ILLINOIS JOURNAL OF MATHEMATICS
Volume 33, Number 1, Spring 1989

THE NUMBER OF mod p $A(p)$-SPACES

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Dedicated to Professor Hirosi Toda on his 60-th birthday

Section 1

It was proved in [5] that there are only finitely many possible homotopy types for the space of a mod p Finite associative H-space of a given rank. The character of the proof given there suggests that one might be able to replace the strict associativity assumption by the weaker hypothesis that the space supports an $A(p)$-structure and reach the same conclusion. We show how to refine the argument of [5] using the results of [7] to establish this.

We work at a fixed odd prime p; at the prime 2 the matters discussed here are either vacuous or known. We lose no generality in assuming that all spaces are simply connected as the general case of Theorem 1.1. below follows from this special case (see paragraph 5 of [5]). In addition all spaces are assumed to be p-localizations of finite connected CW-complexes, [4, 10]; thus an $A(k)$-space [8], [9] means a mod p Finite $A(k)$-space, etc.

Let X be an $A(p)$-space. The rationalization of X is of the form $\prod K(Q, 2n_i - 1)$, $1 \leq i \leq r$ where $r = r(X)$ is the rank of X, the n_i are ordered so that $1 < n_1 \leq n_2 \leq \cdots \leq n_r$ and the dimension of X is $d(X) = \Sigma(2n_i - 1)$.

THEOREM 1.1. *Let X be a* mod p *Finite $A(p)$-space of rank r. Then there are only finitely many possible homotopy types for the space X.*

There are infinitely many possible homotopy types for the space of an $A(k)$-space of given rank where $k < p$: one can construct these to taking cartesian products of odd dimensional spheres localized at p (see the proof of Theorem 17 of [9, part 1]). In the opposite direciton, Theorem 1.1 with "$A(p)$-space" replaced by "$A(k)$-space" where $k > p$ follows quite easily by combining results of [5] and [7]. The case $k = p$ is a little less obvious.

Received March 17, 1987.

In general the properties of $A(p)$-spaces are not transparent. If a product of spheres at the prime 3 supports an $A(4)$-structure, then the spheres have dimensions 1, 3 or 5 (see [6]); but if a product of spheres supports a homotopy associative H-multiplication at the prime 3, it is not known if the set of possible dimensions is bounded as the rank increases [3]. More generally it is not known if every $A(p)$-space has the homotopy type of a loop space although the spectacular advances made in the last decade in understanding Finite mod p loop spaces use the $A(\infty)$-structure in an essential manner, see for example [1].

In Section 2 we make some deductions from results in [5] and [7] and complete the proof of Theorem 1.1. in Section 3.

Section 2

We assume that X as in Theorem 1.1 is given a finite mod p cellular structure of p-local cells giving a 'homology decomposition' and that the maps defining the $A(p)$-structure, M_i: $K_1 \times X^i \to X$, are cellular [9].

Theorem 1.1 is established once it has been shown that there is a constant B depending only on r such that $d(X) < B$ (see [2]). We associate with each X a strictly increasing finite sequence $\{L_k\}$ for $1 \leq k \leq r$ where $L_k = \Sigma(2n_i - 1)$, $1 \leq i \leq k$, and use the sequence to place X into one of two families. X is in the first family if for each $k < r$, $n_{k+1} \leq p^3 L_k$; it is in the second if for some $k < r$, $n_{k+1} > p^3 L_k$ and then t is defined to be the largest such k and we set $q = n_{t+1}$. It is sufficient to show that $n_1 < C$ in the first situation and $q < D$ in the second, where C and D are constants depending only on the rank.

We will concentrate exclusively on the second family where $q > p^3 L_t$ until the final paragraph of this section. So W, the L_t skeleton of X, inherits from X an $A(p)$-structure using Lemma 2.6 of [5]. The $A(p)$-space W has rank t, dimension L_t and the inclusion r: $W \to X$ is an $A(p)$-map. The rational cohomology ring of X, $H^*(X, Q)$, is $E(x_1, x_2, \ldots, x_r)$, an exterior algebra over Q on primitive generators $\{x_i\}$ with dimensions $\{2n_i - 1\}$, and

$$H^*(W, Q) = E(x_1', x_2', \ldots, x_t')$$

where $x_i' = r^*(x_i)$ and $r^*(x_i) = 0$ for $i > t$. Similar results hold for the $Z/2Z$ graded complex K-theory of X and W with Q_p-coefficients;

$$K^*(X) = E(u_1, u_2, \ldots, u_r),$$

an exterior algebra over Q_p on odd dimensional generators $\{u_i\}$, and

$$K^*(W) = E(u_1', u_2', \ldots, u_t')$$

where $u_i' = r^*(u_i)$ for $i \le t$ and $r^*(u_i) = 0$ otherwise, as proved in Corollary 2.4 of [5].

