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ON THE COMPLEX GRASSMANN MANIFOLD

BY

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In this paper we prove the following theorem.

THEOREM. *Let M be a compact complex analytic variety of dimension mn having the following properties*:

(i) *The cohomology ring of M has the form*

$$H^*(M,\mathbf{Z}) = \sum_{k=0}^{mn} \sum_{\substack{a_1 \geq a_2 \geq \cdots \geq a_n, \\ a_1 + a_2 + \cdots + a_n = k}} \mathbf{Z} c_{a_1 \cdots a_n}$$

with $c_{0 \cdots 0} = 1$ and $c_{1\,0 \cdots 0} \in H^2(M,\mathbf{Z})$, and the product rule is subject to the formula

$$c_{a_1 \cdots a_n} \cdot c_{h\,0 \cdots 0} = \sum c_{b_1 \cdots b_n}$$

where $\Sigma b_i = h + \Sigma a_i$ and $b_1 \geq a_1 \geq b_2 \geq a_2 \cdots \geq b_n \geq a_n \geq 0$.

(ii) *There exists a holomorphic vector bundle F of rank n on M whose cohomology class is*

$$c(F) = 1 + c_1(F) + \cdots + c_n(F),$$

with $c_i(F) = c_{i\,0 \cdots 0}$; moreover the determinantal bundle $\overset{n}{\wedge} F$ is positive with fundamental class $c(\overset{n}{\wedge} F) = c_{1\,0 \cdots 0}$.

(iii) $\dim H^0(M, F) = m + n$.

Then M is complex analytically homeomorphic to the complex projective Grassmann manifold $G(n, m+n)$.

The Kodaira imbedding theorem [3] states that if M is a compact complex analytic variety with a Kähler metric such that the associated exterior form

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belongs to the cohomology class of an integral 2-cocycle, then M can be imbedded as a non-singular algebraic variety in a projective space. Hirzebruch and Kodaira subsequent to the proof of the Kodaira Imbedding Theorem proved a rigidity theorem for the projective space [2]. Namely, they showed that if in addition to the properties of the variety M in the Kodaira imbedding theorem the cohomology ring of M has the form

$$H^*(M, \mathbf{Z}) = \mathbf{Z} + \mathbf{Z}g + \cdots + \mathbf{Z}g^n$$

where n is the dimension of M and $g \in H^2(M, \mathbf{Z})$, and there exists a positive holomorphic divisor D on M whose cohomology class $c(D) = g$ and the dimension of

$$H^0(M, \nu D) = \binom{n+\nu}{n},$$

then M is complex analytically homeomorphic to the complex projective space $\mathbf{P}^n(\mathbf{C})$; see Theorem 6 in [2]. The theorem we have stated at the start of this introduction is a rigidity theorem for the Grassmann manifold.

The crucial argument in the proof of the Hirzebruch-Kodaira theorem fails in the Grassmannian case since their proof involves the irreducibility of algebraic cycles in M whose cohomology class is g^s $(0 \leq s \leq n)$. In this paper the algebraic cycles involved have more complicated cohomology classes, consequently the Hirzebruch-Kodaira proof does not generalize to the Grassmannian case. In our proof we use a lemma on the imbedding degrees of irreducible varieties in projective space, which is an interesting result on its own (Lemma 2.1). This lemma makes it possible to prove that the base of the linear system involved in the imbedding is empty, which is the essentional step in the proofs of both the projective case [2, Theorem 6] and ours.

1. Preliminaries

Let F be a vector bundle of rank n over a compact complex analytic variety M of dimension mn. Let $\varphi_1, \ldots, \varphi_{m+n}$ be linearly independent sections of F over M. Choose a sufficiently fine covering $\{U_j\}$ of M and let the sections φ_ν at $x \in U_j$ be represented by

$$x \to \begin{bmatrix} \varphi^j_{1\nu}(x) & \cdots & \varphi^j_{n\nu}(x) \end{bmatrix}^t$$

with respect to some holomorphic frame for F over the open set U_j, where $[*]^t$ is the transpose of the matrix $[*]$, and $\varphi^j_{1\nu}(x), \ldots, \varphi^j_{n\nu}(x)$ are the fibre coördinates of φ_ν at $x \in U_j$. When $x \in U_j \cap U_k$, these fibre coördinates are subject to the patching rule

