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STRICTLY (CO)SINGULAR OPERATORS, THE MEYER-KÖNIG ZELLER PROPERTY, AND SUMS OF BANACH SPACES[1]

BY

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1. Introduction

Let Y be a subspace of a Banach space Z. The present work examines theorems of the following two related types: (i) $X = Z$ whenever $X + Y = Z$ with X an appropriate subspace of Z; (ii) $W \cap Z$ is closed in W whenever $W \cap Z \subset Y$ with W an appropriate subspace of a universe containing Z. An example of a type (i) theorem is a result of Bennett [2, Theorem 16]: If l_1 is compactly embedded in the BK space Z and if X is a separable FK space containing $\{\delta^n\}$ and satisfying $X + l_1 = Z$, then $X = Z$. The Meyer-König and Zeller theorem ([5] and [6]) is a theorem of type (ii): If W is an FK space such that $W \cap l_\infty \subset c_0$, then $W \cap l_\infty$ is closed in W. Further examples of type (i) results were given by the author in [9] and [10], and additional type (ii) theorems were established by Bennett in [1]. The two types were shown to be dual in a very special setting by the author in [10].

The principal goal of the following is to relate type (i) theorems to strictly cosingular operators, to connect type (ii) theorems with w^* strictly singular operators, and to thereby conclude the duality of the types in a general setting. Some old results, obtained frequently using gliding humps techniques, are established using Banach space theory. For instance the Meyer-König and Zeller theorem follows from the fact that c_0 contains no infinite dimensional w^*-closed subspace of l_∞.

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2. Preliminaries

Let H be a vector space with a Hausdorff (not necessarily linear) topology. A *BH space* is a subspace X of H which is a Banach space such that the injection of X into H is continuous. Details about BH spaces may be found in [12]. The closed graph theorem shows that if X and Y are BH spaces with $X \subset Y$, then the topology of X contains the relative topology of Y. Thus, BH topologies are unique. Also, X is closed in Y if and only if their norms are equivalent on X.

Let ω denote the space of all complex sequences with the topology of pointwise convergence. A *BK space* is a BH space with $H = \omega$.

If X and Y are BH spaces, then $X + Y$ is a BH space under

$$\|z\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y\colon x \in X,\, y \in Y,\, z = x + y\}.$$

Details about sums of BH spaces may be found in [12].

The following familiar BK spaces will occur in the sequel:

$$l_p = \left\{x\colon \|x\|_p^p = \sum |x_k|^p < \infty\right\}, \qquad 1 \le p < \infty.$$

$$l_\infty = \{x\colon \|x\|_\infty = \sup|x_k| < \infty\}.$$

$$c = \{x\colon \lim x_k \text{ exists}\} \text{ with norm of } l_\infty.$$

$$c_0 = \{x\colon \lim x_k = 0\} \text{ with norm of } l_\infty.$$

$$bs = \left\{x\colon \|x\|_{bs} = \sup_n \left|\sum_{k=1}^n x_k\right| < \infty\right\}.$$

$$cs = \left\{x\colon \lim_n \sum_{k=1}^n x_k \text{ exists}\right\} \text{ with norm of } bs.$$

$$bv = \left\{x\colon \|x\|_{bv} = |x_1| + \sum_{k=1}^\infty |x_{k+1} - x_k| < \infty\right\}.$$

$$bv_0 = bv \cap c_0.$$

Let ϕ devote the span of $\{\delta^n\}$ in ω where $\delta_k^n = 0$ for $k \neq n$, $\delta_n^n = 1$.

If X is a BK space containing ϕ, let $X^f = \{\{g(\delta^n)\}\colon g \in X^*\}$. Then X^f is a BK space with appropriate norm. If ϕ is dense in X, then X^f is just the dual of X.

For x, $y \in \omega$ and $W \subset \omega$, let $xy = \{x_n y_n\}$ and $xW = \{xw\colon w \in W\}$.

All maps between Banach spaces will be assumed to be bounded and linear. If $\{x_n\}$ is a sequence in a Banach space, let $[x_n]$ denote the closed linear span.

Let Y and Z be Banach spaces and let $T: Y \to Z$. Following Kato, T is called *strictly singular* if T is an isomorphism into Z on no infinite dimensional closed subspace of Y. Following Pelczynski, T is *strictly cosingular* if the existence of surjections A and B of Y and Z respectively onto a Banach space E with $A = BT$ implies that $\dim E < \infty$. The required properties of strictly (co)singular maps may be found in [7] and [11]. In particular, T is strictly cosingular if and only if T^* is w^* strictly singular; i.e., T^* is a w^*-isomorphism on no infinite dimensional w^*-closed subspace of Z^*.

