.1}$$

where U' has finite length and the arrow $\rightarrowtail\!\!\!\!\twoheadrightarrow$ denotes an isomorphism. We prove this before using it.

Let a presentation $f\colon P \twoheadrightarrow U$ be given. As in (3.2.1) there is a decomposition $U = U' \oplus P_2$ with P_2 projective and U' of finite length. Since P_2 is projective, the composite surjection $P \twoheadrightarrow U \twoheadrightarrow P_2$ splits, giving a decomposition $P = P' \oplus P''$ with $f\colon P'' \to f(P'')$ an isomorphism. Moreover, we have $U = U' \oplus f(P'')$. This decomposition, together with $P = P' \oplus P''$ expresses f in the form (3.5.1).

Now let g be another presentation $P \twoheadrightarrow U$. We get decompositions $f = f' \oplus f''$ and $g = g' \oplus g''$ corresponding to decompositions of the form (3.5.1). So it suffices to show that $f' \sim g'$ and $f'' \sim g''$. For the second of these, it suffices to show that P'' is unique up to isomorphism. This holds since P'' is isomorphic to U modulo its torsion submodule U'. (See the remark below (3.2.1).)

To see that $f' \sim g'$ it suffices to show that P' and U' are unique up to isomorphism and U' is uniquely presentable by P'. Uniqueness of U' holds since it is the torsion submodule of U. For uniqueness of P', note that $P'/\ker f \cong U'$, which has finite length. So, by (3.2.9), $\text{u-rk}(P') = \text{u-rk}(\ker f)$ which is ≥ 2 by hypothesis.

We have two decompositions of the form (3.5.1), one corresponding to f, and one corresponding to g. Since P' has u-rk ≥ 2, we can apply the Drozd Cancellation Lemma for maximal orders, 2.10, to conclude uniqueness of P'. Finally, we again use the fact that $\text{u-rk}(P') \geq 2$, together with the fact that U' has finite length, to conclude, from the case of the theorem already proved, that U' is uniquely presented by P'. This completes the proof of the theorem. □

3.6 COROLLARY (Nakayama's question). *Let Γ be a* PID *that is module-finite over its center, and let* **A** *and* **B** *be matrices of equal size over Γ that present isomorphic left Γ-modules*: $\Gamma^n/(\Gamma^m\mathbf{A}) \cong \Gamma^n/(\Gamma^m\mathbf{B})$. *If* **A** *has rank ≥ 2, then* $\mathbf{A} \sim \mathbf{B}$.

Proof. First we show that Γ is a maximal ring-order algebra over its center R. Since Γ is a PID and is module-finite over R, it suffices to show that R is a Dedekind domain; and this is done in [RS '74].

Now, α = right multiplication by **A** yields a free resolution

$$\Gamma^n \xrightarrow{\alpha} \Gamma^m \xrightarrow{f} U \tag{3.6.1}$$

of $U = \Gamma^n/(\Gamma^m\mathbf{A})$. Similarly **B** yields another free resolution (β, g) of U. By our Unique Presentability Theorem we have $f \sim g$. If α is one-to-one, then so is β, and we have $\alpha \sim \beta$ as desired.

If α is not one-to-one, first get decompositions $\alpha \sim \alpha_0 \oplus \alpha_1$ and $\beta \sim \beta_0 \oplus \beta_1$ in which $\alpha_0 = \beta_0 = 0$ and α_1 and β_1 are one-to-one (for example, by diagonalizing **A** and **B**). Then apply the previously considered case to α_1 and β_1. □

3.7 COROLLARY. *Let f: $P \twoheadrightarrow U$ be a presentation of a module over a maximal ring-order algebra Γ in a semisimple artinian ring A. Then* $\mathrm{pres}_f(P, U) = \{0\}$.

Proof. We can suppose that Γ is a maximal order in a simple artinian ring, and Γ is not itself artinian. By Corollary 1.12, it suffices to show that the group $\mathscr{P} = \mathrm{pres}_{f\oplus f}(P \oplus P, U \oplus U)$ equals $\{0\}$. Moreover, full stability holds in $\mathrm{gen}(f^2)$, so $\mathscr{P}$ consists of actual presentation classes.

If f is an isomorphism, so is $f \oplus f$, so $\mathscr{P}$ obviously equals zero. Otherwise

$$\text{u-rk}(\ker(f \oplus f)) \geq 2,$$

so our Unique Presentability Theorem shows that $\mathscr{P} = \{0\}$. $\square$

3.8 LEMMA. *Let M be a Γ-module, where Γ is a maximal ring-order algebra in a simple artinian ring. Then $\mathscr{G}(M) = \{0\}$ if M has finite length. Otherwise* $\mathscr{G}(M) \cong \mathscr{G}(\Gamma)$.

Proof. We can suppose that Γ is not artinian, and use the notation in Remarks 3.2. If M has finite length as a Γ-module, then it has finite length as an R-module by Lemma 3.1 and the fact that R has Krull dimension 1. Hence M is the direct sum of its localizations at maximal ideals of R. Since the isomorphism class of M is determined by its localizations, we have $\mathscr{G}(M) = \{0\}$.

Now suppose that M does not have finite length.

Let M' be a uniform projective Γ-module. All projective Γ-modules are direct sums of uniform modules, by (3.2.2); and all uniform projective Γ-modules are in the same genus, by (3.2.3). So $\mathscr{G}(M) \cong \mathscr{G}(M')$ by Corollary 1.7. In particular, $\mathscr{G}(\Gamma) \cong \mathscr{G}(M')$, so $\mathscr{G}(M) \cong \mathscr{G}(\Gamma)$. $\square$

3.9 PROPOSITION. *Let*

$$K \rightarrowtail P \xrightarrow{f} U$$

be a presentation of a Γ-module, where $P \neq 0$, and Γ is a non-artinian maximal ring-order algebra in a simple artinian ring. Then:

(i) $\mathscr{G}(f) \cong \mathscr{G}(\Gamma) \times \mathscr{G}(\Gamma)$ *unless $K = 0$ or U has finite length.*

(ii) $\mathscr{G}(f) \cong \mathscr{G}(\Gamma)$ *if $K = 0$ or U has finite length.*

Proof. After replacing f by f^2 we can assume (Cor. 1.12) that full stability holds in $\mathrm{gen}(f)$. We get a homomorphism γ: $\mathscr{G}(f) \to \mathscr{G}(P) \times \mathscr{G}(U)$ by letting $\gamma[g] = ([S], [V])$ for any presentation

$$L \rightarrowtail S \xrightarrow{g} V$$

in $\mathrm{gen}(f)$. By Corollary 3.7, γ is a monomorphism.

Now suppose that U does not have finite length. Then, by Lemma 3.8, $\mathcal{G}(P) \cong \mathcal{G}(\Gamma) \cong \mathcal{G}(U)$. Now suppose, in addition, that $K \neq 0$. To complete the proof of (i), it suffices to show that γ is a surjection.

By (3.5.1) there are decompositions

$$(3.9.1) \qquad P = P_1 \oplus P_2 \text{ and } U = U' \oplus P'' \quad (U' \text{ of finite length})$$

where $f(P_1) = U'$ and f takes P_2 isomorphically onto P''. Since U does not have finite length, P'' and P_2 are nonzero, and since K is nonzero, so is P_1. Since all projective Γ-modules are direct sums of uniform modules, and since all uniform modules are in the same genus, Lemma 1.2 shows that by suitably varying the nonzero module P'' we get an arbitrary isomorphism class in $\operatorname{gen}(U)$. Then, by suitably varying the nonzero module P_1 we get an arbitrary isomorphism class in $\operatorname{gen}(P)$. This completes the proof of (i).

The proof of (ii) is obtained by letting $P_1 = 0$ or $P_2 = 0$ in the proof of (i). $\square$

4. Restricted genus of a presentation

In this section we introduce the restricted genus, $\operatorname{r.gen}(f)$, consisting of all elements of $\operatorname{gen}(f)$ that become equivalent over Γ. Our main result uses the associated restricted genus class group $\operatorname{r.}\mathcal{G}(f)$ to construct a Mayer-Vietoris-like sequence that shows that $\mathcal{G}(f)$ is built from $\mathcal{G}(\Gamma)$ and $\mathbf{K}_1(\Gamma/I\Gamma)$, where I is a conductor ideal from Γ to Λ (Theorem 4.8). As a consequence, we obtain an explicit formula for $\operatorname{pres}_f(P, U)$ in certain situations (Theorem 4.11). Actual computations with these results are delayed until the next two sections.

4.1 *Notation*. Throughout this section, let Λ be an indecomposable, non-artinian ring-order R-algebra in a semisimple artinian ring $A = Q(R)\Lambda$, and let Γ be a maximal R-order in A such that $\Gamma \supseteq \Lambda$.

Since a Λ-*lattice* means a (finitely generated) submodule K of a free Λ-module, we can always view K as a Λ-submodule of an A-module $AK = Q(R)K$ which, in turn, we can identify with $A \otimes_\Lambda K = Q(R) \otimes_R K$ whenever convenient. Hence we can view the Λ-endomorphism ring $E(K)$ as a subring of the A-endomorphism ring $E(Q(R)K)$.

Let K and L be Λ-lattices. Then *every* Λ-*homomorphism* f: $K \to L$ *extends naturally to a* Γ-*homomorphism* $\Gamma K \to \Gamma L$ (because $1 \otimes f$ is an A-homomorphism $Q(R)K \to Q(R)L$ whose restriction to K coincides with f). Hence, as with orders over Dedekind domains, we write $E(K) \subseteq E(\Gamma K)$, and make other, similar identifications without further comment.

Consider presentations of Λ-modules

$$(4.1.1) \qquad K \rightarrowtail P \overset{f}{\twoheadrightarrow} U, \qquad L \rightarrowtail S \overset{g}{\twoheadrightarrow} V.$$

The *restricted genus* r.gen(f) means the collection of all presentations $g \in$ gen(f) such that

$$(1 \otimes f\colon \Gamma \otimes_\Lambda P \twoheadrightarrow \Gamma \otimes_\Lambda U) \sim (1 \otimes g\colon \Gamma \otimes_\Lambda S \twoheadrightarrow \Gamma \otimes_\Lambda V)$$

as presentations of Γ-modules.

We use the notation gen(K, P) for the genus of the inclusion map $K \subseteq P$, and we often identify gen(f) with gen(K, P).

Then r.gen(f), which we also call r.gen(K, P), becomes the collection of all inclusion maps $L \subseteq S$ in gen(K, P) such that $(\Gamma L \subseteq \Gamma S) \sim (\Gamma K \subseteq \Gamma P)$ as inclusions of Γ-modules. The proof of this uses the fact that $\Gamma \otimes_\Lambda P = \Gamma P$ (canonical isomorphism) since P is projective.

We caution the reader that, usually, $\Gamma \otimes K \not\cong \Gamma K$ for Λ-lattices K. Of course, isomorphism holds if K is projective.

To resolve a conflict between two potentially different meanings of r.gen(K, P), let i be the inclusion map $K \subseteq P$. We claim that we can identify r.gen(K, P) with the collection of all inclusion maps $j\colon L \subseteq S$ in gen(K, P) such that $1 \otimes i \sim 1 \otimes j$, where $\otimes$ refers to $\Gamma \otimes_\Lambda (\dots)$.

To prove this, it suffices to show that $\Gamma \otimes K \cong \Gamma K \oplus W$ for some Γ-module W that is annihilated by a regular element of R (hence hom$(W, \Gamma K) = 0$) and whose Γ-isomorphism class is determined by the Λ-genus of K. (Recall that $\Gamma \otimes P = \Gamma P$.)