We consider the K-theory of X and associated spaces filtered with the Chern character, or equivalently, the rational γ-filtration as in Section 3 of [5]. As $r\colon W \to X$ is an $A(p)$-map, it induces maps of projective spaces $r\colon WP^k \to XP^k$ for $k \le p$ (see [9]). (It will be clear from the context precisely what is intended where similar notations are used for different maps.) The exact sequence of the pair (WP^k, WP^{k-1}) for $k \le p$ implies that $H^i(WP^k, Q) = 0$ for $i > k(L_t + 1)$; the case $k = p$ is Corollary 2.8(b) of [5]. So for $k \le p$,

(2.1) $r^*\colon H^*(XP^k, Q) \to H^*(WP^k, Q)$ is an isomorphism in dimensions $\le k(L_t + 1)$ and zero otherwise.

We will apply (2.1) using the fact that $\mathrm{ch} \otimes Q$ induces isomorphisms

$$K^*(XP^k) \otimes Q \cong H^*(XP^k, Q) \quad \text{and} \quad K^*(WP^k) \otimes Q \cong H^*(WP^k, Q)$$

as follows. The first non-zero class in $H^*(XP^k, Q)$ of dimension greater than $k(L_t + 1)$ has dimension at least $2q$ (in fact, precisely $2q$ as is shown in the proof of Proposition 2.3). So if $z \in K^0(XP^k)$ is not a torsion class (which we will prove is always true) and $r^*(z) = 0$ in $K^0(WP^k)$, then z lies in filtration $2q$. If one applies this argument for $k = 1$ when $XP^1 = SX$ and $WP^1 = SW$, then the choice of the generators $\{u_i\}$ for $K^*(X)$ ensures that their suspensions lie in filtration $2q$ for $i > t$ and any non trivial linear combination $\Sigma\alpha_i u_i$ for $1 \le i \le t$, when suspended, lies infiltration less than $2q$. (We will not halve the indices on the filtered groups as was done in [5].)

We consider for $k \le p$ the diagram in (2.2) of exact sequences; see (3.3) of [7].

(2.2)

$$\begin{array}{ccccccccccc}
\longrightarrow \tilde{K}^{*-1}(XP^{k-1}) & \xrightarrow{p_k^*} & \tilde{K}^{*-1}(E_X^k) & \xrightarrow{\delta_k} & \tilde{K}^*(XP^k) & \xrightarrow{i_{k-1}^*} & \tilde{K}^*(XP^{k-1}) & \xrightarrow{p_k^*} & \tilde{K}^*(E_X^k) & - \\
\downarrow & & \downarrow{\scriptstyle s^*} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle s^*} & \\
\longrightarrow \tilde{K}^{*-1}(WP^{k-1}) & \xrightarrow[p_k^*]{} & \tilde{K}^{*-1}(E_W^k) & \xrightarrow[\delta_k]{} & \tilde{K}^*(WP^k) & \xrightarrow[i_{k-1}^*]{} & \tilde{K}^*(WP^{k-1}) & \xrightarrow[p_k^*]{} & \tilde{K}^*(E_W^k) & -
\end{array}$$

Here $i_{k-1}\colon XP^{k-1} \to XP^k$ is the inclusion and E_X^k is the total space of the quasi-fibration $p_k\colon E_X^k \to XP^{k-1}$ as is usual. The vertical homomorphisms are induced from $r\colon W \to X$. The space E_X^k has the homotopy type of the k-fold join of X with itself and an isomorphism

$$\Delta^{(k)}\colon \left(\tilde{K}^*(X) \otimes \cdots \otimes \tilde{K}^*(X)\right)^j \to \tilde{K}^{j+k-1}\left(E_X^k\right)$$

is defined after the proof of Lemma 2.2 of [5]. The element

$$\Delta^{(k)}(z_1 \otimes z_2 \otimes \cdots \otimes z_k)$$

is denoted by $z_1 * z_2 * \cdots * z_k$ where $z_i \in \tilde{K}^*(X)$.