$$\begin{bmatrix} \varphi^j_{1\nu}(x) \\ \vdots \\ \varphi^j_{n\nu}(x) \end{bmatrix} = \begin{bmatrix} e^{jk}_{11}(x) & \cdots & e^{jk}_{1n}(x) \\ \vdots & & \vdots \\ e^{jk}_{n1}(x) & \cdots & e^{jk}_{nn}(x) \end{bmatrix} \begin{bmatrix} \varphi^k_{1\nu}(x) \\ \vdots \\ \varphi^k_{n\nu}(x) \end{bmatrix}$$

where the matrices $[e^{jk}_{\alpha\beta}]$ are the transition matrices defining the vector bundle F. Let $G^*(n, m+n)$ be the Grassmann manifold of n planes in $(m+n)$-dimensional space. A point of $G^*(n, m+n)$ is represented by an $n \times (m+n)$ matrix (κ_{ij}) of rank n, and two such matrices $(\kappa_{ij}), (\eta_{ij})$ represent the same point if there is a nonsingular $n \times n$ matrix σ such that $\sigma(\kappa_{ij}) = (\eta_{ij})$. We define the structure sheaf of rings $\mathcal{O}_{G^*(n, m+n)}$ via the Plücker imbedding

$$\pi\colon G^*(n, m+n) \to \mathbf{P}^{\binom{m+n}{n}-1}.$$

If (κ_{ij}) is an $n \times (m+n)$ matrix representing the point κ of $G^*(n, m+n)$, let $\kappa_{i_1 \cdots i_n}$ be the $(i_1, \ldots, i_n)$-th Plücker coördinate of κ. Associate with each $\kappa \in G^*(n, m+n)$ the holomorphic divisor $D_\kappa = (\varphi^j_\kappa)_j$, with

$$\varphi^j_\kappa(x) = \sum_{1 \le i_1 < \cdots < i_n \le m+n} \kappa_{i_1 \cdots i_n} \varphi^j_{i_1 \cdots i_n}(x)$$

for $x \in U_j$, where

$$\kappa_{i_1 \cdots i_n} = \det \begin{bmatrix} \kappa_{1 i_1} & \cdots & \kappa_{1 i_n} \\ \vdots & & \vdots \\ \kappa_{n i_1} & \cdots & \kappa_{n i_n} \end{bmatrix}$$

and

$$\varphi^j_{i_1 \cdots i_n}(x) = \det \begin{bmatrix} \varphi^j_{1 i_1}(x) & \cdots & \varphi^j_{1 i_n}(x) \\ \vdots & & \vdots \\ \varphi^j_{n i_1}(x) & \cdots & \varphi^j_{n i_n}(x) \end{bmatrix}.$$

We note that $\{D_\kappa \mid \kappa \in G^*(n, m+n)\}$ is a complete linear system $|D_\kappa|$ and by the hypothesis of our theorem $c(D_\kappa) = c(\overset{n}{\Lambda} F) = c_{1\,0\,\cdots\,0}$.

Let $B = \bigcap_\kappa D_\kappa$ be the base of the linear system $|D_\kappa|$. Thus B consists of the points $x \in U_j \subset M$ such that $\varphi^j_{i_1 \cdots i_n}(x) = 0$ for all $i_1, \ldots, i_n$, $1 \le i_1 < \cdots < i_n \le m+n$. Let $x \in M - B$, $x \in U_j$. For $\nu = 1, \ldots, m+n$, consider the vectors of meromorphic functions

$$f^j_\nu(x) = \left[\frac{\varphi^j_{\nu 2 \cdots n}(x)}{\varphi^j_{1 \cdots n}(x)} \cdots \frac{\varphi^j_{1 \cdots n-1\, \nu}(x)}{\varphi^j_{1 \cdots n}(x)} \right]^t.$$