Since no element of ΓK is annihilated by any regular element of R, it is a projective Γ-module. (Write Γ as a direct sum of maximal orders in simple artinian rings. None of these has an R-submodule of finite length, by (2.1.2). Now apply Lemma 3.1(iii) and (3.2.1).) The canonical surjection $\tau\colon \Gamma \otimes K \twoheadrightarrow \Gamma K$ therefore splits, yielding the summand W. To see that $W = \ker \tau$ is annihilated by a regular element of R, note that $I \cdot \ker \tau = 0$, where I is any conductor ideal from Γ to Λ, as in Notation 2.2. Since $IW = 0$ and R has Krull dimension 1, W has finite length as an R-module. Hence its Γ-isomorphism class is determined locally.

To set up some notation for our double-coset description of r.gen(f), we fix a decomposition of Γ-modules:

(4.1.2) $\Gamma P = X \oplus Y$ where $X \supseteq \Gamma K$ and ${}_R(X/\Gamma K)$ has finite length.

To see that such a decomposition exists, let $X = (Q(R)K) \cap \Gamma P$. Then no nonzero element of $(\Gamma P)/X$ is annihilated by a regular element of R. So $\Gamma P/X$ is Γ-projective, as in our discussion of $\Gamma \otimes K$. Hence the canonical surjection $\Gamma P \twoheadrightarrow \Gamma P/X$ splits, yielding Y. Since ${}_R(X/\Gamma K)$ is annihilated by regular elements of R, it has finite length.

Now let $L \subseteq S$ be Λ-lattices such that L and S are full Λ-lattices in $Q(R)X$ and $Q(R)P$ respectively, that is,

$$(4.1.3)\quad Q(R)L = Q(R)X\,(= Q(R)K) \quad \text{and} \quad Q(R)S = Q(R)P$$

Let $E(X_\pi, \Gamma P_\pi)$ be the ring of all endomorphisms of the Γ_π-module ΓP_π that take X_π into itself, and $E^*(X_\pi, \Gamma P_\pi)$ the group of units of $E(X_\pi, \Gamma P_\pi)$. (See Notation 2.2.)

For $\alpha \in E^*(X_\pi)$, let $L \cdot \alpha$ be the unique full Λ-lattice M in $Q(R)X$ such that $M_\pi = L_\pi \alpha$ and $M_{\mathfrak{m}\rho} = L_{\mathfrak{m}\rho}$ for every maximal ideal $\mathfrak{m} \notin \pi$. (Here $\mathfrak{m}\rho$ denotes "regular localization" as in 2.3.) This is possible by (2.4) and the Strong Consistency Theorem 2.6. Note that this makes sense even if $L \not\subseteq X_\pi$ because we always consider such an α to be an element of $E^*(Q(R)X)$. Similarly, for $\alpha \in E^*(\Gamma P_\pi)$, we define $S \cdot \alpha$ by replacing S_π by $S_\pi \alpha$ and leaving $S_{\mathfrak{m}\rho}$ unchanged when $\mathfrak{m} \notin \pi$. Finally, for L, S as in (4.1.3) we set

$$(4.1.4)\qquad (L, S) \cdot \alpha = (L \cdot \alpha, S \cdot \alpha) \quad \text{for } \alpha \in E^*(X_\pi, \Gamma P_\pi).$$

Thus we have defined a group action of $E^*(X_\pi, \Gamma P_\pi)$ on all inclusions $L \subseteq S$, where L and S satisfy (4.1.3). In particular, $L \cdot \alpha$ and $S \cdot \alpha$ again satisfy (4.1.3), and

$$(4.1.5)\qquad (L, S) \cdot \alpha \cdot \beta = (L, S) \cdot (\alpha\beta)$$

as is easily checked locally at π and at $\mathfrak{m}\rho$, $\mathfrak{m} \notin \pi$.

For readers familiar with [G '87] we note that, in the notation of that paper, $P_\alpha = P \cdot \alpha$, but usually $L_\alpha \neq L \cdot \alpha$ if $\Gamma L_\pi \neq X_\pi$.

4.2 Lemma. *Suppose that $\Gamma P/\Gamma K$ is uniquely presentable by ΓP (as Γ-modules). Then*

$$(K, P) \cdot \alpha \in \text{r.gen}(K, P) \quad \text{for all } \alpha \in E^*(X_\pi, \Gamma P_\pi)$$

Proof. To see that the inclusions (K, P) and $(K, P) \cdot \alpha$ are in the same genus, check separately at π (where the desired equivalence is α) and at $\mathfrak{m}\rho$, $\mathfrak{m} \notin \pi$, where the two inclusions become the same inclusion. Thus the crux of the lemma is to show $(\Gamma K, \Gamma P) \sim \Gamma((K, P) \cdot \alpha) = (\Gamma K, \Gamma P) \cdot \alpha$.

As before, the equality is checked locally, at π and at $\mathfrak{m}\rho$, $\mathfrak{m} \notin \pi$. For the equivalence assertion we claim it suffices to verify

$$(4.2.1)\qquad \Gamma P/\Gamma K \cong \Gamma P \cdot \alpha/\Gamma K \cdot \alpha$$

To see this sufficiency, first note that ΓP equals $\Gamma P \cdot \alpha$, as is easily checked locally, then invoke the unique presentability hypothesis.

To prove (4.2.1) first note that

$$(4.2.2) \quad \Gamma P/\Gamma K \cong X/\Gamma K \oplus Y \quad \text{and} \quad \Gamma P/\mathrm{K} \cdot \alpha \cong X/\Gamma K \cdot \alpha \oplus Y.$$

The first isomorphism follows from (4.1.2). For the second isomorphism, first check locally that $\Gamma K \cdot \alpha \subseteq X_\pi$, remembering that $\alpha \in E^*(X_\pi)$. Since $X/\Gamma K$ is annihilated by a regular element of R (Lemma 3.1(iii)), so is $X/\Gamma K \cdot \alpha$ (check locally); hence ${}_R(X/\Gamma K \cdot \alpha)$ has finite length. Since R-modules of finite length are the direct sum of their localizations at maximal ideals of R, the desired Γ-isomorphism $X/\Gamma K \cong X/\Gamma K \cdot \alpha$ can be checked locally, where it becomes obvious because, at π, α acts as an automorphism of X_π, while at $\mathfrak{m} \notin \pi$, α can be ignored. □

4.3 Lemma. *Keep the notation of* 4.1.
(i) $\Gamma(P \cdot \alpha) = \Gamma P$ *for every* $\alpha \in E^*(\Gamma P_\pi)$.
(ii) $E^*(K_\pi, P_\pi) \subseteq E^*(\Gamma K_\pi, \Gamma P_\pi) \subseteq E^*(X_\pi, \Gamma P_\pi)$.

Proof. By (4.1.2) there is a regular element $d \in R$ such that

$$(4.3.1) \qquad dX \subseteq \Gamma K.$$

Statement (i) of the lemma is verified locally, first at π, then at regular localizations $\mathfrak{m}\rho$, $\mathfrak{m} \notin \pi$.

Consider (ii). The first inclusion is obvious. Take $x \in X_\pi$ and $\alpha \in E^*(\Gamma K_\pi, \Gamma P_\pi)$. Since $(x)\alpha \in \Gamma P_\pi = X_\pi \oplus Y_\pi$, we have $(x)\alpha = w + y$ with $w \in X_\pi$ and $y \in Y_\pi$. Multiplying by the element d in (4.3.1) puts both terms $(dx)\alpha$ and dw into X_π. Hence $dy = 0$. Since d is regular in R. hence is a unit in $A = Q(R)\Gamma$, and since Y is Γ-projective, we get $y = 0$. So $\alpha \in E(X, \Gamma P_\pi)$. Giving the same treatment to α^{-1} that was just given to α completes the proof. □

Let τ (for "torsion") denote the set of all maximal ideals $\mathfrak{m} \notin \pi$ such that $X_{\mathfrak{m}\rho} \neq \Gamma K_{\mathfrak{m}\rho}$. The set τ is finite, by (4.1.2).

4.4 Double Coset Theorem. *Let f be as in* (4.1.1), *and suppose that $\Gamma P/\Gamma K$ is uniquely presentable by ΓP. Then the correspondence $\alpha \to (K, P) \cdot \alpha$ yields a bijection between the set of double cosets*

$$(4.4.1) \quad E^*(K_\pi, P_\pi) \backslash E^*(X_\pi, \Gamma P_\pi) / \left[E^*(X, \Gamma P) \cap E^*(\Gamma K_{\tau\rho}, \Gamma P_{\tau\rho})\right]$$

and the set of equivalence classes in r.gen$(K, P) =$ r.gen(f).

Proof. Suppose $(K, P) \cdot \alpha \sim (K, P) \cdot \beta$, and let γ be such an equivalence. Then

$$(4.4.2) \qquad (K, P) \cdot \beta = ((K, P) \cdot \alpha)\gamma.$$

Left-multiplying (4.4.2) by Γ and using Lemma 4.3(i) shows that $\gamma \in E(\Gamma P)$. Giving γ^{-1} the same treatment then shows that $\gamma \in E^*(\Gamma P)$.

Next we show that $\gamma \in E^*(X)$. Localizing (4.4.2) at π gives

$$(4.4.3) \qquad K_\pi \beta = K_\pi \alpha\gamma.$$

Since ${}_R(X/\Gamma K)$ has finite length, it is annihilated by a regular element of R, by Lemma 3.1. Therefore $QX = Q\Gamma K = QK$, where $Q = Q(R)$. Left-multiplying (4.4.3) by Q and remembering that π is a regular localization (Lemma 2.4) therefore shows $QX\beta = QX\alpha\gamma$. Since α and β are automorphisms of X_π, hence of QX, we have $QX = QX\gamma$. Hence $X\gamma \subseteq (QX) \cap \Gamma P = X$ (since X is a direct summand of ΓP). Similarly, $X\gamma^{-1} \subseteq X$, so $\gamma \in E^*(X)$, hence $\gamma \in E^*(X, \Gamma P)$.

We claim that γ belongs to the group $\mathscr{E} = E^*(X, \Gamma P) \cap E^*(\Gamma K_{\tau\rho}, \Gamma P_{\tau\rho})$ at the extreme right of (4.4.1). When $\mathfrak{m} \notin \pi$ we have $((K, P) \cdot \alpha)_{\mathfrak{m}\rho} = (K, P)_{\mathfrak{m}\rho} = ((K, P) \cdot \beta)_{\mathfrak{m}\rho}$. So (4.4.2), together with the fact that $\Lambda_{\tau\rho} = \Gamma_{\tau\rho}$, shows

$$(4.4.4) \qquad \gamma_{\mathfrak{m}\rho} \in E^*\left(K_{\mathfrak{m}\rho}, P_{\mathfrak{m}\rho}\right) = E^*\left(\Gamma K_{\mathfrak{m}\rho}, \Gamma P_{\mathfrak{m}\rho}\right) \quad \text{when } \mathfrak{m} \notin \pi.$$

This verifies, locally, that $\gamma \in E^*(\Gamma K_{\tau\rho}, \Gamma P_{\tau\rho})$, hence $\gamma \in \mathscr{E}$.

Returning to (4.4.3), we see that $\alpha\gamma\beta^{-1} \in E^*(K_\pi, P_\pi)$ as claimed in the theorem.

Conversely, if $\alpha\gamma\beta^{-1} \in E^*(K_\pi, P_\pi)$ with $\gamma \in \mathscr{E}$, then reading backwards establishes (4.4.2), so γ is an equivalence, as desired.

