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OPERATORS INTERPOLATING BETWEEN RIESZ POTENTIALS AND MAXIMAL OPERATORS

BY

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1. Introduction

Let λ be normalized Lebesgue measure on either the unit ball or the unit sphere in $\mathbf{R}^n$ and write λ_r for the dilate of λ defined by

$$\langle f, \lambda_r \rangle = \int_{\mathbf{R}^n} f(rx)\, d\lambda(x), \quad r > 0.$$

Suppose $1 \leqq p \leqq q \leqq \infty$, $1 \leqq s \leqq \infty$ and suppose f is continuous with compact support. When λ is the measure on the ball, define

$$S_{p,q,s} f(x) = \left[\int_0^\infty |r^{n/p - n/q} \lambda_r * f(x)|^s \frac{dr}{r} \right]^{1/s},$$

$$S_{p,q,\infty} f(x) = \sup_{r>0} r^{n/p-n/q} |\lambda_r * f(x)|.$$

When λ is the measure on the sphere, define operators $T_{p,q,s}$ and $T_{p,q,\infty}$ analogously. For nonnegative f, both $S_{p,q,1} f$ and $T_{p,q,1} f$ are multiples of the Riesz potential $I_\alpha(f)$ when $\alpha = n/p - n/q$. Hence $S_{p,q,1}$ and $T_{p,q,1}$ are bounded from $L^p (= L^p(\mathbf{R}^n))$ to L^q whenever $1 < p < q < \infty$. On the other hand, $S_{p,q,\infty}$ and $T_{p,q,\infty}$ are maximal operators, weighted to allow the possibility of $L^p - L^q$ boundedness. Indeed, $S_{p,p,\infty}$ is the Hardy-Littlewood maximal operator and therefore bounded on L^p for $1 < p \leqq \infty$, while $T_{p,p,\infty}$ is the spherical maximal operator, now known to be bounded on L^p when $n/(n-1) < p \leqq \infty$ (see [7], [2]). In general, and especially when $s = 2$, the functions $S_{p,q,s} f$ and $T_{p,q,s} f$ are reminiscent of g-functions. The purpose of this paper is to begin the study of the following question:

For what values of p, q, and s is $T_{p,q,s}$ bounded from L^p to L^q?

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The corresponding problem for $S_{p,q,s}$ is not difficult. We give its solution in the short §2. In §3 we tell what we know for the operators $T_{p,q,s}$. There are necessary conditions which may be sufficient and sufficient conditions which fall short of the necessary conditions. In §4 we apply one of the results of §3 to study the mapping properties of the convolution operator defined by a certain singular measure on $\mathbf{R}^3$. From now on the statement "$S_{p,q,s}$ is bounded" will mean that $S_{p,q,s}$ is bounded from L^p to L^q, and similarly for $T_{p,q,s}$.

2. The operators $S_{p,q,s}$

THEOREM 1. *The operator $S_{p,q,s}$ is bounded exactly when one of the following holds*:

(a) $1 < p < q < \infty$ *and* $1 \leqq s < \infty$,
(b) $1 < p \leqq q \leqq \infty$ *and* $s = \infty$,
(c) $1 < p \leqq s < \infty$ *and* $q = \infty$,
(d) $p = 1$, $q = s = \infty$.

Proof. In order to apply complex interpolation we view the operators $S_{p,q,s}$ not as sublinear operators from L^p to L^q but as linear operators from L^p into the mixed normed spaces $L^q_{dx}(L^s_{dr/r})$. Define