3. Sums of Banach spaces and strictly cosingular operators

Let Y and Z be BK spaces with $\phi \subset Y \subset Z$. Following [10] let the relation $Y < Z$ indicate that $X = Z$ whenever X is a BK space containing ϕ such that $X + Y = Z$. Theorem 1 in [9] shows that $c_0 < l_\infty$ and Theorem 3 in [8] shows that $l_1 < Z$ if l_1 is weakly compactly embedded in Z. The principal result of the present section establishes the equivalence of $Y < Z$ and the fact that the natural injection of Y into Z is strictly cosingular. First a red herring in the definition of $Y < Z$ is removed. The removal allows an easy extension to the setting of Banach spaces, permits a perturbation theory argument, and yields Banach space proofs of many of the results of [10] without resorting to a primitive version of Section 4.

3.1 LEMMA. *Let Y and Z be BK spaces with $Y \subset Z$ and ϕ dense in Y. The following are equivalent*:

(i) *$X = Z$ whenever X is a BK space containing ϕ such that $X + Y = Z$.*

(ii) *$\dim Z/X < \infty$ whenever X is a BK space such that $X + Y = Z$.*

Proof. (ii) $\Rightarrow$ (i). X is closed in Z since $\dim Z/X < \infty$. The closure of ϕ in Y is then contained in the closure of ϕ in X so $Y \subset X$.

(i) $\Rightarrow$ (ii). There exists $z \in \omega$ with $z_n \neq 0$ for all n such that $zl_1 \subset Z$ and the natural injection is compact. Now $\phi \subset X + zl_1$ so $X + zl_1 = Z$ by hypothesis. The proof of Lemma 6(i) in [10] shows that $\dim Z/X < \infty$. ∎

DEFINITION. Let Z be a Banach space and let Y be a BZ space. Then $Y < Z$ if $\dim Z/X < \infty$ whenever X is a BZ space such that $X + Y = Z$.

Clearly, $Z < Z$ if and only if $\dim Z < \infty$.

3.2 THEOREM. *Let Y and Z be Banach spaces and let T be a map from Y into Z. The following are equivalent*:

(i) *T is strictly cosingular.*

(ii) *The natural injection of TY into Z is strictly cosingular.*

(iii) *$TY < Z$.*

Proof. The equivalence of (i) and (ii) is known and easy.

(i) $\Rightarrow$ (iii). Assume that $X + TY = Z$. Let B denote the unit ball of Z. A constant M may be selected such that for each $b \in B$ there exists $x(b) \in X$ and $y(b) \in Y$ satisfying $x(b) + Ty(b) = b$ and $\|x(b)\|_X \leq M$, $\|y(b)\|_Y \leq M$. Define operators A, B, C from $l_1(B)$ into Z by

$$A\lambda = \sum \lambda(b)x(b), \quad B\lambda = \sum \lambda(b)Ty(b), \quad C\lambda = \sum \lambda(b)b.$$

Then $A + B = C$, C is a surjection, and B is strictly cosingular. A result of Vladimirskii [11, Corollary 1] establishes that the range of A has finite codimension in Z. But $Al_1(B) \subset X$.

(iii) $\Rightarrow$ (i). Assume that P and Q are surjections of Y and Z respectively onto a Banach space E and that $P = QT$. If $TY < Z$, then one observes easily that $QTY < QZ$. Thus, $E = PY = QTY < QZ = E$. According to the remark following the definition of $<$, $\dim E < \infty$.

3.3 Corollary. (i) [10, Corollary 7(i)] $l_p < l_q$ *for* $1 \leq p < q \leq \infty$.

(ii) [9, Theorem 1] $c_0 < l_\infty$.

(iii) [8, Theorem 3] *If Z is a BK space containing l_1 and the injection of l_1 into Z is weakly compact, then $l_1 < Z$.*

(iv) [10, Corollary 5] $l_1 < cs$.

Proof. (i) The natural injection of l_p into l_q is strictly cosingular because, for instance, its adjoint is strictly singular.

(ii) The injection map is strictly cosingular according to a result of Pelczynski [7, Proposition 5].

Observe that the injection of c_0 into l_∞ is "w^* strictly singular". For instance, suppose that $S \subset c_0$ is infinite dimensional and w^*-closed in l_∞. Then S is closed in c_0 so S contains a copy of c_0. But then S contains a copy of l_∞, contradicting the separability of S.