Lemma 4.2 proves that $(K, P) \cdot \alpha \in \text{r.gen}(K, P)$. Before proceeding to the converse of this, we note that if an inclusion (L, S) is in the genus of (K, P), then $(L_\pi, S_\pi) \sim (K_\pi, P_\pi)$ (equivalent inclusion maps). To see this, let i and j be the original two inclusion maps. Then, in the notation of Remarks 1.8, the $T_2(\Lambda)$-modules $M(i)$ and $M(j)$ are in the same genus. Hence so are the $T_2(\Lambda_\pi)$-modules $M(i_\pi)$ and $M(j_\pi)$. Since $T_2(\Lambda_\pi)$ is a module-finite algebra over the semilocal ring R_π, the $T_2(\Lambda_\pi)$-modules $M(i_\pi)$ and $M(j_\pi)$ are therefore isomorphic, as claimed.

Now suppose $(L, S) \in \text{r.gen}(K, P)$. Then $(\Gamma L, \Gamma S) \sim (\Gamma K, \Gamma P)$. So we can suppose $(\Gamma L, \Gamma S) = (\Gamma K, \Gamma P)$. In particular, $(L_{\mathfrak{m}\rho}, S_{\mathfrak{m}\rho}) = (K_{\mathfrak{m}\rho}, P_{\mathfrak{m}\rho})$ when $\mathfrak{m} \notin \pi$. By hypothesis $(L, S) \in \text{gen}(K, P)$. So, as shown in the previous paragraph, $(L_\pi, S_\pi) \sim (K_\pi, P_\pi)$. Letting α be such an equivalence we get $(L, S) = (K, P) \cdot \alpha$, completing the proof of the theorem provided we can show that $\alpha \in E^*(X_\pi, \Gamma P_\pi)$. Since $\Gamma L = \Gamma K$, this follows from Lemma 4.3(ii). $\square$

Let δ be an endomorphism of some module, or of some inclusion of modules, and suppose that $\delta^2 = 0$. Then $\varepsilon = 1 + \delta$ is an automorphism of that module or inclusion. We call such an ε an *elementary automorphism*.

4.5 COROLLARY. *$(K, P) \cdot \varepsilon \sim (K, P)$ for (K, P) as in* (4.1.1) *and $\varepsilon = 1 + \delta$ any elementary automorphism of $(X_\pi, \Gamma P_\pi)$.*

Proof. The R-module $X/\Gamma K$ has finite length as an R-module. So does $\Gamma K/K$, since it is annihilated by regular elements of the ring R of dimension 1 (namely, regular elements of any conductor ideal from Γ to Λ). So X/K has finite length, and hence is the direct sum of its localizations at maximal ideals of R. Each of these localizations is annihilated by a power of a maximal ideal of R. Therefore X/K is annihilated by an ideal of the form I^eH where I is the conductor ideal in Notation 2.2, and H is a product of maximal ideals none of which belongs to π. By the Chinese Remainder Theorem, there is an element $s \in H$ such that $s \in R - \bigcup_\pi$. We have

$$I^e sX \subseteq K. \tag{4.5.1}$$

We have $\delta \in E(X_\pi, \Gamma P_\pi)$ so $r\delta \in E(X, \Gamma P)$ for some $r \in R - \cup\, \pi$. Since π is the set of all maximal ideals containing I, rs has a reciprocal t modulo I^e. So $1 - rst \in I^e$. We have

$$\varepsilon = 1 + \delta = (1 + rst\delta)(1 + [1 - rst]\delta). \tag{4.5.2}$$

It now suffices to prove that the factors in this commutative factorization belong to the subgroups from which cosets were formed in (4.4.1) of the Double Coset Theorem.

Since $rst\delta$ belongs to $E(X, \Gamma P)$ and has square zero, we have

$$(1 + rst\delta) \in E^*(X, \Gamma P).$$

Localizing (4.5.1) yields $sX_{\tau\rho} \subseteq K_{\tau\rho}$. Hence

$$(K_{\tau\rho}, P_{\tau\rho})rst\delta \subseteq (X_{\tau\rho}, P_{\tau\rho})s \subseteq (K_{\tau\rho}, P_{\tau\rho})$$

so $(1 + rst\delta) \in E^*(K_{\tau\rho}, P_{\tau\rho})$. Since $\Lambda_{\tau\rho} = \Gamma_{\tau\rho}$ we have

$$(1 + rst\delta) \in E^*(\ldots) \cap E^*(\ldots)$$

as required.

On the other hand,

$$(K_\pi, P_\pi)[1 - rst]\delta \in (I^e X_\pi, I^e \Gamma P_\pi) \subseteq (K_\pi, P_\pi) \quad \text{(by (4.5.1))}$$

so the last factor in (4.5.2) belongs to $E^*(K_\pi, P_\pi)$. $\square$

4.6 LEMMA. *Let W, X be locally free Γ-modules of ranks ≥ 2. Then $\nu E^*(W) = \nu E^*(X)$ in $\mathbf{K}_1(\Gamma_\pi)$, where ν denotes "natural image".*

Proof. If rank $X >$ rank W then W is isomorphic to a direct summand of X, by Lemma 1.2. Since Γ_π has 1 in its stable range, we conclude that $\nu E^*(X) \supseteq \nu E^*(W)$. Hence, if rank $X \geq$ rank $W \geq$ rank W' and $\nu E^*(X) \subseteq \nu E^*(W')$, then we have $\nu E^*(X) = \nu E^*(W)$.

The proof is therefore completed by repeated use of the special case of the lemma in which

$$\text{rank } X = \text{rank } W + 1. \tag{4.6.1}$$

We now prove this special case.

Since W is isomorphic to a direct summand of X, we can suppose that $X = W \oplus V$. Since rank $V = 1 \leq$ (rank X) $- 2$, Lemma 2.12 yields $\delta \in \ker \nu$ (since Γ_π has 1 in its stable range, so that the abstract elementary automorphisms in that lemma become products of actual elementary matrices) with $V\delta = V\alpha$. Thus $\beta = \alpha\delta^{-1}$ has a 2×2 block upper triangular form when viewed with respect to the decomposition $X = W \oplus V$. So

$$\nu(\alpha) = \nu(\beta) = \nu(\beta_{11})\nu(\beta_{22}).$$

We have $\beta_{11} \in E^*(W)$. Since rank $V = 1 <$ rank W, V is isomorphic to a direct summand of W. So $\beta_{22} \in E^*(V)$ can be extended to an automorphism γ of a module $\cong W$ such that $\nu(\gamma) = \nu(\beta_{22})$. Therefore $\nu(\alpha) = \nu(\beta_{11})\nu(\gamma) \in \nu E^*(W)$, as desired. □

4.7 *Notation.* We now set up some notation for use in stating the Mayer-Vietoris-like sequence in the next theorem. As usual, let f be as in (4.1.1). Write $\Gamma = \oplus_k \Gamma_k$ where each Γ_k is a maximal order in a simple artinian ring. Let $\Gamma P = X \oplus Y$ as in (4.1.2) and set

$$\Gamma(X) = \bigoplus \{\Gamma_k | \Gamma_k X \neq 0\} \quad \text{and} \quad \Gamma(Y) = \bigoplus \{\Gamma_k | \Gamma_k Y \neq 0\}. \tag{4.7.1}$$

Let r.$\mathcal{G}(f)$ be the subgroup of $\mathcal{G}(f)$ consisting of all $[g]$ with $g \in$ gen(f) such that $[1 \otimes g] = [1 \otimes f]$ in $\mathcal{G}(1 \otimes f)$, where $\otimes$ refers to $\Gamma \otimes_\Lambda (\ldots)$. When full stability holds in gen(f) and gen($1 \otimes f$), r.$\mathcal{G}(f)$ consists of all equivalence classes $[g]$ of presentations $g \in$ r.gen(f). As in Notation 2.2, I denotes a conductor ideal from Γ to Λ, π denotes the set of maximal ideals of R that contain I; and we set

$$\bar{\Gamma} = \Gamma/I\Gamma, \quad \bar{\Gamma}(X) = \Gamma(X)/I \cdot \Gamma(X),$$
$$\bar{\Gamma}(Y) = \Gamma(Y)/I \cdot \Lambda(Y) \quad \text{and} \quad \bar{\Lambda} = \Lambda/I\Lambda.$$

4.8 THEOREM. *For every presentation f of a Λ-module, as in* (4.1.1), *there is an exact sequence*

$$(4.8.1)\quad E^*(K_\pi, P_\pi) \times \mathbf{K}_1(\Gamma(X) \oplus \Gamma(Y)) \xrightarrow{\nu} \mathbf{K}_1(\overline{\Gamma}(X) \oplus \overline{\Gamma}(Y)) \xrightarrow{\sigma} \mathcal{G}(f) \xrightarrow{\gamma} \mathcal{G}(\Gamma(X)) \times \mathcal{G}(\Gamma(Y))$$

in which $\operatorname{im}\sigma = \mathrm{r}.\mathcal{G}(f)$.

Proof. Recall that $\mathcal{G}(f) = \mathcal{G}(f^d)$ (canonical isomorphism) for every d. Thus, replacing f by f^2, we can assume that full stability holds in gen(f) and gen($1 \otimes f$) where $\otimes$ refers to $\Gamma \otimes_\Lambda (\ldots)$ (see Corollary 1.12). In what follows we shall use even larger values of d to achieve further simplification. Note that this never changes $\Gamma(X)$ or $\Gamma(Y)$.

Consider the decomposition $\Gamma P = X \oplus Y$ in (4.1.2). Replacing f by a suitable f^d, and applying our Stable Invariant Factor Theorem 3.3 to each Γ_k (which is a maximal order in a simple artinian ring, and is not artinian by (2.1.2) and Lemma 3.1), we get a pair of compatible *external* direct sum decompositions of the form (4.8.2) and (4.8.3), in which each T_i is a two-sided ideal, $\neq 0$, of some Γ_k and $X_i \in \text{gen}(\Gamma_k)$. Moreover, each $Y_j \in$ gen(some Γ_k).

$$(4.8.2)\qquad \Gamma P = X_1 \oplus \cdots \oplus X_r \oplus Y_1 \oplus Y_2 \oplus \cdots \oplus Y_s,$$

$$(4.8.3)\qquad \Gamma K = T_1 X_1 \oplus \cdots \oplus T_r X_r$$

with $X = X_1 \oplus X_2 \oplus \cdots \oplus X_r$ and $Y = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_s$.

Moreover, by doubling the previous d if necessary, we can ensure that each distinct ideal T_i that appears in (4.8.3) appears there *at least twice*.

Finally, identify each $(X_i)_\pi$ and $(Y_j)_\pi$ with the $(\Gamma_k)_\pi$ to which it is isomorphic. Then $(T_i X_i)_\pi = (T_i)_\pi$. We now have

$$E(X_\pi, \Gamma P_\pi) = \oplus_k E(\Gamma_k X_\pi, \Gamma_k P_\pi)$$

where each ring $E(\Gamma_k X_\pi, \Gamma_k P_\pi)$ is a 2×2 block lower triangular matrix ring over $(\Gamma_k)_\pi$, acting by right multiplication on the direct sum of those terms in (4.8.2) that are $(\Gamma_k)_\pi$-modules. The blocks in this 2×2 matrix ring result from viewing $\Gamma_k P$ as (X-summands) $\oplus$ (Y-summands), and the $(1,2)$-blocks of zeros result from the fact that $\Gamma_k X_\pi \to \Gamma_k X_\pi$ in $E(\Gamma_k X_\pi, \Gamma_k P_\pi)$.

Thus we can view each element $\alpha \in E^*(X_\pi, \Gamma P_\pi)$ as a tuple of 2×2 block lower triangular matrices $\alpha(k)$, one for each coordinate ring $(\Gamma_k)_\pi$ of Γ_π.

We are now ready to begin dealing with the maps ν, σ, γ in (4.8.1).