The corollary to Theorem B of [7] implies that $K^*(X)$ is an exterior algebra on odd dimensional $A(k-1)$-primitive classes for $k \leq p$. Let S: $K^*(X) \to K^*(SX)$ be the suspension isomorphism. Theorem A of [7] for $k < p$ implies that

$$\begin{aligned} K^*(XP^k) &= Q_p[v_1, v_2, \ldots, v_r]^{k+1} \oplus S_k, \\ K^*(WP^k) &= Q_p[v_1', v_2', \ldots, v_t']^{k+1} \oplus S_k', \end{aligned} \tag{2.3}$$

where $Q_p[\ \ ]^{k+1}$ denotes a polynomial algebra truncated at height $k+1$, S_k is defined below and is a free Q_p-module and an ideal such that $S_k \cdot \tilde{K}^*(XP^k) = 0$, and $i_2^* i_3^* \ldots i_{k-1}^*(v_i) = S(u_i)$ for $1 \leq i \leq r$. Any choice of $\{v_i\}$ satisfying the last equation will serve in (2.3). The summand S_k is the submodule of $\tilde{K}(XP^k)$ spanned by elements of the form $\delta_k(z_1 * z_2 * \cdots * z_k)$ where $z_i \in \tilde{K}(X)$ and at least one z_i is decomposable. The composition

$$\tilde{K}^*(E_X^{k-1}) \xrightarrow{\delta_{k-1}} \tilde{K}^{*+1}(XP^{k-1}) \xrightarrow{p_k^*} \tilde{K}^{*+1}(E_X^k)$$

is given explicitly in Proposition 2.4 of [7].

$$\begin{aligned} &p_k^* \delta_{k-1}(z_1 * z_2 * \cdots * z_{k-1}) \\ &\quad = \sum (-1)^j z_1 * \cdots * \left(\sum z_j' * z_j''\right) * \cdots * z_{k-1}, \end{aligned} \tag{2.4}$$

$1 \leq j \leq k-1$, where $z_j \in \tilde{K}(X)$ and $M^*(z_j) = z_j \otimes 1 + \Sigma z_j' \otimes z_j'' + 1 \otimes z_j$ is induced from the H-space multiplication M: $X \times X \to X$.

A useful first application of (2.4) is that $K^0(XP^p)$ is torsion free. We consider

$$\begin{array}{c} \to \tilde{K}^{-1}(XP^{p-1}) \xrightarrow{p_p^*} \tilde{K}^{-1}(E_X^p) \xrightarrow{\delta_p} \tilde{K}^0(XP^p) \longrightarrow \tilde{K}^0(XP^{p-1}) \to \\ \uparrow{\scriptstyle \delta_{p-1}} \\ \tilde{K}^{-1}(E_X^{p-1}) . \end{array}$$

If $\tilde{K}^0(XP^p)$ has p-torsion, there exists $v \in \tilde{K}^{-1}(XP^{p-1})$ not divisible by p with $p_p^*(v) = pw \neq 0$, since $\tilde{K}^0(XP^{p-1})$ is torsion free. For dimensional reasons, $v \in S_{p-1}$. We have a contradiction using (2.4) or Lemma 3.5 of [7]. (We do not assert that $K^*(XP^p)$ has no torsion.)

The fact that $K^0(XP^p)$ is torsion free enables us to extend from $k < p$ to $k \leq p$ a corollary to a result of Thomas [11].

(2.5). Let $z_{i_1}, z_{i_2}, \ldots, z_{i_k} \in \tilde{K}^0(XP^k)$ satisfy $i_2^* i_3^* \ldots i_{k-1}^*(z_{i_j}) = S(u_{i_j})$ for each j, $1 \leq j \leq k$. Then

$$\delta_k(u_{i_1} * u_{i_2} * \cdots * u_{i_k}) = z_{i_1} z_{i_2} \ldots z_{i_k}.$$

This follows from the corresponding result in rational cohomology (see Theorem 4.1 of [7]) and the fact that

$$\mathrm{ch}\colon K^0(XP^k) \to H^{\mathrm{even}}(XP^k, Q)$$

is a monomorphism for $k \leq p$.

We now strengthen (2.3).

LEMMA 2.1. *For $k < p$, the homomorphism $r^*\colon K^*(XP^k) \to K^*(WP^k)$ is surjective. The classes $\{v_i\}$ and $\{v_i'\}$ in* (2.3) *can be chosen so that $v_i' = r^*(v_i)$ for $i \leq t$ and $r^*(v_i) = 0$ for $i > t$, and $r^*(S_k) = S_k'$.*

Proof. As $XP^1 = SX$, the case $k = 1$ follows from Corollary 2.4 of [5]. We argue by induction using (2.2) and the fact that

$$s^*\colon K^*(E_X^k) \to K^*(E_W^k)$$

is surjective for all k. Naturality implies that we may define v_i' by $v_i' = r^*(v_i)$ for $i \leq t$. For $i > t$ the induction hypothesis implies that by adding to v_i an element in the image of δ_k, we can ensure that $r^*(v_i) = 0$. The definitions imply that $r^*(S_k) = S_k'$.