Thus $f_1^j(x) = [1 \quad 0 \quad \cdots \quad 0]^t$, $f_2^j(x) = [0 \quad 1 \quad 0 \quad \cdots \quad 0]^t, \ldots, f_n^j(x) = [0 \quad \cdots \quad 0 \quad 1]^t$. By our positivity hypothesis $c(\overset{n}{\wedge} F)$ has a Hodge form as a representative, so by Kodaira's imbedding theorem [3, Theorem 4], M is a nonsingular algebraic variety imbedded in a projective space $\mathbf{P}^l(\mathbf{C})$ where $x_0, \ldots, x_l$ are the homogeneous coördinates of the point $x \in M$. The meromorphic functions

$$\varphi^j_{1 \cdots \mu-1\, \nu\, \mu+1 \cdots n}(x)/\varphi^j_{1 \cdots n}(x)$$

are rational functions (Chow, [1, Theorem 1]). Thus there exist polynomials $\Phi_{i_1 \cdots i_n}$ of some degree d such that

$$\frac{\varphi^j_{1 \cdots \mu-1\, \nu\, \mu+1 \cdots n}(x)}{\varphi^j_{1 \cdots n}(x)} = \frac{\Phi_{1 \cdots \mu-1\, \nu\, \mu+1 \cdots n}}{\Phi_{1 \cdots n}(x)}.$$

If $\xi = (1, \xi_1, \ldots, \xi_l)$ is a generic point of M and $f\colon M - B \to \mathbf{P}^N(\mathbf{C})$ is given by

$$x \to (f_1(x), \ldots, f_{n+m}(x))$$

where $(f_1(x), \ldots, f_{n+m}(x))$ is an $n \times (m+n)$ matrix with ν-th column

$$f_\nu(x) = \left[\frac{\Phi_{\nu 2 \cdots n}(x)}{\Phi_{1 \cdots n}(x)} \quad \frac{\Phi_{1 \nu 3 \cdots n}(x)}{\Phi_{1 \cdots n}(x)} \quad \cdots \quad \frac{\Phi_{1 \cdots n-1\, \nu}(x)}{\Phi_{1 \cdots n}(x)}\right]^t.$$

and

$$N = \binom{m+n}{n} - 1,$$

then $(f_1(\xi), \ldots, f_{m+n}(\xi))$ is a generic point of the subvariety V of the Grassmannian $G(n, m+n) \subset \mathbf{P}^N(\mathbf{C})$.

2. Degree of imbedding

Let V be an irreducible subvariety of the projective space $\mathbf{P}^N(\mathbf{C})$. Then $\deg V$, the degree of imbedding of V in $\mathbf{P}^N(\mathbf{C})$, is the number of points counting multiplicities of the intersection of V with the generic linear complementary dimensional subspace of $\mathbf{P}^N(\mathbf{C})$.

The Chow ring $A(G(n, m+n))$ of the Grassmann manifold is freely generated over $\mathbf{Z}$ by the Schubert cycles $\omega_{i_1 \cdots i_n}$ where $i_1 \cdots i_n$ run over all integers which satisfy the relation

$$m \geq i_1 \geq \cdots \geq i_n \geq 0.$$

As an immediate consequence of the above definition, $\deg G(n, m+n)$ is the coefficient β in $(\omega_{1\,0\,\cdots\,0})^{mn} = \beta\omega_{m\,\cdots\,m}$.

We need the following lemma to show that the dimension of $\overline{f(M-B)} = V$ is mn.

LEMMA 2.1. *Consider the Plücker map*

$$\pi: G(n, m+n) \to \mathbf{P}^N(\mathbf{C})$$

where

$$N = \binom{m+n}{n} - 1,$$

and an irreducible subvariety V of $G(n, m+n)$ such that

(i) *V is not in any linear subspace of $\mathbf{P}^N$ and*

(ii) $\dim V < \dim G(n, m+n)$.

Then $\deg V > \deg G(n, m+n)$.

Proof. If V is a zero-dimensional subvariety of the Grassmannian $G(n, m+n)$ and is in none of the hyperplanes in $\mathbf{P}^N$, then V belongs to an equivalence class of cycles $\alpha\omega_{m\,\cdots\,m}$ with $\alpha > \beta = \deg G(n, m+n)$. Now suppose V is of codimension r in $G(n, m+n)$ and satisfies condition (i). Then V belongs to the equivalence class of cycles

$$\text{(I)} \qquad \sum_{i_1 + \cdots + i_n = r} \alpha_{i_1\,\cdots\,i_n}\omega_{i_1\,\cdots\,i_n}$$

where $\omega_{i_1\,\cdots\,i_n}$ does not have the factor $\omega_{1\,0\,\cdots\,0}$. We have

$$\omega_{mn-r\,0\,\cdots\,0} \cdot \sum_{i_1 + \cdots + i_n = r} \alpha_{i_1\,\cdots\,i_n}\omega_{i_1\,\cdots\,i_n} = \left(\sum_{i_1 + \cdots + i_n = r} \alpha_{i_1\,\cdots\,i_n}\right)\omega_{m\,\cdots\,m},$$

and this intersection class represents a cycle of dimension zero whose underlying varieties satisfy condition (i). Thus by the first observation in this proof

$$\text{(II)} \qquad \sum_{i_1 + \cdots + i_n = r} \alpha_{i_1\,\cdots\,i_n} > \beta = \deg G(n, m+n).$$