DEFINITION OF ν. This denotes "natural image in $\mathbf{K}_1$" of the appropriate ring. So the definition of ν on the factor $\mathbf{K}_1(\Gamma(X) \oplus \Gamma(Y))$ is clear. To define ν on $E^*(K_\pi, P_\pi)$, recall that $E^*(K_\pi, P_\pi) \subseteq E^*(X_\pi, \Gamma P_\pi)$, by Lemma 4.3.

Hence, for $\alpha \in E^*(K_\pi, P_\pi)$, we can use the 2×2 block triangular form of each $E(\Gamma_k X_\pi, \Gamma_k P_\pi)$ to define

(4.8.4) $\quad \nu(\alpha)(k) = \text{natural image of } (\alpha(k)_{11}, \alpha(k)_{22}) \text{ in } \mathbf{K}_1(\overline{\Gamma}(X) \oplus \overline{\Gamma}(Y))$

By the local freeness of X_π and Y_π (due to using f^d in place of the original f) this definition suffices for the present proof.

However, for use in applications of the theorem, we give a more general definition of "natural image" in (4.8.4) that shows that $\mathrm{im}(\nu)$ is independent of the value of d that produces local freeness, and can be applied to the original $E^*(K_\pi, P_\pi)$, even when X_k and Y_k are not locally Γ_k-free. Each ring $(\Gamma_k)_\pi$ is a full matrix ring over a (noncommutative) PID $\Delta_k(\pi)$, as explained in Notation 2.8. Therefore $\alpha(k)_{11}$ and $\alpha(k)_{22}$ can be viewed as right multiplications by matrices over $\Delta_k(\pi)$, and hence have natural images in $\mathbf{K}_1(\Delta_k(\pi)) = \mathbf{K}_1((\Gamma_k)_\pi)$ which has a natural image in $\overline{\Gamma}_k$. This is the natural image that we use in (4.8.4). Since $\Delta_k(\pi)$ has 1 in its stable range, the image of ν is independent of the uniform ranks of $\Gamma_k X_\pi$ and $\Gamma_k Y_\pi$ whenever they are nonzero.

Definition of γ. For $[g\colon S \twoheadrightarrow V] \in \mathscr{G}(f)$ let $\gamma'([g]) = [1 \otimes g]$ where $\otimes$ refers to $\Gamma \otimes_\Lambda (\dots)$. Then $\ker \gamma' = \mathrm{r.}\mathscr{G}(f)$. By using Proposition 3.9 separately for each Γ_k we get an isomorphism $\gamma''\colon\ \mathscr{G}(1 \otimes f) \cong \mathscr{G}(\Gamma(X)) \times \mathscr{G}(\Gamma(Y))$. Then let $\gamma = \gamma''\gamma'$.

Definition of σ. We do this by means of the following commutative diagram, in which $\sigma'(\alpha)$ is the equivalence class of the inclusion $(K, P) \cdot \alpha$, κ and ω denote "natural image in $\mathbf{K}_1$" and σ'' and σ are the unique homomorphisms that make the diagram commute.

(4.8.5)
$$\begin{tikzcd} E^*(X_\pi, \Gamma P_\pi) \arrow[r, two heads, "\kappa"] \arrow[dr, "\sigma'"'] & \mathbf{K}_1(\Gamma_\pi(X) \oplus \Gamma_\pi(Y)) \arrow[r, two heads, "\omega"] \arrow[d, "\sigma''"] & \mathbf{K}_1(\overline{\Gamma}(X) \oplus \overline{\Gamma}(Y)) \arrow[dl, "\sigma"] \\ & \mathrm{r.}\mathscr{G}(K, P) & \end{tikzcd}$$

By our Double Coset Theorem, $\mathrm{im}(\sigma')$ is the restricted genus of (K, P) as desired. We claim that σ' is a group homomorphism.

We want to prove that $(K, P) \cdot \alpha \oplus (K, P) \cdot \beta \sim (K, P) \cdot (\alpha\beta) \oplus (K, P)$ for every α, β. By Corollary 4.5 it suffices to prove

(4.8.6)
$$\alpha \oplus \beta = (\alpha\beta \oplus 1)\varepsilon$$

for some ε in $F = E((X_\pi, \Gamma P_\pi) \oplus (X_\pi, \Gamma P_\pi))$ that is a product of elementary automorphisms in F. We can view F as the ring of 2×2 matrices over $H = E(X_\pi, \Gamma P_\pi)$. Since H is a module-finite algebra over the semilocal ring R_π, one is in the stable range of H. Therefore (4.8.6) reduces to the well-known (and easily proved) formula $\operatorname{diag}(\alpha, \beta) = (\operatorname{diag}(\alpha\beta, 1))\varepsilon$ that holds in the 2×2 matrix ring over any ring H with one in its stable range, where α and β are units of H and ε is a product of elementary automorphisms that come from elementary column operations.

To define the homomorphism κ in (4.8.5) we view each $\alpha \in E^*(X_\pi, \Gamma P_\pi)$ as a tuple of block lower triangular matrices $\alpha(k)$, as explained above, and let $\kappa(\alpha)$ be the natural image, in $\mathbf{K}_1(\Gamma_\pi(X) \oplus \Gamma_\pi(Y))$, of the matrix pair $(\alpha(k)_{11}, \alpha(k)_{22})$. Since $\Gamma_\pi(X)$ and $\Gamma_\pi(Y)$ are rings with 1 in their stable range, κ is a surjection.

To show the existence of a unique map σ'' making the first triangle in diagram (4.8.5) commute, it now suffices to check that $\ker(\kappa) \subseteq \ker(\sigma')$. Each nonzero $(\Gamma_k)_\pi$-module $\Gamma_k X_\pi$ and $\Gamma_k Y_\pi$ is free of rank ≥ 2. So, by the "stable range 1" condition in the previous paragraph, $\ker(\kappa)$ is generated by coordinatewise elementary matrices [B '68, p. 240]. Every such matrix is an element of $\ker(\sigma')$ by Lemma 4.5. Thus σ'' exists.

To complete the definition of σ, and the proof of the theorem, it now suffices to establish the following facts. The following sequence is exact:

$$(4.8.7)\quad E^*(K_\pi, P_\pi) \times \mathbf{K}_1(\Gamma(X) \oplus \Gamma(Y)) \xrightarrow{\nu} \mathbf{K}_1(\Gamma_\pi(X) \oplus \Gamma_\pi(Y)) \xrightarrow{\sigma''} \mathrm{r}.\mathcal{G}(f)$$

where ν again denotes "natural image in $\mathbf{K}_1$", the map ω in (4.8.5) is a surjection, and $\ker\omega \subseteq \ker\sigma''$, so that σ'' induces the required map σ in (4.8.1).

Our Double Coset Theorem 4.4 gives $\ker\sigma' = E^*(K_\pi, P_\pi)H$, where

$$(4.8.8)\qquad H = E^*(X, \Gamma P) \cap E^*\big(\Gamma K_{\tau\rho}, \Gamma P_{\tau\rho}\big)$$

so the description (4.8.7) of $\ker(\sigma'')$ will be proved if we show that for each k we have

$$(4.8.9)\quad \kappa H(k) = \kappa E^*(X, \Gamma P)(k) = \nu\mathbf{K}_1\big(\Gamma_k(X) \oplus \Gamma_k(Y)\big)$$
$$\big(\text{natural image in coordinate } k \text{ of } \mathbf{K}_1\big(\Gamma_\pi(X) \oplus \Gamma_\pi(Y)\big)$$

For the first equality it suffices to establish the inclusion $\supseteq$.

For $\alpha \in E^*(X, \Gamma P)$ we have $\kappa(\alpha)(k) = (\alpha(k)_{11}, \kappa\alpha(k)_{22})$, so we can suppose that $\alpha(k)_{12} = 0$. It suffices to find $\beta \in E^*(X) \cap E^*(\Gamma K)$ such that, for each k, $\kappa(\beta(k)) = \kappa(\alpha(k)_{11})$. For then $(\beta(k), \alpha(k)_{22})$ is the desired element of $H(k)$. So fix a value of k.

We use the fact that every T_i that occurs in (4.8.3) occurs at least twice. Let X_i and X_j be a pair of terms in (4.8.3) that are Γ_k-modules and such that $T_i = T_j$, which we now all T. We have

$$\kappa(E^*(X)(k)) = \kappa\big(E^*(X_i \oplus X_j)\big)$$

by Lemma 4.6. Take $\beta(k) \in E^*(X_i \oplus X_j)$ such that $\kappa(\beta(k)) = \kappa\alpha(k)_{11}$. Then

$$\beta(k) \in E^*\big(X_i \oplus X_j\big) \subseteq E^*\big(TX_i \oplus TX_j\big) \subseteq E^*(\Gamma K)$$

so β satisfies the required conditions. Thus we now know that (4.8.7) is exact.

The map ω in (4.8.5) is a surjection because the rings involved have 1 in their stable range, and units can be lifted where necessary.

Finally, we show that $\ker \omega \subseteq \ker \sigma''$. Since $\overline{\Gamma}$ has 1 in its stable range, $\ker \omega$ is generated by the images of all 2×2 elementary matrices [B '68, p. 240]. For such an elementary matrix ε we have $(K, P) \cdot \varepsilon \sim (K, P)$ by Lemma 4.5. So $\sigma''\kappa(\varepsilon) = \sigma'(\varepsilon) = 1$, and this completes the proof of the theorem. □

We now begin working toward our explicit formula for $\operatorname{pres}_f(P, U)$. The relation between this group and restricted class groups is given by:

4.9 Lemma. *Let*

$$K \rightarrowtail P \xrightarrow{f} U$$

be a presentation of a Λ-module where P is a progenerator. Then

$$\text{(4.9.1)} \qquad \operatorname{pres}_f(P, U) = \ker(\mathrm{r}.\mathscr{G}(K, P) \to \mathrm{r}.\mathscr{G}(K \oplus \Lambda, P \oplus \Lambda))$$

Proof. Replacing f by f^2 and Λ by Λ^2 we can suppose that full stability holds in $\operatorname{gen}(f)$. Let $(f, 0)$ map $P \oplus \Lambda^2 \twoheadrightarrow U$. Then full stability also holds in $\operatorname{gen}(f, 0)$. We claim that $\operatorname{pres}_f(P, U) = \ker(\mathscr{G}(f) \to \mathscr{G}(f, 0))$.

By Lemma 1.10 every presentation $g\colon P \twoheadrightarrow U$ is in $\operatorname{gen}(f)$; and by Lemma 1.13, we have $(g, 0) \sim (f, 0) = 0 \in \mathscr{G}(f, 0)$. This establishes the inclusion $\subseteq$. For the opposite inclusion, suppose

$$(g, 0) \sim (f, 0) \quad \text{for } (g\colon S \twoheadrightarrow V) \in \mathscr{G}(f).$$

Then, by full stability, $V \cong U$ and $S \oplus \Lambda^2 \cong P \oplus \Lambda^2$. Since P is a progenerator, we have $P^n \cong \Lambda \oplus X$ for some n and X. Hence $S \oplus P^{2n} \cong P^{2n+1}$. Stability in $\operatorname{gen}(P)$ and Lemma 1.3 now give $S \cong P$, completing the proof of the claim.