We will assume that generators $\{v_i\}$ have been chosen as specified in Lemma 2.1. It follows from (2.1) that each v_i for $i > t$ lies in filtration $2q$.

LEMMA 2.2. *Let $v \in \tilde{K}^0(XP^k)$ where $k \leq p$ and suppose that*

$$i_{k-1}^*(v) \in \tilde{K}^0(XP^{k-1})$$

lies in filtration $2q$. Then there exists $z \in \tilde{K}^{-1}(E_X^k)$ such that $v + \delta_k(z)$ lies in filtration $2q$.

Proof. Since $H^i(WP^k, Q) = 0$ for $i > k(L_t + 1)$ and $r^* i_{k-1}^*(v) \in \tilde{K}^0(WP^{k-1})$ lies in filtration $2q$ which is greater than $k(L_t + 1)$, we deduce that $r^* i_{k-1}^*(v) = 0$. So by the exactness of (2.2), there exists $z' \in \tilde{K}^{*-1}(E_W^k)$ with $\delta_k(z') = r^*(v)$. As s^* is surjective, let $s^*(z) = -z'$. Then $r^*(v + \delta_k(z)) = 0$, as required.

We use these lemmata to prove the following key proposition. Recall that $K^0(XP^p)$ is a free Q_p-module.

PROPOSITION 2.3. *There exists $v \in \tilde{K}^0(XP^p)$ of exact filtration $2q$ such that $v^p \neq 0 \bmod p$.*

Proof. Let $S(x_{t+1}) \in H^{2q}(SX, Q)$ be the suspension of the generator x_{t+1}. Then there exists

$$y'_{2q} \in H^{2q}(XP^{p-1}, Q)$$

with $i_2^* i_3^* \dots i_{p-2}^*(y'_{2q}) = S(x_{t+1})$. We consider the exact sequence

$$\to H^{2q}(XP^p, Q) \xrightarrow{i_{p-1}^*} H^{2q}(XP^{p-1}, Q) \xrightarrow{p_p^*} H^{2q}(E_X^p, Q) \to .$$

For dimensional reasons $H^{2q}(E_X^p, Q) = 0$ and so

$$i_{p-1}^*(y_{2q}) = y'_{2q} \quad \text{for some } y_{2q} \in H^{2q}(XP^p, Q).$$

We choose $v' \in \tilde{K}^0(XP^{p-1})$ not divisible by p such that $\mathrm{ch}(v') = y'_{2m}$; so in particular v' has exact filtration $2q$. Therefore by naturality and the fact that ch: $\tilde{K}^*(E_X^p) \to H^*(E_X^p, Q)$ is mono, there exists $v \in \tilde{K}^0(XP^p)$ with $i_{p-1}^*(v) = v'$ and by Lemma 2.2 we can choose v to have filtration $2q$. We need to ensure that $v^p \neq 0 \bmod p$ and this may not be true unless $i_2^* i_3^* \dots i_{p-1}^*(v) \neq 0 \bmod p$. We show that we can modify v to achieve this.

LEMMA 2.4. *There exists $v \in \tilde{K}^0(XP^p)$ of exact filtration $2q$ such that*

$$i_2^* i_3^* \dots i_{p-1}^*(v) \neq 0 \bmod p \quad \text{in } \tilde{K}(SX).$$

Proof. Let v' above be expressed as $\tilde{v} + d + c$ where $\tilde{v} = \Sigma \beta_i v_i$ is a linear combination of $\{v_i\}$ in $\tilde{K}^0(XP^{p-1})$, d is decomposable and a polynomial in $\{v_i\}$ and $c \in S_{p-1}$. If we map v' to $\tilde{K}^0(SX)$, take the Chern character and use the remarks following (2.1), it follows that the summation defining v runs over $i > t$. Let d be expressed explicitly as a polynomial in the form

$$q(v_1, v_2, \dots, v_t) + e,$$

where e is a polynomial in which each monomial is divisible by v_i for some $i > t$. So e has filtration $> 2q$ and as $p_p^*(e) = 0$, e pulls back to $\tilde{K}^0(XP^p)$. So we can assume that $v' = \tilde{v} + q(v_1, \dots, v_t) + c$ is nonzero mod p, has exact filtration $2q$ and pulls back to $\tilde{K}^0(XP^p)$. To establish Lemma 2.4, we must show that $\tilde{v} \neq 0 \bmod p$. We argue by contradiction and assume that $\tilde{v} = 0 \bmod p$. Now $p_p^*(v') = 0$ in $\tilde{K}^0(E_X^p)$ and also