Next we establish the fact that if $(\omega_{1\,0\,\cdots\,0})^r = \Sigma\beta_{i_1\,\cdots\,i_n}\omega_{i_1\,\cdots\,i_n}$ with $r \leq mn$, then

$$\text{(III)} \qquad \sum \beta_{i_1\,\cdots\,i_n} \leq \deg G(n, m+n).$$

To show this we will use induction downward on r. The inequality holds for $r = mn, mn - 1$ since $(\omega_{10\cdots 0})^{mn} = \deg G(n, m+n)\omega_{m\cdots m}$ and

$$\begin{aligned}\deg G(n, m+n)\omega_{m\cdots m} &= (\omega_{10\cdots 0})^{mn-1}\omega_{10\cdots 0}\\ &= \beta_{m\cdots mm-1}\omega_{m\cdots mm-1}\cdot\omega_{10\cdots 0}\\ &= \beta_{m\cdots mm-1}\omega_{m\cdots m}.\end{aligned}$$

For $2 < r \le mn - 2$, if

$$\text{(IV)}\quad (\omega_{10\cdots 0})^r = \sum_{\substack{j_1 i_1 + \cdots + j_d i_d = r,\\ i_1 > i_2 > \cdots > i_d}} \beta_{\underbrace{i_1\cdots i_1}_{j_1\text{ times}}\cdots\underbrace{i_d\cdots i_d}_{j_d\text{ times}}}\omega_{i_1\cdots i_1\cdots i_d\cdots i_d},$$

then

$$(\omega_{10\cdots 0})^{r+1} = \sum_{l_1 k_1 + \cdots + l_e k_e = r+1} \gamma\omega_{\underbrace{k_1\cdots k_1}_{l_1\text{ times}}\cdots\underbrace{k_e\cdots k_e}_{l_e\text{ times}}}$$

where for at least one summand the coefficient γ is the sum of two or more coefficients in (IV). This together with our induction hypothesis implies

$$\sum_{\substack{j_1 i_1 + \cdots + j_d i_d = r,\\ i_1 > i_2 > \cdots > i_d}} \beta_{i_1\cdots i_1\cdots i_d\cdots i_d} \le \sum\gamma \le \deg G(n, m+n).$$

For $r \le 2$ the inequality (II) can be verified directly. Putting together what we have shown thus far: Comparing

$$(\omega_{10\cdots 0})^{mn-r} = \sum\beta_{i_1\cdots i_n}\omega_{i_1\cdots i_n}$$

with the subvariety V of $G(N, m+n)$ which satisfies condition (i) and belongs to the equivalence class of cycles

$$\Omega_{mn-r} = \sum_{i_1 + \cdots + i_n = mn-r} \alpha_{i_1\cdots i_n}\omega_{i_1\cdots i_n},$$

we have obtained the inequality

$$\text{(V)}\qquad \sum\alpha_{i_1\cdots i_n} > \sum\beta_{i_1\cdots i_n}.$$

To complete the proof, we observe that for the cycle Ω_{mn-r} above,

(VI)
$$\Omega_{mn-r}\cdot\omega_{10\cdots 0}=\sum_{j_1+\cdots+j_n=mn-r+1}\left(\sum a_{i_1\cdots i_n;\,j_1\cdots j_n}\alpha_{i_1\cdots i_n}\right)\omega_{j_1\cdots j_n},$$

and

(VII)
$$(\omega_{10\cdots 0})^{mn-r}\cdot\omega_{10\cdots 0}=\sum_{j_1+\cdots+j_n=mn-r+1}\left(\sum a_{i_1\cdots i_n;\,j_1\cdots j_n}\beta_{i_1\cdots i_n}\right)\omega_{j_1\cdots j_n}$$

where the coefficients $a_{i_1\cdots i_n;\,j_1\cdots j_n}$ are the same in (VI) and (VII).