Since $\mathrm{pres}_f(P, U) \subseteq \mathrm{r}.\mathcal{G}(f)$ by Corollary 3.7 to our unique presentability theorem for maximal orders, we have

$$\mathrm{pres}_f(P, U) = \ker(\mathrm{r}.\mathcal{G}(f) \to \mathrm{r}.\mathcal{G}(f, 0)),$$

and the lemma follows. □

We now assume that K is faithful. Then $\Gamma(X) = \Gamma$ in the notation of (4.7.1). In this situation we have:

4.10 LEMMA. *$\nu E^*(K_\pi \oplus \Lambda_\pi, P_\pi \oplus \Lambda_\pi) = \nu E^*(K_\pi, P_\pi)\nu(\mathbf{K}_1(\overline{\Lambda}) \times 1)$, where ν denotes "natural image in $\mathbf{K}_1(\overline{\Gamma} \oplus \overline{\Gamma}(Y))$.*

Proof. Note that $\nu\mathbf{K}_1(\overline{\Lambda}) = \nu\mathbf{K}_1(\Lambda_\pi)$ (natural image in $\mathbf{K}_1(\overline{\Gamma})$) since all three rings have 1 in their stable range. We use the latter form in this proof. We have

$$\Gamma \cdot (P \oplus \Lambda) = X \oplus \Gamma \oplus Y, \tag{4.10.1}$$

$$\Gamma \cdot (K \oplus \Lambda) = \Gamma K \oplus \Gamma. \tag{4.10.2}$$

We establish the lemma by means of two opposite inclusions. The inclusion $\supseteq$ holds because right multiplication on Λ_π by each unit of Λ_π is an element of $E^*(K_\pi \oplus \Lambda_\pi, P_\pi \oplus \Lambda_\pi)$.

For the opposite inclusion, note that we have

$$E(K_\pi \oplus \Lambda_\pi, P_\pi \oplus \Lambda_\pi) = E\left[(K_\pi, P_\pi) \oplus (\Lambda_\pi, \Lambda_\pi)\right].$$

So each element $\alpha \in E^*(K_\pi \oplus \Lambda_\pi, P_\pi \oplus \Lambda_\pi)$ can be considered to be right multiplication by a 2×2 "matrix" with α_{22} an element of $\Lambda_\pi = E(\Lambda_\pi, \Lambda_\pi)$, with α_{12} a map from (K_π, P_π) to $(\Lambda_\pi, \Lambda_\pi)$, and so on.

Let $\beta = \alpha^{-1}$. The (2, 2)-entry of $\alpha\beta$ is $1 = \alpha_{21}\beta_{12} + \alpha_{22}\beta_{22}$. Here, the product $\alpha_{21}\beta_{12}$ as well as the factors α_{22} and β_{22} belong to $E(\Lambda_\pi, \Lambda_\pi) = \Lambda_\pi$, a ring with 1 in its stable range. So there is an expression

$$\alpha_{21}\beta_{12}x + \alpha_{22} = u$$

with $x, u \in \Lambda_\pi$ and u a unit. Therefore we have the elementary column operation

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} 1 & \beta_{12}x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} * & * \\ * & u \end{bmatrix}$$

Now our new matrix α has α_{22} equal to the unit u. So an additional pair of elementary row and column operations puts α into diagonal form without changing its image in $\mathbf{K}_1$, as desired. □

We now give our explicit formula for $\operatorname{pres}_f(P, U)$, with notation as in 4.7. The computations of specific presentation class groups in Section 6 will be applications of this result.

4.11 THEOREM. *Let*

$$K \rightarrowtail P \xrightarrow{f} U$$

be a presentation of a Λ*-module, with* K *faithful, and* P *a progenerator. Then*

(4.11.1) $\operatorname{pres}_f(P, U)$

$$\cong \text{image of } K_1(\overline{\Lambda}) \times 1 \text{ in } \frac{\mathbf{K}_1(\overline{\Gamma} \oplus \overline{\Gamma}(Y))}{\nu E^*(K_\pi, P_\pi)\nu\mathbf{K}_1(\Gamma \oplus \Gamma(Y))}$$

where ν denotes "natural image in $\mathbf{K}_1(\overline{\Gamma} \oplus \overline{\Gamma}(Y))$".

Proof. Applying (4.8.1) gives

(4.11.2) $\mathrm{r}.\mathscr{G}(K, P) \cong \mathbf{K}_1(\overline{\Gamma} \oplus \overline{\Gamma}(Y))/\nu E^*(K_\pi, P_\pi)\nu\mathbf{K}_1(\Gamma \oplus \Gamma(Y))$.

Applying (4.11.2) to the inclusion $(K \oplus \Lambda, P \oplus \Lambda)$ and using Lemma 4.10 we get

(4.11.3) $\mathrm{r}.\mathscr{G}(K \oplus \Lambda, P \oplus \Lambda)$

$$\cong \mathbf{K}_1(\overline{\Gamma} \oplus \overline{\Gamma}(Y))/\nu E^*(K_\pi, P_\pi)\nu\big(\mathbf{K}_1(\overline{\Lambda}) \times 1\big)\nu\mathbf{K}_1(\Gamma \oplus \Gamma(Y))$$

Isomorphism (4.11.1) now follows immediately from Lemma 4.9. □

4.12 COROLLARY. *Let*

$$K \rightarrowtail P \xrightarrow{f} U$$

be a presentation of a Λ*-module, with* K *faithful and* P *a progenerator. Then* $\operatorname{pres}_f(P, U)$ *is a homomorphic image of* $(\overline{\Lambda})^*$.

Proof. $\overline{\Lambda} = \Lambda/I\Lambda$ is artinian, hence has 1 in its stable range. So $\mathbf{K}_1(\overline{\Lambda})$ is a homomorphic image of $\overline{\Lambda}^*$. □

5. Global fields

5.1 *Notation.* In this section R denotes a Dedekind domain whose field of quotients $Q(R)$ is a global field, and Λ denotes an R-order in a semisimple, separable $Q(R)$-algebra $A = Q(R)\Lambda$. As usual, Γ denotes a maximal order in A, containing Λ, and I denotes a nonzero ideal of R such that $I\Gamma \subseteq \Lambda$.

5.2 THEOREM. *There is an integer $n = n(\Lambda)$ such that, for every presentation $f\colon P \twoheadrightarrow U$ of a Λ-module, $|\mathcal{G}(f)| < n$. In particular, $|\mathrm{pres}_f(P, U)| < n$.*

Proof. This is an immediate consequence of the Mayer-Vietoris sequence in Theorem 4.8, since the group at the extreme right is finite (by the Jordan-Zassenhaus Theorem) and $\mathbf{K}_1$ of a finite ring is again finite. $\square$

Theorem 5.2 is most interesting when it applies to actual, rather than stable presentation classes, that is, when full stability holds (see Definitions 1.9). Our results about full stability require an "Eichler condition" which, for the orders considered here, is more general than the Drozd Condition in Theorem 2.18 and its corollary.

5.3 *Eichler condition.* Let U be a Λ-module, with Λ as in Notation 5.1. We say that U satisfies the *Eichler condition* (relative to R) if for each simple A-module S that appears exactly once as a composition factor of the A-module $Q(R) \otimes_R U$, the endomorphism ring $E(S)$ is *not* one of the division algebras B listed in (i) and (ii) below. (We sometimes refer to these division algebras B as *exceptional.*)

(i) (if $Q(R)$ is an algebraic number field) A *totally definite quaternion algebra B*; that is, a division algebra B such that, for all valuations v arising from embeddings of $Q(R)$ in the complex numbers, the completion B_v is isomorphic to the division algebra $\mathbf{H}$ of real quaternions (hence B itself has dimension 4 over its center).

(ii) (if $Q(R)$ is a function field of characteristic $\neq 0$). A division algebra B such that, for every nontrivial valuation v of $Q(R)$ whose associated valuation ring is not a localization of R, the completion B_v is a direct product of noncommutative division algebras.

The Eichler condition places no restriction on composition factors of $Q(R) \otimes_R U$ that occur more than once.

This definition is taken from Swan [Sw '80, p. 174], and yields a sharper cancellation result than some earlier definitions of the Eichler Condition.

Our stability result is:

5.4 JACOBINSKI CANCELLATION THEOREM (presentations). *Let*

$$K \rightarrowtail P \xrightarrow{f} \twoheadrightarrow U$$

be a presentation of a Λ-module, and suppose that K and U satisfy the Eichler condition. Then full direct-sum cancellation holds in the genus of f.

Proof. Let $Q = Q(R)$. The proof of the Drozd Cancellation Theorem 2.16 reduces the proof of the present theorem to showing that direct-sum cancellation holds for locally free E'-modules, where E' is an R-order (defined in the proof of Theorem 2.16) in the semisimple artinian ring $E(QK) \oplus E(Q\overline{U})$.

In the situation that Q is a field of characteristic $\neq 0$, we need to know that the semisimple Q-algebra $E(Q\overline{U})$ is separable over Q. But since $\overline{U}$ is a Λ-*lattice*, the division algebras associated with $E(Q\overline{U})$ are among those associated with $A = Q\Lambda$, and hence their centers are separable over Q.

Now the Eichler Condition for K and U allows the use of the Jacobinski Cancellation Theorem for locally free E'-modules [S '80, 9.3]. □

5.5 JACOBINSKI CANCELLATION THEOREM (modules). *Let U be a Λ-module that satisfies the Eichler condition. Then direct-sum cancellation holds in the genus of U.*

Proof. This is proved in [G '87, 6.7]. Alternatively, copy the proof of the Drozd Cancellation Theorem 2.17 for modules, but use the Jacobinski Cancellation Theorem for Presentations in place of the corresponding Drozd theorem. □

As an immediate consequence, we have:

5.6 THEOREM (full stability). *Let*

$$K \rightarrowtail P \xrightarrow{f} \twoheadrightarrow U$$

be a presentation of a Λ-module, and suppose that K, P, and U satisfy the Eichler condition. Then full stability holds in gen(f).

5.7 COROLLARY. *Suppose no division algebra of $A = Q(R)\Lambda$ is one of the exceptional algebras listed in the Eichler condition. Then full stability holds in* gen(f), *for every presentation f: $P \twoheadrightarrow U$ of every Λ-module.*

Proof. For every simple A-module S, $E(S)$ is among the division algebras of A. Hence K, P, and U satisfy the Eichler condition, and Theorem 5.6 applies. □

5.8 *Examples.* Examples of specific integral group rings Λ of finite groups G to which the preceding corollary applies are $\Lambda = \mathbf{Z}G$ where G has no elements of order 4 [I '76, p. 165, Theorem 10.9], or is a commutative group, or is any group (e.g., a symmetric group) for which $\mathbf{Q}$ is a splitting field. On the other hand, if R is the ring of integers in any number field $Q(R)$ such that

$Q(R)$ has at least one valuation v such that the completion $Q(R)_v$ is the complex numbers, then the corollary applies to $\Lambda = RG$ for *every* finite group G.

To complete this section, we take a brief look at what happens when the Eichler condition fails, hence $\mathrm{pres}_f(P, U)$ might not consist of actual equivalence classes. Briefly: the number of presentations of any U by any P remains finite; but, in the case of number fields, the uniform bound $n = n(\Lambda)$ *never* exists.

5.9 PROPOSITION. *Let*

$$K \rightarrowtail P \xrightarrow{f} U$$

be a presentation of a Λ-module. Then the genus of f is finite.

Proof. It suffices to show that the order E' in the proof of Theorem 2.16 has only finitely many isomorphism classes in the genus of E' itself. This is done by the Jordan-Zassenhaus theorem. □

Let $\#(P \twoheadrightarrow U)$ denote the number of presentations of the Λ-module U by P.

5.10 THEOREM. *Let $Q = Q(R)$ be an algebraic number field, and suppose that some division algebra of $A = Q\Lambda$ is a totally definite quarternion algebra. There is no uniform bound $n = n(\Lambda)$ such that $\#(P \twoheadrightarrow U) \leq n$ for every P and U.*

Proof. We first consider the case that $\Lambda = \Gamma$, a maximal order, and produce an infinite sequence of presentations: $P \twoheadrightarrow U_\alpha$ of presentations of Γ-modules such that

$$\#(P \twoheadrightarrow U_\alpha) \to \infty \quad \text{as } \alpha \to \infty \tag{5.10.1}$$

with P cyclic of uniform rank 1 and each U_α simple.