$$S_z f(x,r) = r^z \lambda_r * f(x), \quad z \in \mathbf{C}.$$

Then the boundedness of $S_{p,q,s}$ is equivalent to the boundedness of $S_{n/p-n/q}$ from L^p to $L^q(L^s)$. Now suppose $1 < p < q < \infty$. The operator $S_{p,q,1}$ is bounded, so $S_{n/p-n/q}$ is bounded from L^p to $L^q(L^1)$. By the interpolation theorem in [1], (a) will follow when we show that $S_{n/p-n/q}$ is bounded from L^p to $L^q(L^\infty)$. But the operators S_{iy} $(y \in \mathbf{R})$ are controlled by the Hardy-Littlewood maximal operator and so are uniformly bounded from L^p to $L^p(L^\infty)$ (even if $p = \infty$). Also, the operators $S_{n/p+iy}$ $(y \in \mathbf{R})$ are uniformly bounded from L^p to $L^\infty(L^\infty)$ since the measures $r^{n/p+iy}\lambda_r$ $(r > 0,\ y \in \mathbf{R})$ are uniformly bounded in the dual of L^p. Thus another application of the mixed norm interpolation theorem finishes the proof of the sufficiency of (a). Along the way, we have also established the sufficiency of (b). The case $s = p$ of (c) can be deduced from Hardy's inequality (p. 196 of [8], for example). The remainder of (c) follows by interpolating with the case $s = \infty$. To see that one of (a)–(c) is necessary for boundedness when $p > 1$, just note that if $1 < p = q \leqq \infty$ and $1 \leqq s < \infty$, the integral defining $S_{p,q,s}f$ will not generally converge. The remaining case occurs when $1 \leqq s < p$ and $q = \infty$. To see that $S_{p,q,s}$ is not bounded here, consider $S_{p,q,s}f$ when

$$f(x) = \begin{cases} |x|^{-n/p} & \text{if } 1 \leqq |x| \leqq N \\ 0 & \text{otherwise} \end{cases}$$

and let $N \to \infty$.

If $p = 1$, then $S_{p,q,s}$ will be bounded if and only if

$$\|S_{p,q,s}\delta_0\|_q < \infty.$$

Condition (d) is necessary and sufficient for this to occur.

3. The operators $T_{p,q,s}$

Mixed norm interpolation arguments like those in §2 show that the collection of points

$$\left\{\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{s}\right) : 0 \leqq \frac{1}{q} \leqq \frac{1}{p} \leqq 1, 0 \leqq \frac{1}{s} \leqq 1, \text{ and } T_{p,q,s} \text{ is bounded}\right\}$$

is a convex set. Convergence of the integral in the definition of $T_{p,q,s}$ requires

(A) $$\frac{1}{q} < \frac{1}{p} \quad \text{unless } s = \infty.$$

To get necessary conditions reflecting the dimension n ($n \geqq 2$), we estimate norms

$$\|T_{p,q,s}f\|_q$$

where f is the characteristic function χ_E of a suitable subset E of $\mathbf{R}^n$. For example, if E is a ball, then

$$\|T_{p,q,s}\chi_E\|_q < \infty$$

implies

(B) $$\frac{n}{p} - n + 1 < \frac{1}{s}.$$

If E is an annulus of inner radius 1 and outer radius $1 + \varepsilon$, then

(1) $$\|T_{p,q,s}\chi_E\|_q = O(\|\chi_E\|_p)$$

leads to

(C) $$\frac{1}{p} \leqq \frac{n}{q} + \frac{1}{s}.$$

Finally, if

$$E = [0,1]^{n-1} \times [0, \varepsilon],$$

then for $x = (x_1, \ldots, x_n)$ with $x_i \in [\frac{1}{2}, \frac{3}{4}]$ for $1 \leqq i \leqq n-1$ and $x_n > \varepsilon^{-1}$, say, we have

$$T_{p,q,s}\chi_E(x) \sim \left[\left(x_n^{n/p - n/q - n+1} \right)^s \frac{\varepsilon}{x_n} \right]^{1/s}.$$

Thus, if $1 \leqq s < \infty$, (1) leads to

$$\varepsilon^{1/s} \left[\int_{\varepsilon^{-1}}^{\infty} x_n^{[(n/p - n/q - n+1)s - 1]q/s} dx_n \right]^{1/q} = 0(\varepsilon^{1/p}).$$