$$p_p^*(q(v_1, v_2, \dots, v_t)) = 0.$$

Therefore $p_p^*(c) = -p_p^*(\tilde{v})$. But p_p^* restricted to S_{p-1} is a monomorphism (see the proof of Proposition 4.2 $(B)_p$ of [7]; the missing hypotheses are not needed for this) and has torsion free cokernel (Lemma 3.5 of [7]). Therefore $c = 0 \mod p$. Also $r^*(v') = 0$ and $r^*(v) = q(v_1', v_2', \ldots, v_t') + r^*(c)$, using Lemma 2.1. As $r^*(c) = 0 \mod p$, $q(v_1', v_2', \ldots, v_t') = 0 \mod p$ and so $q(v_1, v_2, \ldots, v_t) = 0 \mod p$. Thus $v' = 0 \mod p$, which is false. So $\tilde{v} \neq 0 \mod p$ as required.

Let v be as in Lemma 2.4. The lemma following completes the proof of Proposition 2.3.

LEMMA 2.5. *In $K^0(XP^p)$, $v^p \neq 0 \mod p$.*

Proof. By (2.5), $\delta_p(u * u * \cdots * u) = v^p$ where $i_2^* i_3^* \ldots i_{p-1}^*(v) = S(u)$. If $v^p = 0 \mod p$, there exists $c \in \tilde{K}^{-1}(XP^{p-1})$ with $p_p^*(c) = u * u * \cdots * u \mod p$, as the groups are torsion free. For dimensional reasons $c \in S_{p-1}$ and the formula of (2.4) shows immediately that this is impossible.

We return in this final paragraph of Section 2, to the case when X lies in the first family, that is $n_{k+1} \leq p^3 L_k$ for all $k < r$. Much simpler arguments than those given above ensure that:

PROPOSITION 2.6. *There exists $v \in \tilde{K}^0(XP^p)$ of exact filtration $2n$, such that $v^p \neq 0 \mod p$.*

Section 3

We consider $M = K^0(XP^p)$ as a filtered ring with the rational γ-filtration. M is a free Q_p-module of dimension less than $2^{(p+1)r(X)}$ by Corollary 2.8(a) of [5] applied to X. We again use Corollary 2.8(b) of [5] to deduce that $M_{2n} = 0$ for $2n > p(d(X) + 1)$.

PROPOSITION 3.1. *Let $\dim_{Q_p} M < l$ and $M_{2pnm+2} = 0$, where m is a constant. Assume that there exists $v \in M_{2n}$ with $v^p \neq 0 \mod p$. Then $n < C(l, m)$, a constant depending upon l and m.*

This is Proposition 3.2 of [5] using the fact that $\psi^p(v) = v^p \mod p$.

Let X be a member of the first family, that is, $n_2 \leq p^3 L_1$, $n_3 \leq p^3 L_2, \ldots,$ $n_r \leq p^3 L_{r-1}$. Then as in the proof of Theorem 3.1 of [5], we deduce that

$$L_r < \left(1 + 4p^3\right)^{r-1} 2n_1 \quad \text{and} \quad p(L_r + 1) < 4\left(1 + 4p^3\right)^{r-1} pn_1.$$

We let $l = 2^{(p+1)r}$, $2m = 4(1 + 4p^3)^{r-1}$, v be as in Proposition 2.6 and $n = n_1$ and deduce that $n_1 < C(l, m) = C$ as required. So $d(X)$ is bounded.

In general we argue by induction on rank, rank one being covered by the case above. We assume that if the rank is less than r, then the dimension is less than $E(r)$. We need only consider X in the second family and by the induction hypothesis, $d(W) = L_t < E(r)$. Then

$$d(X) = L_r < \left(1 + 4p^3\right)^{r-2} 2q + E(r)$$

and so

$$p(L_r + 1) < p\left(\left(1 + 4p^3\right)^{r-2} 2q + E(r) + 1\right).$$

As all terms in this last expression are bounded except for q, we can choose a constant m so that $p(L_r + 1) < 2pqm$. We again apply Proposition 3.1 with $l = 2^{(p+1)r}$, m as chosen, v as in Proposition 2.3 and $n = q$. So $q < C(l, m) = D$. So $d(X)$ is bounded by $B(r)$ in all cases and the proof of Theorem 1.1 is completed.

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