Since the maximal order Γ is a direct sum of maximal orders in the simple artinian ring-direct summands of $A = Q\Gamma$, we can suppose that Γ is itself an order in the simple artinian ring A whose associated division ring is the given totally definite quaternion algebra D. Moreover, Γ is Morita equivalent to a maximal order Δ in D, and we claim it suffices to consider the case $\Gamma = \Delta$.

Let P be the projective Δ-module that satisfies (5.10.1). The image P' of P under the Morita equivalence is clearly projective and of uniform rank 1. Thus it only remains to show that P' is cyclic when $\Gamma \neq \Delta$. But, in this situation ($\Gamma \neq \Delta$) *every* uniform projective Γ-module P' is cyclic, because ${}_\Gamma\Gamma$ is isomorphic to the direct sum of at least 2 uniform projective modules one of which can be chosen arbitrarily. Thus P' is isomorphic to a direct summand of Γ, hence is cyclic.

Let $Z = Z(\Delta)$, the center of Δ.

Since D is a division algebra of characteristic zero and dimension 4 over its center $Q(Z)$, we have $D = Q(Z)[i, j, k]$ where $i^2 = a$, $j^2 = b$, $k = ij = -ji$, and a and b are nonzero elements of Z.

Let $\mathfrak{M} = \{M_\alpha\}$ be any infinite sequence of maximal ideals of Z such that the characteristic of the finite field Z/M_α approaches ∞ as $\alpha \to \infty$. We can suppose that none of these fields has characteristic 2. Let $\Delta' = Z[i, j, k]$. After deleting a finite number of terms of $\mathfrak{M}$, we can suppose that $\overline{\Delta}_\alpha = \Delta/(M_\alpha\Delta)$ equals $(Z/M_\alpha)[i, j, k]$ for all α, and i, j, k are nonzero elements of $\overline{\Delta}_\alpha$. Thus $\overline{\Delta}_\alpha$ is a quaternion, hence simple, algebra over the field Z/M_α, and is noncommutative since Z/M_α has characteristic $\neq 2$. So $\overline{\Delta}_i$ is a 2×2 matrix ring over Z/M_α. Let U_α be the simple $\overline{\Delta}_\alpha$-module. We prove (5.10.1) by showing that $\#(\Delta \twoheadrightarrow U_\alpha) \to \infty$.

We have $U_\alpha \cong \Delta/L_\alpha$ for every maximal left ideal L_α containing M_α. Moreover, $(L_\alpha \rightarrowtail \Delta \twoheadrightarrow U_\alpha) \sim (L_\beta \rightarrowtail \Delta \twoheadrightarrow U_\beta)$ if and only if there is a unit u of Δ such that $L_\alpha u = L_\beta$.

The number of maximal left ideals L_α containing M_α equals the number $|Z/M_\alpha| + 1$ of maximal left ideals of $\overline{\Delta}_\alpha$, hence $\to \infty$ as $\alpha \to \infty$. Hence it suffices find a bound, independent of α, to the number of left ideals in each set $\{L_\alpha u \mid u \in \Delta^*\}$.

Since $L_\alpha v = L_\alpha$ for every $v \in Z^*$, the existence of this bound follows from the known finiteness of the index $[\Delta^* : Z^*]$ for maximal orders in totally definite quaternion algebras. (Sketch of proof: Let N denote the reduced norm from Δ to Z. A topological argument shows $N(\delta) = 1$ for only finitely many $\delta \in \Delta$ because of the total positive definiteness of D. Then $[\Delta^* : Z^*] \leq |\ker N|[N(\Delta^*) : NZ^*] \leq |\ker N|[Z^* : (Z^*)^2]$ which is finite since Z^* is finitely generated.) This completes the case $\Lambda = \Gamma$.

Now let Λ be any R-order such that one of the division algebras of $A = Q\Lambda$ is a totally definite quarternion algebra D. We have $\Gamma = \oplus_k \Gamma_k$ with each Γ_k a maximal order in a simple algebra A_k, and we can suppose that the division algebra associated with A_1 is D. We showed, above, that there exists a cyclic projective Γ_1-module P of uniform rank 1, and a sequence of simple Γ_1-modules U_α such that (5.10.1) holds. Since Γ is a maximal order, we have $\Gamma = P \oplus S$ for some left ideal S. This gives an infinite sequence of Γ-modules $V_\alpha = U_\alpha \oplus S$ for which $\#(\Gamma \twoheadrightarrow V_\alpha) \to \infty$.

Let z be any nonzero (rational) integer such that $z\Gamma \subseteq \Lambda$. After deleting a finite number of terms of the sequence $\{U_\alpha\}$ we have $zU_\alpha \neq 0$ hence $zU_\alpha = U_\alpha$ for every α. For each of the above presentations $f\colon \Gamma \twoheadrightarrow V_\alpha$ we have $f(\Lambda) \supseteq f(z\Gamma) \supseteq U_\alpha$ so, letting f' be the restriction of f to Λ we have

(5.10.2) $\quad f'(\Lambda) = U_\alpha \oplus S_f \quad$ with S_f a Λ-lattice and $zS \subseteq S_f \subseteq S$.

Since $\Lambda/z\Lambda$ is a finite ring, only finitely many modules S_f arise in this way, as α varies. So, passing to a subsequence of $\{U_\alpha\}$, we can suppose they are all equal; say $S_f = S'$ for all f.

We complete the proof by showing $\#(\Lambda \twoheadrightarrow U_\alpha \oplus S') \to \infty$ as $\alpha \to \infty$. To do this it suffices to show that $f' \sim g'$ implies $f \sim g$. The nontrivial part of this is to show that every Λ-automorphism φ of $W_\alpha = U_\alpha \oplus S'$ can be extended to a Γ-automorphism of $\Gamma W_\alpha = U_\alpha \oplus S$. this is well known in the analogous situation where U_α is a lattice, and false for modules in general; but we prove it is true here.

We can think of φ as left multiplication by a 2×2 upper triangular "matrix" where $\varphi_{11} \in E^*({}_\Lambda U_\alpha)$, etc. To see that $\varphi_{11} \in E^*({}_\Gamma U_\alpha)$, take $x \in U_\alpha$, $\gamma \in \Gamma$, and let z be the integer defined above. Then

$$z\varphi_{11}(\gamma x) = \varphi_{11}(z\gamma x) = z\gamma\varphi_{11}(x)$$

since $z\Gamma \subseteq \Lambda$. Since multiplication by z is one-to-one on U_α, we have $\varphi_{11} \in E^*({}_\Gamma U_\alpha)$. Next consider $\varphi_{22} \in E^*({}_\Lambda S')$. Since S' is a Λ-lattice and $S = \Gamma S'$, we can extend φ_{22} to an element of $E^*({}_\Gamma S)$. We deal with φ_{12} by combining the previous two situations: Choose an integer y such that $yz \equiv 1$ modulo the annihilator of U_α. Then (for $x \in S'$ and $\gamma \in \Gamma$) define $\varphi_{21}(\gamma x) = y\varphi_{12}(z\gamma x)$. This completes the proof of the theorem. □

We conclude this section with the nonuniqueness example, promised in the introduction, related to Nakayama's matrix problem.

5.11 *Example.* Let Δ be the ring of integral quaternions,

$$\Delta = \mathbf{Z}[i, j, k, (1 + i + j + k)/2],$$

which is a PID (in fact, Euclidean). Then Δ is a maximal order in a totally definite quaternion algebra. By (5.10.2) we can choose left ideals $\Delta x_1, \Delta x_2, \dots, \Delta x_n$ of Δ (with n as large as we wish, by suitable choice of α) such that the presentations

$$\Delta x_\nu \rightarrowtail \Delta \twoheadrightarrow U_\alpha$$

(α fixed) are all inequivalent.

Hence the 1×1 matrices $[x_\nu]$ are all inequivalent despite the fact that they all present the same left Δ-module U_α.

6. Commutative case

When Λ is a commutative ring-order, full stability (see Definitions 1.9) holds in every genus of presentations of Γ-modules, by Corollary 2.19. Thus elements of $\mathscr{G}(f)$ are always actual equivalence classes of presentations of modules.

We ask what kinds of groups can occur as $\text{pres}_f(P, U)$ in this situation. Perhaps the most surprising of our results is that, if Λ is a finitely generated algebra over an algebraically closed field of characteristic 0, then $\text{pres}_f(P, U) = 0$ whenever U has finite length; that is, every Λ-module of finite length is uniquely presented by every projective module that presents it. Moreover, this becomes false if the field has nonzero characteristic.

6.1 *Notation.* Throughout this section we assume that $\Lambda = R$ is an indecomposable, nonartinian ring-order. Then the maximal order Γ becomes the normalization of R, and $\Gamma = \oplus_k \Gamma_k$ where each Γ_k is a Dedekind domain. Let

$$K \rightarrowtail P \xrightarrow{f} U \tag{6.1.1}$$

be a presentation of an R-module. Throughout this section we assume

$$K \text{ (hence } P\text{) is a faithful } R\text{-module.} \tag{6.1.2}$$

Before proceeding, we note that (6.1.2) *always holds if R is an integral domain or U has finite length*. Here is a sketch of the proof of the assertion involving U of finite length. Since R is indecomposable and of Krull dimension 1, every projective R-module P is a direct sum of faithful ideals of R. On the other hand, since U has finite length, $dU = 0$ for some regular element d of R, by (2.1.2). Hence $K \supseteq dP \cong P$, so K is faithful, too.

As in (4.1.2), there is a decomposition

$$\Gamma P = X \oplus Y \quad \text{where } X \supseteq \Gamma K \text{ and } {}_R(X/\Gamma K) \text{ has finite length.} \tag{6.1.3}$$

Let $\bar{\Gamma} = \Gamma/I$ and $\bar{R} = R/I$. As in Notation 2.2, π denotes the (finite) set of maximal ideals of R that contain I. We write $\nu(\Gamma^*)$ for the natural image of the group Γ^* of units of Γ in the group $\bar{\Gamma}^*$ of units of $\bar{\Gamma}$. Recall that $E(f) = E(K, P)$ denotes the endomorphism ring of f, which consists of all elements of $\alpha \in E(P)$ such that $K\alpha \subseteq K$.

Next, we define the natural image $\nu E^*(K_\pi, P_\pi)$ of $E^*(K_\pi, P_\pi)$ in $\bar{\Gamma}^*$. Take

$$\alpha \in E^*(K_\pi, P_\pi).$$

Then

$$\alpha \in E^*(\Gamma K_\pi, \Gamma P_\pi) \subseteq E^*(X_\pi, \Gamma P_\pi)$$

by Lemma 4.3. So $\alpha = \oplus_k \alpha(k)$ where each $\alpha(k) \in E^*(\Gamma_k X_\pi, \Gamma_k P_\pi)$. Moreover, each $\Gamma_k X_\pi$ is a free module over the principal ideal ring $(\Gamma_k)_\pi$. Hence each $\alpha(k)$ can be viewed as a 2×2 block lower triangular matrix, acting by right multiplication, where

$$\alpha(k)_{11} \in E^*(\Gamma_k X_\pi) \quad \text{and} \quad \alpha(k)_{22} \in E^*(\Gamma_k Y_\pi).$$

We can therefore define $\nu(\alpha)$ by

(6.1.4) $\nu(\alpha)(k) = \text{image of } (\det \alpha(k)_{11}, \det \alpha(k)_{22}) \text{ in } \overline{\Gamma}_k^* \times \Gamma_k^*$

Thus we have $\nu(\alpha) \in \overline{\Gamma}^* \times \overline{\Gamma}(Y)^*$ where $\overline{\Gamma}(Y)$ denotes the direct sum of those $\overline{\Gamma}_k$ such that $\Gamma_k Y \neq 0$.