For $T_{p,q,s}$ to be bounded, (B) must hold. It follows that the integral above converges and that

$$\text{(D)} \qquad \frac{n+1}{p} - n + 1 \leqq \frac{2}{s} + \frac{n-1}{q}.$$

One can check directly that (D) must also hold when $s = \infty$ (and $T_{p,q,s}$ is bounded). Figure 1 represents in the case $n \geqq 3$ the intersection of the subset B of

$$\left\{ \left(\frac{1}{p}, \frac{1}{q}, \frac{1}{s} \right) : 0 \leqq \frac{1}{q} \leqq \frac{1}{p} \leqq 1, 0 \leqq \frac{1}{s} \leqq 1 \right\}$$

defined by (A)–(D) with the plane $1/s = 0$. Figure 2 represents the projection of the intersection of B with the plane $1/q = 1/s$ onto the $1/p - 1/q$ plane. The sufficient conditions which we will establish can be explained as follows. Theorem 2 implies that $T_{p,q,\infty}$ is bounded whenever $(1/p, 1/q)$ lies strictly above the line through $(0,0)$ and $((n-1)/n, 1/n)$ in the region of Figure 1. Theorem 3 shows that in the case $n = 2$, $T_{p,q,q}$ is bounded whenever $(1/p, 1/q)$ is in the region of Figure 2 except perhaps on the open segment from $(\frac{1}{2}, \frac{1}{6})$ to $(\frac{2}{3}, \frac{1}{3})$. Of course other sufficient conditions follow by interpolating these results with the boundedness of $T_{p,q,1}$ whenever $1 < p < q < \infty$. The corollary after Theorem 2 is an example.

THEOREM 2. *Suppose $n \geqq 3$ and let p' denote the exponent dual to p. Then $T_{p,p',\infty}$ is bounded whenever $n/(n-1) < p \leqq 2$.*

Proof. This is an easy application of complex interpolation in the mixed norm setting as in the proof of Theorem 1. Adopt the notations of [7] and fix p with $n/(n-1) < p \leqq 2$. Let