Thus by (V) we conclude that

(VIII)
$$\sum_{j_1+\cdots+j_n=mn-r+1}\left(\sum a_{i_1\cdots i_n;\,j_1\cdots j_n}\alpha_{1_1\cdots i_n}\right) > \sum_{j_1+\cdots j_n=mn-r+1}\left(\sum a_{i_1\cdots i_n;\,j_1\cdots j_n}\beta_{i_1\cdots i_n}\right).$$

Similarly an inequality of type (VIII) can be established for

$$\Omega_{mn-r}\cdot(\omega_{10\cdots 0})^{s}\quad\text{and}\quad(\omega_{10\cdots 0})^{mn-r+s}$$

with $0\le s\le r$. Hence when $s=r$,

$$\Omega_{mn-r}\cdot(\omega_{10\cdots 0})^{r}=\alpha\omega_{m\cdots m}\quad\text{with } \alpha>\deg G(n,m+n).$$

This completes the proof of the lemma.

3. The theorem

PROPOSITION 3.1. *Let f, M, B, V, m, n, N have the same meaning as in Section 1. Then*

$$\dim\overline{f(M-B)}=\dim V=mn.$$

Proof. Consider

$$M-B\xrightarrow{f}V\subset G(n,m+n)\subset\mathbf{P}^N.$$

Let $\beta\colon\tilde{M}\to M$ be the blowing up of M with center the base B. Then f extends to the regular map $\tilde{f}\colon\tilde{M}\to V$. If $\dim V=r$, take r hyperplanes

$H_1, \dots, H_r$ in $\mathbf{P}^N$ in general position so that $H_1 \cap H_2 \cap \cdots \cap H_r$ is transversal to V. Thus $V \cap H_1 \cap \cdots \cap H_r$ is zero-dimensional and the number of these points, counting multiplicities, is the degree of V in $\mathbf{P}^N$. Let K_V^* be the restriction of the determinantal bundle K_G^* of $G(n, m+n)$ to V. Then

$$f^*K_V^* = \overset{n}{\wedge} F|_{M-B}.$$

Let $\tilde{s}_i$ be the sections of $\tilde{f}^*K_V^*$ corresponding to the hyperplanes H_i. Then

$$\tilde{f}^{-1}(V \cap H_1 \cap \cdots \cap H_r) \subseteq \bigcap_{i=1}^{r} (\tilde{s}_i)_0,$$

where $(\tilde{s}_i)_0$ is the divisor of zeros of $\tilde{s}_i$ on $\tilde{M}$. Each component of $\bigcap_{i=1}^{r}(\tilde{s}_i)_0$ has positive intersection number with a suitable self intersection of the line bundle $\tilde{f}^*K_V^*$. If we sum up these intersection numbers, we get a number $\tilde{d} \geq \deg V$. If we assume $\dim V < mn$, by Lemma 2.1, we get $\tilde{d} > \deg G(n, m+n)$. This is a contradiction because $\tilde{d}$ must equal the intersection number of the mn-fold self intersection of $\tilde{f}^*K_V^*$ which must equal $\deg G(n, m+n)$. Thus $\dim V = mn$.

Let $\phi_{\mu\nu} = \Phi_{1 \cdots \mu-1\, \nu\, \mu+1 \cdots n}(x)/\Phi_{1 \cdots n}(x)$ where the right hand side is the rational function defined in Section 1.

PROPOSITION 3.2. *The mn rational functions*

$$\{\phi_{\mu\nu}\}_{\substack{\mu=1,\dots,n;\\ \nu=n+1,\dots,m+n}}$$

are algebraically independent.