The above notation and assumptions remain in force throughout this section.

6.2 Proposition. *For every f as in* (6.1.1) *and* (6.1.2) *we have*

$$\operatorname{pres}_f(P, U) \cong \overline{R}^*/D$$

where

(6.2.1) $D = \{a \in \overline{R}^* \mid (a, 1)$, *considered as an element of* $\overline{\Gamma}^* \times \overline{\Gamma}(Y)^*$ *belongs to the subgroup* $\nu E^*(K_\pi, P_\pi)\nu[\Gamma^* \times \Gamma(Y)^*]\}$

Proof. Recall that every faithful, projective R-module is a progenerator (because, when localized at any maximal ideal, it becomes free and nonzero). Since $\overline{\Gamma}$ is an artinian ring, it has 1 in its stable range. So $\mathbf{K}_1(\overline{\Gamma}) \cong \overline{\Gamma}^*$ via the determinant. Thus the proposition is an immediate consequence of Theorem 4.11. □

6.3 Corollary. *Suppose that U has finite length. Then*

$$\operatorname{pres}_f(P, U) \cong \overline{R}^*/\nu E^*(K_\pi, P_\pi)\nu(R^*).$$

Moreover, $\nu(\alpha) = \nu \det(\alpha)$ *for* $\alpha \in E^*(K_\pi, P_\pi)$.

Proof. When U has finite length we have $\Gamma(Y) = 0$. So

$$D = \overline{R}^* \cap \nu E^*(K_\pi, P_\pi)\nu\Gamma^*$$

Again, since $\Gamma(Y) = 0$, we have $\nu E^*(K_\pi, P_\pi) \subseteq \nu \det E^*(P_\pi)$ (ν the natural image in $\overline{\Gamma}^*$). Since R is an indecomposable ring, all projective R-modules have rank. Hence P_π is a free R_π-module, so $\nu \det E^*(P_\pi) \subseteq \overline{R}^*$. Thus it suffices to show

$$\overline{R}^* \cap \nu\Gamma^* = \nu R^*$$

For the nontrivial inclusion $\subseteq$, take $u \in \Gamma^*$ such that $\bar{u} \in \bar{R}^*$. Then $u \in R + I = R$. Similarly, $u^{-1} \in R$, so u is a unit of R, as claimed. □

Since scalar multiplications by units of R_π belong to $E^*(K_\pi, P_\pi)$ and P_π is a free R_π-module, as mentioned above, we have:

6.4 LEMMA. *Suppose that U has finite length. Then $\nu E^*(K_\pi, P_\pi)$ contains $(R^*)^{n\,\mathrm{pwr}}$, the set of n^{th} powers of elements of $\bar{R}^*$, where $n = \mathrm{rank}(P)$.*

An immediate consequence of this and Corollary 6.3 is:

6.5 THEOREM. *Suppose U has finite length. Then $\mathrm{pres}_f(P, U)$ is a torsion group whose exponent divides the rank of P (even if all residue fields of R are infinite).*

Remark. In simpler language, this theorem states that if f and g are presentations $P \twoheadrightarrow U$ with U of finite length and P of rank n, then $f^n \sim g^n$.

6.6 THEOREM. *Suppose R is a finitely generated algebra over an algebraically closed field F, and*

(i) *F has characteristic zero; or*
(ii) *F has characteristic $p \neq 0$, and p does not divide $n = \mathrm{rank}(P)$; or*
(iii) *$\bar{R}$ has no nilpotent elements $\neq 0$.*

Then $\mathrm{pres}_f(P, U) = 0$ whenever U has finite length.

Proof. In view of Corollary 6.3 and Lemma 6.4, it suffices to show that $(\bar{R}^*)^{n\,\mathrm{pwr}} = \bar{R}^*$ when any of the hypotheses is satisfied. By algebraic closure of F, we have $(F^*)^{n\,\mathrm{pwr}} = F^*$. This property is lifted to $\bar{R}$ in [WW '87, Lemma 2.3], provided (i), (ii), or (iii) holds. □

Wiegand and Wiegand used their lemma, quoted above, to prove a cancellation result for torsionfree modules over certain rings. As in their situation, we show that the result can fail when the characteristic is nonzero and divides the rank of P. The crux of the construction of this and other examples is to find a presentation (6.1.1) for which we know detailed information about $\nu E^*(K_\pi, P_\pi)$. We do this next.

6.7 *Construction.* Let Γ be a direct sum of Dedekind domains, none of which is a field. Suppose that Γ has two maximal ideals $\mathfrak{m} \neq \mathfrak{n}$, such that $\Gamma/\mathfrak{m}^e \cong \Gamma/\mathfrak{n}^e$ as rings, for some e.

Let $\theta, \psi\colon \Gamma \twoheadrightarrow V$ be surjective ring homomorphisms with kernels $\mathfrak{m}^e$ and $\mathfrak{n}^e$ respectively, so that $V \cong \Gamma/\mathfrak{m}^e \cong \Gamma/\mathfrak{n}^e$. Let

$$R = \{r \in \Gamma \mid \theta(r) = \psi(r)\}. \tag{6.7.1}$$

Then R is a ring-order, $Q(R) = Q(\Gamma)$,

(6.7.2) the conductor from Γ to R is $I = \mathfrak{m}^e \cap \mathfrak{n}^e$, $\overline{R} \cong V$, and π is the one-element set $\{\mathfrak{p} = R \cap \mathfrak{m} \cap \mathfrak{n}\}$

To prove the statements in (6.7.2), first verify that the conductor is as described, then note that $\overline{R} = R/I$ is the "diagonal" subring of $V \oplus V$.

We claim that, for every $n \geq 1$ there is a presentation (6.1.1) of an *indecomposable* R-module U of finite length, P is free, and

$$\operatorname{pres}_f(P, U) \cong \overline{R}^* \Big/ \left[(\overline{R}^*)^{n\,\mathrm{pwr}} \nu(R^*) \right] \quad \text{where } n = \operatorname{rank}(P) \tag{6.7.3}$$

In view of Corollary 6.3, it suffices to define f and then prove that

$$\nu E^*(K_\pi, P_\pi) = (\overline{R}^*)^{n\,\mathrm{pwr}}. \tag{6.7.4}$$

Note that (6.7.4) shows that Lemma 6.4 says as much as can be said about the general group $\nu E^*(K_\pi, P_\pi)$.

For later use we note that

$$\mathfrak{m}^e \cap R = I = \mathfrak{n}^e \cap R. \tag{6.7.5}$$

Choose elements b and c in Γ such that

$$b \in \mathfrak{m}^e - \mathfrak{m}^{e+1},\ b \notin n, \qquad c \in \mathfrak{n}^e - \mathfrak{n}^{e+1},\ b \notin \mathfrak{m}. \tag{6.7.6}$$

Consider the following elements of $\bigoplus^n R$:

$$x^{(1)} = (b, c, 0, 0, \ldots, 0), \quad x^{(2)} = (0, b, c, 0, \ldots, 0), \ldots,$$
$$x^{(n-1)} = (0, 0, \ldots, 0, b, c)$$

We claim that the following R-module presentation, where f is the natural homomorphism, satisfies (6.7.4) and U is indecomposable of finite length.

$$K \rightarrowtail P = \bigoplus{}^n R \xrightarrow{f} U = \left(\bigoplus{}^n R\right) / \left(IP + \Sigma_i (R x^{(i)})\right) \tag{6.7.7}$$

The module U has finite length because I contains regular elements of R. Since (6.7.4) and indecomposability of U can be verified after localizing at π, we can now suppose that $\mathfrak{m}$ and $\mathfrak{n}$ are the only maximal ideals of Γ, and $R = R_\pi$. Then (6.7.6) gives $\mathfrak{m}^e = \Gamma b$ and $\mathfrak{n}^e = \Gamma c$.

Let $\alpha \in E(K, P) \subseteq E(P)$. Then α is right multiplication by a matrix, which we also call α, whose entries α_{ij} are elements of R.

To complete the proof of (6.7.4), it suffices to show that $\alpha_{ij} \in I$ when $i \neq j$, and that $\alpha_{11} \equiv \alpha_{22} \equiv \cdots \equiv \alpha_{nn} (\text{mod } I)$, since R is now local with maximal ideal $\mathfrak{p} \supseteq I$.

To express the relation $K\alpha \subseteq K$, let E_i denote the row vector whose ith entry is 1 and whose other entries are zero. Then $x^{(i)} = bE_i + cE_{i+1}$. Hence there exist elements $r_{ij} \in R$ such that

$$(6.7.8) \quad (bE_i + cE_{i+1})\alpha = r_{i1}(bE_1 + cE_2) + r_{i2}(bE_2 + cE_3) + \cdots$$
$$+ r_{i(n-1)}(bE_{n-1} + cE_n) + IP$$

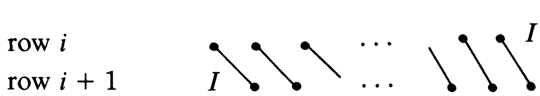
View this as a matrix relation with entries in Γ, remembering that $IP = \oplus^n \Gamma bc$. Reading (6.7.8) modulo c, and then cancelling b, which becomes a unit in $\Gamma/\Gamma c$ we get

$$(6.7.9) \qquad (\alpha_{i1}, \alpha_{i2}, \ldots) \equiv (r_{i1}, r_{i2}, \ldots, r_{i(n-1)}, 0) \quad (\text{mod } \mathfrak{n}^e).$$

Then reading (6.7.8) modulo b and cancelling c gives

$$(6.7.10) \quad (\alpha_{(i+1)1}, \alpha_{(i+1)2}, \ldots) \equiv (0, r_{i1}, r_{i2}, \ldots, r_{(n-1)}) \quad (\text{mod } \mathfrak{m}^e)$$

Since (6.7.9) is a congruence with elements of R on both sides, we can use (6.7.5) to conclude that the two sides are congruent modulo I. The same holds for (6.7.10). The conclusion of this is that rows i and $i + 1$ of α have the following form:



where entries connected by straight lines are congruent modulo I and the corner entries are elements of I.

Since this is true for every i, α is a "striped" matrix, and this concludes the proof that $\alpha_{ij} \in I$ when $i \neq j$, and $\alpha_{11} \equiv \alpha_{22} \equiv \cdots \equiv \alpha_{nn} (\mathrm{mod}\, I)$. Hence (6.7.4) holds.

To see that U is indecomposable, note first that $K \subseteq (\mathrm{rad}\, R)P$, so f is a projective cover of U. Hence any decomposition of U can be lifted to a decomposition of f, hence of the inclusion (K, P). But $E(K, P)$ contains no idempotent endomorphisms, because its striped form shows that $E(K, P)$ modulo an ideal contained in its radical is isomorphic to the local ring V. □

6.8 *Example* (See Theorem 6.6). Let F be any field of characteristic $p \neq 0$. There exists an integral domain R, finitely generated F-algebra, and a presentation (6.1.1) of an R-module of finite length such that $\mathrm{pres}_f(P, U) \neq \{0\}$.