$$\varepsilon = \frac{1}{2}\left(n - \frac{p}{p-1} \right).$$

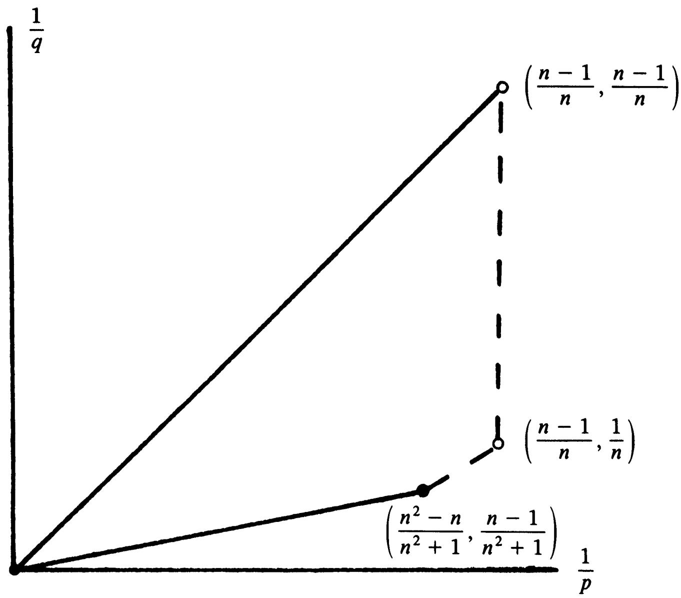

FIG. 1

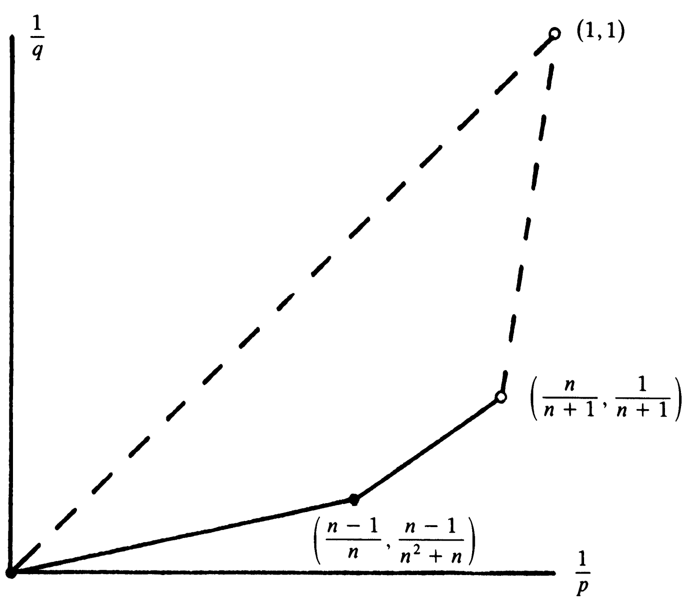

FIG. 2

Then $\varepsilon > 0$. Put

$$\alpha(z) = 1 + \left(\frac{n}{2} - \varepsilon\right)\left(\frac{z}{n} - 1\right)$$

and define

$$T_z f(x, r) = r^z\left(M_r^{\alpha(z)} f\right)(x).$$

Then Theorem 2 of [7] gives the boundedness of the operators T_z from L^2 to $L^2(L^\infty)$ when $\operatorname{Re} z = 0$. When $\operatorname{Re} z = n$ the operators T_z are bounded from L^1 to $L^\infty(L^\infty)$. After interpolating it follows that $T_{n/p - n/p'}$ is bounded from L^p to $L^{p'}(L^\infty)$. This is the desired result since

$$\alpha\left(\frac{n}{p} - \frac{n}{p'}\right) = 0$$

by choice of ε.

COROLLARY. *For $n \geqq 2$, $(n/(n+1), 1/(n+1))$ is a limit of points $(1/p, 1/q)$ such that $T_{p,q,q}$ is bounded.*

Proof. As a consequence of [2] (when $n = 2$) and Theorem 2 (when $n \geqq 3$), there are always points $(1/p, 1/q)$ close to $((n-1)/n, 1/n)$ such that $T_{p,q,\infty}$ is bounded. Interpolating judiciously with the fact that $T_{p_2, p_2', 1}$ is bounded for p_2 slightly larger than 1 yields the corollary.

THEOREM 3. *Fix $n = 2$ and suppose $(1/p, 1/q)$ is a point in the region of Figure 2 not on the open segment joining $(\frac{1}{2}, \frac{1}{6})$ and $(\frac{2}{3}, \frac{1}{3})$. Then $T_{p,q,q}$ is bounded.*

Proof. By interpolation and the corollary above it is enough to show that $T_{2,6,6}$ is bounded. Let

$$\mathbf{R}^3_+ = \{(x, r) \colon x \in \mathbf{R}^2,\ r > 0\}$$

and define an operator T taking functions on $\mathbf{R}^2$ to functions on $\mathbf{R}^3_+$ by

$$Tf(x, r) = r^{1/2}\lambda_r * f(x).$$

Then the boundedness of $T_{2,6,6}$ is equivalent to the boundedness of T from L^2 to $L^6(\mathbf{R}^3_+)$, where the measure on $\mathbf{R}^3_+$ is given by $dx\,dr$ (the restriction of three-dimensional Lebesgue measure) and not by $dx\,dr/r$. This latter boundedness is equivalent to that of TT^* from $L^{6/5}(\mathbf{R}^3_+)$ to $L^6(\mathbf{R}^3_+)$. A computation yields

$$T^* g(x) = \int_0^\infty \lambda_s * g(\cdot, s)(x) s^{1/2}\, ds,$$

and so

$$TT^*g(x,r) = r^{1/2}\lambda_r * (T^*g)(x) = r^{1/2}\int_0^\infty \lambda_r * \lambda_s * g(\cdot, s)(x) s^{1/2}\,ds.$$

A Jacobian computation shows that for $f \geqq 0$,

$$\int_{\mathbf{R}^2} f\,d(\lambda_r * \lambda_s)$$
$$= 4\int_{\{|r-s|<|y|<r+s\}} f(y)\big[\big[(r+s)^2 - |y|^2\big]\big[|y|^2 - (r-s)^2\big]\big]^{-1/2} dy.$$