If Z satisfies the hypothesis of (iii) or if $Z = cs$, then ϕ may be assumed dense in Z. Now $Z^f \subset c$, so the injection of Z^f into l_∞ is w^* strictly singular. Therefore, the injection of l_1 into Z is strictly cosingular. ∎

Observe that the BK hypothesis in 3.3(iii) cannot be dropped. It is easy to find a weakly compact non strictly cosingular injection of l_1 into a Banach space.

4. The Meyer-König Zeller property and strictly singular operators

A direct dualization of the condition $Y < Z$ yields a property which arose in summability theory. For instance, assume that X, Y, and Z are BK spaces with basis $\{\delta^n\}$, $X + Y = Z$, and $\dim Z/X = \infty$. Then $Z^f = X^f \cap Y^f$. It follows that Z^f is not closed in X^f, for otherwise $X = Z$. Therefore, the injection of Z^f into Y^f violates the following condition.

DEFINITION. Let U be a subspace of a Banach space Z. The injection of U into Z has the *Meyer-König Zeller property* (abbreviated *$U \to Z$ has MKZ*) if $W \cap Z$ is closed in W for each Hausdorff vector space H for which Z is a BH space and for each BH space W satisfying $W \cap Z \subset U$.

The Meyer-König Zeller theorem [5], [6] states that if W is an FK space such that $W \cap l_\infty \subset c_0$, then $W \cap l_\infty$ is closed in W. This is essentially the assertion that $c_0 \to l_\infty$ has MKZ.

The main results of the present section relate the MKZ property to w^* strictly singular operators in the context of adjoints of maps in separable spaces and thus to strictly cosingular operators. Theorem 4.3 substantially generalizes a result of the author [10, Theorem 2]. Technical lemmas designed for enhancing convergence in certain Hausdorff vector spaces H to dual BH spaces are required.

The first lemma is a simple consequence of compactness of the ball in finite dimensional spaces.

4.1 LEMMA. *Assume that Y is a separable Banach space with $[y_n] = Y$ and that S is a finite dimensional subspace of Y^*. There exists a positive integer r such that for each $g \in S$,*

$$\sup\{|g(y)| : \|y\| \leq 1, y \in [y_1, \ldots, y_r]\} \geq \tfrac{1}{2}\|g\|_{Y^*}.$$

4.2 LEMMA. *Assume that Y is a separable Banach space with $[y_n] = Y$ and that $\{g_n\} \subset Y^*$ satisfies $g_n(y_i) = 0$ for $i < n$. Then $\{g_n\}$ has a subsequence $\{h_n\}$ with the following property:*

If $\{t_k\} \in \omega$ and if there exists $h \in Y^*$ such that

$$h(y_n) = \sum_{k=1}^{n} t_k h_k(y_n) \left(= \sum_{k=1}^{\infty} t_k h_k(y_n) \right) \quad \text{for all } n,$$

then the series $\Sigma t_k h_k$ has bounded sections in Y^* and hence is ω^*-convergent.

Proof. Let $h_1 = g_1$, $r_1 = 1$, and assume h_n and a positive integer r_n have been chosen. Using 4.1, choose a positive integer $r_{n+1} > r_n$ such that

$$\text{(1)} \qquad \sup\{|h(y)| : \|y\| \leq 1, y \in [y_1, \ldots, y_{r_{n+1}}]\} \geq \tfrac{1}{2}\|h\|_{Y^*}$$

for all $h \in [h_1, \ldots, h_n]$. Let $h_{n+1} = g_{r_{n+1}+1}$.

Assume that the inductively defined subsequence $\{h_n\}$ does not have the desired property. This means that there exists $\{t_k\} \in \omega$ and $h \in Y^*$ such that

$$h(y_n) = \sum_{k=1}^{n} t_k h_k(y_n) \quad \text{for all } n$$

and $\Sigma t_k h_k$ fails to have bounded sections in Y^*.

Choose $s_n \uparrow \infty$ such that

$$\left\| \sum_{k=s_n+1}^{s_{n+1}} t_k h_k \right\| > 2^n \left\| \sum_{k=1}^{s_n} t_k h_k \right\| > 2^n \tag{2}$$

for all n. Using (1), choose

$$z_n \in \left[y_1, \ldots, y_{r_{s_{n+1}+1}} \right]$$

with $\|z_n\| \leq 1$ such that

$$\left| \sum_{k=s_n+1}^{s_{n+1}} t_k h_k(z_n) \right| \geq \tfrac{1}{3} \left\| \sum_{k=s_n+1}^{s_{n+1}} t_k h_k \right\|. \tag{3}$$