In Construction 6.7, take $\Gamma = F[x]$, the polynomial ring,

$$\mathfrak{m}^p = \Gamma x^p, \quad \mathfrak{n}^p = \Gamma(x-1)^p.$$

Then (6.7.3), with $n = p$, gives

$$\mathrm{pres}_f(P, U) \cong \overline{R}^* \Big/ \Big[(\overline{R}^*)^{p\,\mathrm{pwr}} F^* \Big] = \mathscr{P} \text{ (say)} \tag{6.8.1}$$

By (6.7.2) we have $\overline{R} \cong F[x]/\langle x^p \rangle$. Every unit of the right-hand side has the form $\alpha + t$ where $\alpha \in F$ and t has constant term zero. Therefore $(\alpha + t)^p = \alpha^p$. Therefore the right-hand side of (6.8.1) reduces to $\overline{R}^*/F^* \neq \{1\}$.

6.9 *Example*. When U has finite length, the torsion group $\mathrm{pres}_f(P, U)$ can be infinite. It can also be finite and nonzero.

The nonzero group $\overline{R}^*/F^*$ at the end of the previous example is infinite if F is an infinite field, and is finite if F is a finite field.

6.10 *Example*. Let $R = ZG$, the integral group ring of a cyclic group of prime order $p \neq 2$. We show:

(6.10.1) The set of integers $|\mathrm{pres}_f(P, U)|$, for U of finite length (hence finite) is the set of all divisors of $(p-1)/2$.

Let $\Gamma = Z \oplus Z[\zeta]$ (ζ a primitive pth root of unity) and $V = Z/pZ$. It is well known that there exist ring homomorphisms $\theta\colon Z \twoheadrightarrow V$ and $\psi\colon Z[\zeta] \twoheadrightarrow V$ such that the ring $R = \{(x, y) \in \Gamma\colon \theta(x) = \psi(y)\}$ is isomorphic to ZG. The details are written out in [L '81, 1.1, 1.13]. Since we can consider θ and ψ to be homomorphisms from Γ onto V, the results of Construction 6.7 apply to $R = ZG$.

We claim that $\nu(R^*) = \{\pm 1\}$. This holds since the map: $R \twoheadrightarrow V$ whose kernel is the conductor ideal I factors through the coordinate ring Z, whose units are ± 1.

Thus by (6.7.3), $\mathrm{pres}_f(P, U)$ is a cyclic group whose order divides $(p-1)/2$. The fact that every divisor can actually occur is now an immediate consequence of varying the value of n in (6.7.3). □

It is interesting to see what happens to $\mathrm{pres}_f(P, U)$ when we take direct sums of presentations of modules of finite length. At one extreme, we already know (Corollary 1.12) that the presentation group of the direct sum of any number of copies of a single presentation f is isomorphic to $\mathrm{pres}_f(P, U)$. Another extreme is given by:

6.11 THEOREM. *Let $P_i \twoheadrightarrow U_i$ $(i = 1, 2, \ldots)$ be a finite number of presentations of R-modules of finite length, and suppose that the greatest common divisor of the numbers $n_i = \operatorname{rank} P_i$ is* 1. *Then $U = \bigoplus_i U_i$ is uniquely presentable by $P = \bigoplus_i P_i$.*

Proof. $\nu E^*(K_\pi, P_\pi) \supseteq$ each $\nu E^*((K_i)_\pi, (P_i)_\pi)$ which, by Lemma 6.4, contains all n_ith powers of elements of $\overline{R}^*$. So by the given relative primeness, we have that $\nu E^*(K_\pi, P_\pi) = \overline{R}^*$. Corollary 6.3 now gives $\mathrm{pres}_f(P, U) = \{0\}$. □

6.12 COROLLARY. *Let $P_i \twoheadrightarrow U_i$ $(i = 1, 2, \ldots)$ be a finite number of presentations of R-modules of finite length, and suppose that some P_i has rank* 1. *Then $U = \bigoplus_i U_i$ is uniquely presentable by $P = \bigoplus_i P_i$.*

We close this paper by showing that the situation for presentations of modules of infinite length is completely different: $\mathrm{pres}_f(P, U)$ can contain elements of both finite and infinite orders, and can be nonzero even if R is an algebra over a field of characteristic zero.

6.13 *Example.* Let k be any field with at least four elements and characteristic $\neq 2$. We show that, for a suitable integral domain R that is a finitely generated k-algebra, there is a presentation f of a torsionfree R-module U, namely $U = \Gamma$, such that

$$\mathrm{pres}_f(R^2, \Gamma) \cong k^*. \tag{6.13.1}$$

In particular, if k is not an algebraic extension of a finite field, this always has elements of both finite and infinite orders.

Let R be a ring-order $\neq \Gamma$ such that, for some $t \in \Gamma$,

$$\Gamma = R + Rt. \tag{6.13.2}$$

We will further restrict the choice of R as this example proceeds. In this example, *I denotes the full conductor ideal from* Γ *to* R. Let f be the following presentation of Γ by a free R-module of rank 2:

(6.13.3) $K \rightarrowtail P = Ru \oplus Rv \xrightarrow{f} \Gamma$ where $f(u) = 1$ and $f(v) = t$.

We begin by proving

$$K = I(tu - v). \tag{6.13.4}$$

The inclusion $\supseteq$ follows immediately from (6.13.3). For the opposite inclusion suppose $0 = f(xu + yv) = x - yt$. Then $yt \in R$. Since $y \in R$, (6.13.2) shows that $y\Gamma \subseteq R$. Since I is the full conductor ideal, this gives $y \in I$. Hence

$$xu + yv = -ytu + yv \in I(tu - v)$$

as claimed.

The next step is to find a decomposition $\Gamma P = X \oplus Y$ where $X \supseteq K$ and X/K has finite length. In view of (6.13.4) the following decomposition accomplishes this:

$$\Gamma P = \Gamma(tu - v) \oplus \Gamma u. \tag{6.13.5}$$

Now take $\alpha \in E(K, P) \subseteq E(\Gamma P)$. With respect to decomposition (6.13.5), α is right multiplication by a 2×2 lower triangular matrix, say

$$\alpha = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix} \quad \text{with } a, b, d \in \Gamma. \tag{6.13.6}$$

We seek a necessary and sufficient condition that $\alpha \in E^*(K, P)$.

To begin, we determine when $P\alpha \subseteq P$. This is equivalent to $u\alpha$ and $v\alpha \in P$. The expressions for u and v in decomposition (6.13.5) are

$$u = 0(tu - v) + u \quad \text{and} \quad v = -(tu - v) + tu.$$

Therefore

$$u\alpha = [0 \quad 1]\begin{bmatrix} a & 0 \\ b & d \end{bmatrix} = [b \quad d]$$

and

$$v\alpha = [-1 \quad t]\begin{bmatrix} a & 0 \\ b & d \end{bmatrix} = [-a + bt \quad dt].$$

In other words, $u\alpha = b(tu - v) + du$ and $v\alpha = (-a + bt)(tu - v) + dtu$.

Equating coefficients shows that $P\alpha \subseteq P = Ru \oplus Rv$ if and only if

(6.13.7)
$$b \in R, \quad bt + d \in R, \quad -a + bt \in R \text{ and } (-a + bt + dt)t \in R$$

Now we make some additional restrictions on the ring R. Let Γ be a Dedekind domain. Suppose the following condition holds.

(6.13.8) There exists a ring homomorphism ν from Γ onto $F \oplus F$, for some ring F. Let $R = \nu^{-1}\operatorname{diag}(F \oplus F)$. Choose $t \in \Gamma$ such that $\nu(t) = (1,0)$. (The conductor ideal is thus $I = \ker \nu$.)

Here, $\operatorname{diag}(F \oplus F)$ denotes the set of elements of the form (x, x) in $F \oplus F$. R is a ring-order. In fact R is a Dedekind-like ring, in the sense of [L '85]. To verify (6.13.2), note that $R \supseteq \ker(\nu)$. So it suffices to verify that $\nu(\Gamma) = \nu(R) + \nu(R) \cdot (1,0)$, which is obvious. Thus the characterization (6.13.7) of the condition $P\alpha \subseteq P$ applies to R.

For *local* such rings R we now obtain a quite explicit description of the matrices $\alpha \in E^*(K, P)$. This will suffice for our needs, since we will apply the description to R_π rather than to R itself.

For $x \in \Gamma$ let $\nu(x) = (x_1, x_2)$.

Since $\ker(\nu) \subseteq R$, conditions (6.13.7) are equivalent to the conditions obtained by applying ν to them and determining when the resulting elements belong to $\operatorname{diag}(F \oplus F)$. After some straightforward manipulations we see that $P\alpha \subseteq P$ if and only if $\nu(\alpha)$ has the form

(6.13.9)
$$\nu(\alpha) = \begin{bmatrix} (a_1, a_2) & 0 \\ (a_1 - a_2, a_1 - a_2) & (a_2, a_1) \end{bmatrix}.$$

As a check on these computations note that the determinant of $\nu(\alpha)$ belongs to $\nu(R) = \operatorname{diag}(F \oplus F)$; and it is a unit of $\nu(R)$ if and only if each a_i is a unit. Since we are supposing that R is local, this makes $\det(\alpha)$ a unit of R. Then α is invertible; moreover its inverse also has the form (6.13.9), so $P\alpha = P$. We also have $K\alpha = K$ by (6.13.4), since I is an ideal of Γ and a is a unit of Γ. Thus (6.13.9), with a_1 and a_2 units, *displays the general form of all* $\alpha \in E^*(K, P)$, when R is local.

We now further specialize R and Γ. Let $\Gamma = k[x]$, the polynomial ring. Let F be any local, finite dimensional k-algebra such that there is a surjective k-algebra homomorphism $\nu\colon \Gamma \twoheadrightarrow F \oplus F$. Let R and t be as in (6.13.8).

Since the conductor from Γ to R is the ideal $I = \ker(R \twoheadrightarrow \operatorname{diag}(F \oplus F))$ of R, the set π of maximal ideals of R containing I consists of the single maximal ideal I.

Localizing at π we see that the three rings Γ_π, R_π, and $F \oplus F$ and the element t satisfy the conditions in (6.13.8). Therefore the matrices $\alpha \in E^*(K_\pi, P_\pi)$ are those matrices such that $\nu(\alpha)$ has the form displayed in (6.13.9).

We now show that

$$\text{pres}_f(P, \Gamma) \cong F^*/k^* \tag{6.13.10}$$

using Proposition 6.2. We have $\bar{R}^* = \nu(R)^* = \text{diag}(F \oplus F)^* \cong F^*$. It therefore suffices to show that the group D in (6.2.1) is $\text{diag}(k \oplus k)^*$.

We have $\Gamma(Y) = \Gamma$ since $Y = \Gamma u$. (See (6.13.5) and the paragraph preceding it.) Moreover $\Gamma^* = k[x]^* = k^*$. We write elements of $\bar{\Gamma} = F \oplus F$ as ordered pairs, so the elements of $\bar{\Gamma}^* \oplus \bar{\Gamma}^*$ in (6.2.1) become 4-tuples. Thus an element $(g, g) \in \bar{R}^* = \text{diag}(F \oplus F)^*$ belongs to D if and only if there is an expression

$$(g, g, 1, 1) = (a_1, a_2, a_2, a_1) \cdot (h, h, j, j) \tag{6.13.11}$$

where h and j belong to k and (a_1, a_2) appears in a matrix of the form (6.13.9). Looking at the last two coordinates shows that each a_i belongs to k and $a_1 = a_2$. So $g \in \text{diag}(k \oplus k)^*$, hence $D \subseteq \text{diag}(k \oplus k)^*$. For the opposite inclusion, we can take each $a_i = g$ and $h = j = 1$.

Finally, let F to be the k-algebra $k \oplus k$. Since k has at least four elements. $\Gamma = k[x]$ can be mapped onto $F \oplus F = k \oplus k \oplus k \oplus k$ by taking the kernel to be generated by the product of four linear polynomials. So we can apply (6.13.10), with $F = k \oplus k$, getting (6.13.1). □

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