Proof. The $n \times (m+n)$ matrix

$$(f_1(\xi) \quad \cdots \quad f_{m+n}(\xi))$$

is a generic point of the variety

$$V \subseteq G(n, m+n) \subseteq \mathbf{P}^N(\mathbf{C}).$$

Applying the Plücker map π: $G(n, m+n) \to \mathbf{P}^N(\mathbf{C})$ to this generic point we get the point

$$(1, \dots, \phi_{\mu\nu}(\xi), \dots) \in \mathbf{P}^N(\mathbf{C}),$$

the generic point of the imbedding of V in $\mathbf{P}^N(\mathbf{C})$. The entries of

$$(1, \dots, \phi_{\mu\nu}(\xi), \dots)$$

except the first entry are linear, quadratic, cubic, etc., in the $\phi_{\mu\nu}(\xi)$. Since this is the generic point of V, and V by Proposition 3.1 has dimension mn, the linear entries in the generic point, namely

$$\{\phi_{\mu\nu}\}_{\substack{\mu=1,\ldots,n;\\ \nu=n+1,\ldots,m+n}}$$

are algebraically independent.

Proof of the theorem. Let k be a positive integer and $\mathscr{L}$ the vector space of homogeneous polynomials

$$\Psi(y) = \Psi(y_0,\ldots,y_N)$$

of degree k over $\mathbf{C}$ in the indeterminates $y_0,\ldots,y_N$ where

$$N = \binom{m+n}{n} - 1.$$

Let $\{\Psi_0,\ldots,\Psi_L\}$ be a basis of $\mathscr{L}$. Then dimension of the vector space $\mathscr{L}$ is $L+1$, where

$$L = \binom{N+k}{N} - 1.$$

Thus

$$\Sigma\colon y \to (\Psi_0(y),\ldots,\Psi_L(y))$$

is a biregular map of $\mathbf{P}^N(\mathbf{C})$ into $\mathbf{P}^L(\mathbf{C})$. Let

$$(1,\ldots,\phi_{\mu\nu}(\xi),\ldots) \in \mathbf{P}^N(\mathbf{C})$$

be the generic point of the imbedding of V in $\mathbf{P}^N(\mathbf{C})$ where, as in Proposition 3.2, the entries of $(1,\ldots,\phi_{\mu\nu}(\xi),\ldots)$ except the first entry are linear, quadratic, $\ldots$, of degree n, in the $\phi_{\mu\nu}(\xi)$. We set

$$\psi_\tau(x) = \Psi_\tau(1,\ldots,\phi_{\mu\nu}(x),\ldots)$$

and associate with each ψ_τ the section

$$\psi_\tau\colon x \to \psi_\tau(x)$$

of the complex line bundle $(\overset{n}{\wedge}F)\otimes\cdots\otimes(\overset{n}{\wedge}F)$ where the tensor product is k-fold. Since by Proposition 3.2, the $\phi_{\mu\nu}(x)$, $\nu = n+1,\ldots,m+n$; $\mu = 1,\ldots,n$ are algebraically independent, the meromorphic functions

$$\psi_\tau(1,\ldots,\phi_{\mu\nu}(x),\ldots)$$

are linearly independent. Thus the $L+1$ sections $\psi_0, \ldots, \psi_L$ of $(\overset{n}{\wedge} F)^k$ are linearly independent. Since by our hypothesis

$$\dim \Gamma\left(\overset{n}{\bigwedge} F\right)^k = \binom{N+k}{N} = L+1,$$

these sections $\psi_0, \ldots, \psi_L$ form a basis of $\Gamma(\overset{n}{\wedge} F)^k$. By the hypothesis of the theorem the Chern class $c(\overset{n}{\wedge} F)$ of the determinantal bundle $\overset{n}{\wedge} F$ is positive. Therefore for k sufficiently large,

$$\psi\colon x \to (\psi_0(x), \ldots, \psi_L(x))$$

is a biregular map of M into $\mathbf{P}^L(\mathbf{C})$. Since for each $x \in M$ at least one $\psi_\tau(x)$ is nonzero, then at least one $\phi_{\mu\nu}(x)$ is nonzero. Thus the base B of the linear system $|D_\kappa|$ is empty, and the map

$$f\colon x \to (f_1(x), f_2(x), \ldots, f_{m+n}(x))$$

is a regular map of M into the Grassmannian Manifold $G(n, m+n)$. It follows now from the commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{f} & G(n, m+n) & \xrightarrow{\pi} & \mathbf{P}^N(\mathbf{C}) \\ & \searrow^{\psi} & & \swarrow^{\Sigma} & \\ & & \mathbf{P}^L(\mathbf{C}) & & \end{array}$$

and the biregularity of ψ, that f is biregular. Since $\dim M = \dim V = mn$, f is a biregular map of M onto $G(n, m+n)$. This completes the proof of the theorem.

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