Now let $u_n = \Sigma_{k=1}^{n} z_k / k^2$ so

$$u_n \in \left[y_1, \ldots, y_{r_{s_{n+1}+1}} \right].$$

Note that $k > s_{n+1}$ implies that $h_k = g_{r_k+1} = 0$ on $[y_1, \ldots, y_{r_k}]$ and that $r_k \geq r_{s_{n+1}+1}$. Thus

$$h_k(u_n) = 0 \quad \text{for } k > s_{n+1}. \tag{4}$$

Also, $i < n$ implies

$$z_i \in \left[y_1, \ldots, y_{r_{s_n+1}} \right].$$

Furthermore, $k \geq s_n + 1$ implies that $r_k + 1 > r_{s_n+1}$, so $h_k(z_i) = g_{r_k+1}(z_i) = 0$. Therefore,

$$h_k(u_n) = h_k\left(\frac{1}{n^2} z_n \right) \quad \text{for } k \geq s_n + 1. \tag{5}$$

Applying (2), (3), (4), and (5),

$$
\begin{aligned}
|h(u_n)| &= \left|\sum_{k=1}^{\infty} t_k h_k(u_n)\right| \\
&= \left|\sum_{k=1}^{s_{n+1}} t_k h_k(u_n)\right| \\
&\geq \left|\sum_{k=s_n+1}^{s_{n+1}} t_k h_k(u_n)\right| - \left|\sum_{k=1}^{s_n} t_k h_k(u_n)\right| \\
&= \left|\sum_{k=s_n+1}^{s_{n+1}} t_k h_k\left(\frac{1}{n^2} z_n\right)\right| - \left|\sum_{k=1}^{s_n} t_k h_k(u_n)\right| \\
&\geq \frac{1}{3n^2}\left\|\sum_{k=s_n+1}^{s_{n+1}} t_k h_k\right\| - \|u_n\|\left\|\sum_{k=1}^{s_n} t_k h_k\right\| \\
&\geq \left(\frac{2^n}{3n^2}\right) - \sum_{k=1}^{n} \frac{1}{k^2}\Bigg)\left\|\sum_{k=1}^{s_n} t_k h_k\right\| \\
&\to \infty \quad \text{as } n \to \infty.
\end{aligned}
$$

The latter contradicts the fact that $\{u_n\}$ is bounded in Y. ■

4.3 THEOREM. *Let Y be a separable Banach space with $[y_n] = Y$ and let U be a subspace of Y^*. The following two conditions are equivalent*:

(i) *$U \to Y^*$ has MKZ*;

(ii) *U contains no infinite dimensional w^*-closed subspace of Y^*.*

If in addition U is a dual Banach space w^*-continuously embedded in Y^*, then (i) *and* (ii) *are equivalent to*:

(iii) *The injection of U into Y^* is strictly singular.*

Proof. (ii) ⇒ (i). Assume that Y^* and W are BH spaces with $W \cap Y^* \subset U$ and $W \cap Y^*$ is not closed in W.

Observe first that there exists $\{f_n\} \subset W \cap Y^*$ such that $\|f_n\|_{Y^*} = 1$, $\|f_n\|_W = O(2^{-n})$, and $\{f_n\}$ is w^*-basic in Y^*. For instance, the BH space

$$U_n = \{f \in W \cap Y^*\colon f(y_i) = 0 \quad \text{for } i < n\}$$

has finite codimension in $W \cap Y^*$ and hence is a BH space with the norm of $W \cap Y^*$ and is not closed in W. Thus, there exists $g_n \in U_n$ such that $\|g_n\|_{U_n} = 1$ and $\|g_n\|_W < 2^{-n}$. But $\{g_n\}$ is bounded away from zero in Y^* since $g_n \to 0$ in W. Also, $g_n \to 0$ w^* in Y^*, so a subsequence $\{f_n\}$ is w^*-basic according to a result of Johnson and Rosenthal [4, Theorem III.1]. Appropriate normalization establishes the claim.

Now assume that $\Sigma t_n f_n$ converges w^* in Y^*. Then since $\Sigma t_n f_n$ is sectionally bounded,

$$\begin{aligned} |t_n| &= \|t_n f_n\|_{Y^*} \\ &\leq \left\| \sum_{k=1}^{n} t_k f_k \right\|_{Y^*} + \left\| \sum_{k=1}^{n-1} t_k f_k \right\|_{Y^*}, \end{aligned}$$

so $t \in l_\infty$. Therefore, $\Sigma t_n f_n \in W \cap Y^* \subset U$, so U contains the w^*-closed span of $\{f_n\}$. The latter is clearly infinite dimensional.