Thus

$$\begin{aligned} TT^*g(x,r) &= 4\iint_{\{s>0,\ |r-s|<|x-y|<r+s\}} g(y,s)(rs)^{1/2}\big[\big[(r+s)^2 - |x-y|^2\big] \\ &\quad \times \big[|x-y|^2 - (r-s)^2\big]\big]^{-1/2} dy\,ds \\ &= 2\iint_{\{s>0,\ |r-s|<|x-y|<r+s\}} g(y,s) \\ &\quad \times \left[\frac{1}{(r+s)^2 - |x-y|^2} + \frac{1}{|x-y|^2 - (r-s)^2}\right]^{1/2} dy\,ds \\ &\leqq 2\iint_{\{s>0,\ |x-y|<r+s\}} g(y,s)\big[(r+s)^2 - |x-y|^2\big]^{-1/2} dy\,ds \\ &\quad + 2\iint_{\{s>0,\ |r-s|<|x-y|\}} g(y,s)\big[|x-y|^2 - (r-s)^2\big]^{-1/2} dy\,ds. \end{aligned}$$

Define kernels K_1 and K_2 on $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ by

$$K_1(y,s) = \begin{cases} (s^2 - |y|^2)^{-1/2} & \text{if } |y| < |s| \\ 0 & \text{if } |y| \geqq |s|, \end{cases}$$

$$K_2(y,s) = \begin{cases} (|y|^2 - s^2)^{-1/2} & \text{if } |y| > |s| \\ 0 & \text{if } |y| \leqq |s|. \end{cases}$$

It is enough to show that, for $i = 1, 2$, convolution with K_i defines a bounded operator from $L^{6/5}(\mathbf{R}^3)$ to $L^6(\mathbf{R}^3)$. To do so we adopt the notation of [3]. Thus we will use x_1, x_2, x_3 for coordinates in $\mathbf{R}^3$ and write $P(x)$ for either

$$x_1^2 - x_2^2 - x_3^2 \quad \text{or} \quad x_1^2 + x_2^2 - x_3^2.$$

In the first case $P_+^{-1/2}$ corresponds to K_1, in the second to K_2. We will use complex interpolation to show that convolution with $P_+^{-1/2}$ defines a bounded operator from $L^{6/5}$ to L^6. To this end, define a family of convolution operators by

$$T_z f = P_+^z * f/\Gamma(z+1)\Gamma(z+\tfrac{3}{2}).$$

By the considerations of Chapter 3 (Section 2.2) of [3], T_z is an entire family of convolution operators. If $\operatorname{Re} z = 0$, the functions P_+^z are uniformly bounded, and so the operators T_z are bounded from L^1 to L^∞. If $\operatorname{Re} z = -\frac{3}{2}$, the functions P_+^z have uniformly bounded Fourier transforms (p. 365 of [3]), and so the operators T_z are bounded on L^2. The interpolation theorem in [6] is applicable. It follows that $T_{-1/2}$ is bounded from $L^{6/5}$ to L^6. This completes the proof of Theorem 3.

4. An application

Let λ_r be the uniform probability measure on the circle in $\mathbf{R}^2$ with center the origin and radius r. Define a measure μ on $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ by

$$\int_{\mathbf{R}^3} f\,d\mu = \int_0^\infty \int_{\mathbf{R}^2} f(x,r)\,d\lambda_r(x)\,dr.$$

Then μ is concentrated on a cone in $\mathbf{R}^3$. Let T be the operator on functions on $\mathbf{R}^3$ given by convolution with μ.

THEOREM 4. *The operator T is bounded from $L^p(\mathbf{R}^3)$ to $L^q(\mathbf{R}^3)$ if and only if*

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{3} \quad \text{and} \quad \frac{6}{5} < p < 2.$$

Proof. If T is bounded from L^p to L^q, homogeneity considerations show that

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{3}.$$

If f is the characteristic function of a ball, then $\|Tf\|_q$ can be finite only when $q > 2$ (and so $p > 6/5$ if $1/q = 1/p - 1/3$). It then follows from duality that $p < 2$ if T is bounded from L^p to L^q. To prove the converse, fix nonnegative functions f and g on $\mathbf{R}^3$. Using $\|\cdot\|_p$ to denote an L^p norm on either $\mathbf{R}^2$ or $\mathbf{R}^3$, we estimate

$$\begin{aligned}\langle f, \mu^* g\rangle &= \int_{-\infty}^{\infty}\int_0^{\infty}\int_{\mathbf{R}^2} f(x, t+r)\lambda_r * g(\cdot, t)(x)\,dx\,dr\,dt\\ &\leq \int_{-\infty}^{\infty}\int_0^{\infty}\|f(\cdot, t+r)\|_{q'}\|\lambda_r * g(\cdot, t)\|_q\,dr\,dt\\ &\leq \int_{-\infty}^{\infty}\left[\int_0^{\infty}\left(\|f(\cdot, t+r)\|_{q'} r^{1/s-2/p+2/q}\right)^{s'} dr\right]^{1/s'}\\ &\quad\times\left[\int_0^{\infty}\|r^{2/p-2/q}\lambda_r * g(\cdot, t)\|_q^s \frac{dr}{r}\right]^{1/s} dt.\end{aligned}$$

Thus T will be bounded if we can find $s \in (1, \infty)$ such that the estimates (2) and (3) below are valid:

$$\left[\int_{-\infty}^{\infty}\left[\int_0^{\infty}\left(\|f(\cdot, t+r)\|_{q'} r^{1/s-2/p+2/qs'}\,dr\right)\right]^{p'/s'} dt\right]^{1/p'} \leqq C\|f\|_{q'}, \tag{2}$$

$$\left[\int_{-\infty}^{\infty}\left[\int_0^{\infty}\|r^{2/p-2/q}\lambda_r * g(\cdot, t)\|_q^s \frac{dr}{r}\right]^{p/s} dt\right]^{1/p} \leqq C\|g\|_p. \tag{3}$$

(The symbol C denotes a positive constant which may increase from line to line.)

If $s' < q' < p' < \infty$, let C be a bound for the one-dimensional Riesz potential of order $s'/q' - s'/p'$ as a mapping from $L^{q'/s'}(\mathbf{R})$ to $L^{p'/s'}(\mathbf{R})$. Then if

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{3},$$

we have

$$s'\left(\frac{1}{s} - \frac{2}{p} + \frac{2}{q}\right) = -1 + \frac{s'}{q'} - \frac{s'}{p'},$$

so

$$\begin{aligned}&\left(\int_{-\infty}^{\infty}\left[\int_0^{\infty}\|f(\cdot, t+r)\|_{q'}^{s'} r^{(1/s-2/p+2/q)s'}\,dr\right]^{p'/s'} dt\right)^{s'/p'}\\ &\quad\leq C\left[\int_{-\infty}^{\infty}\|f(\cdot, t)\|_{q'}^{q'}\,dt\right]^{s'/q'}.\end{aligned}$$

This gives (2). To obtain (3) we start from the fact that the convolution operator defined by the measure λ ($= \lambda_1$) is bounded from $L^{3/2}(\mathbf{R}^2)$ to $L^3(\mathbf{R}^2)$ (see, for example [4] or the lemma in [5]). Thus the following estimate holds for functions h on $\mathbf{R}^2$:

$$\sup_{r>0} \left\| r^{2(2/3-1/3)} \lambda_r * h \right\|_3 \leq C\|h\|_{3/2}.$$

On the other hand, the case $(1/p, 1/q) = (\frac{1}{2}, \frac{1}{6})$ of Theorem 3 yields the estimate

$$\left[\int_0^\infty \left\| r^{2(1/2-1/6)} \lambda_r * h \right\|_6^6 \frac{dr}{r} \right]^{1/6} \leq C\|h\|_2.$$

Interpolating these estimates shows that if $1/q = 1/p - 1/3$ and $\frac{3}{2} < p < 2$, then there is some $s > q$ such that

$$\left[\int_0^\infty \left\| r^{2(1/p-1/q)} \lambda_r * h \right\|_q^s \frac{dr}{r} \right]^{1/s} \leq C\|h\|_p.$$

This yields (3), and so T is bounded from L^p to L^q whenever $1/q = 1/p - 1/3$ and $\frac{3}{2} < p < 2$. Duality and one more interpolation complete the proof of the theorem.

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