(i) $\Rightarrow$ (ii). Suppose that S is w^*-closed in Y^*, $S \subset U$, and $\dim S = \infty$. There exists $\{f_n\} \subset S$ such that $\|f_n\| = 1$ and $f_n(y_i) = 0$ for $i < n$. Let $\{h_n\}$ be a subsequence of $\{f_n\}$ guaranteed by 4.2.

Now Y^* may be considered a BK space by identifying $f \in Y^*$ with the scalar sequence $\{f(y_i)\}$. Let W be the space of all sequences $w = \{w_i\}$ of the form

$$w_i = \sum_k k t_k h_k(y_i)$$

where $t \in l_\infty$. Then W is a BK space under an obvious identification as a quotient of l_∞. For each k let $v_k = \{h_k(y_i)\}$. In W, $\|v_k\| \leq 1/k$, so $W \cap Y^*$ is not closed in W.

Finally, suppose that $w \in W \cap Y^*$ where $w_i = \Sigma_k k t_k h_k(y_i)$ with $t \in l_\infty$. According to 4.2, $\Sigma_{k=1}^n k t_k h_k$ is bounded in Y^*, hence w^*-convergent. Therefore, $w \in S \subset U$, so $W \cap Y^* \subset U$. It follows that $U \to Y^*$ is not MKZ.

The implication (iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Assume that $S \subset U \subset Y^*$, S is closed in Y^*, and $\dim S = \infty$. There exists $\{f_n\} \subset S$ such that $\|f_n\|_{Y^*} = 1$ and $f_n \to 0$ w^* in Y^*. Again using [4, Theorem III.1] one may assume that $\{f_n\}$ is w^*-basic in Y^*.

Suppose that $\Sigma t_n f_n$ is w^*-convergent in Y^*. Then $\{\Sigma_{k=1}^n t_k f_k\}$ is bounded in Y^*, hence in S, hence in U. Assume that $f \in U$ is a w^*-cluster point in U. Then f is a w^*-cluster point in Y^*, so $\Sigma_{k=1}^n t_k f_k \to f$ w^* in Y^*. Therefore, U contains the w^*-closed span of $\{f_n\}$ in Y^*. ∎

4.4 COROLLARY. (i) (*Meyer-König and Zeller*, [5] *and* [6]) *If W is a BK space with $W \cap l_\infty \subset c_0$, then $W \cap l_\infty$ is closed in W.*

(ii) (*Devos* [3]) *For $1 \leq p < q \leq \infty$, if W is a BK space with $W \cap l_q \subset l_p$, then $W \cap l_q$ is closed in W.*

Proof. As observed in the proof of 3.3, c_0 contains no infinite dimensional w^*-closed subspace of l_∞, so (i) is established. The injection of l_p into l_q is actually strictly singular, so (ii) follows. ∎

One should note that the Meyer-König and Zeller theorem for instance holds in the context of *FK* spaces. According to [10, Lemma 4], 4.4 and 4.6 below extend easily to the *FK* space setting.

Further applications are obtained by the following variation on part of 4.3.

4.5 THEOREM. *Let Y be a separable Banach space and let U be a subspace of Y^*. If U contains no infinite dimensional closed subspace of Y^*, then the injection of U into the closure of U in Y^* has MKZ.*

Proof. As in the first part of the proof of 4.3, one may arrange $W \cap \bar{U} \subset U$ and $W \cap \bar{U}$ not closed in W. The same sequence $\{f_n\}$ is basic in Y^*. If $\Sigma t_n f_n$ converges in Y^*, then $\Sigma t_n f_n \in W \cap \bar{U} \subset U$, so U contains $[f_n]$. The latter is infinite dimensional. ■

Note that "closed" may not be replaced by "w^*-closed" in 4.5. For instance, c_0 contains no infinite dimensional w^*-closed subspace of l_∞, but $c_0 \to c_0$ is surely not MKZ.

4.6 COROLLARY (Bennett [1]). (i) *If W is a BK space with $W \cap cs \subset l_1$, then $W \cap cs$ is closed in W.*

(ii) *If W is a BK space with $W \cap c_0 \subset bv_0$, then $W \cap c_0$ is closed in W.*

(iii) *If W is a BK space with $W \cap c_0 \subset \bigcup_{p \geq 1} l_p$, then $W \cap c_0$ is closed in W.*

Proof. The injection of l_1 into $bs(= bv_0^*)$ is strictly singular because l_1 and cs have no isomorphic closed infinite dimensional subspaces. Thus, $l_1 \to \bar{l}_1(= cs)$ has MKZ. Part (ii) follows similarly.

To prove (iii) it suffices to show that $\bigcup_{p \geq 1} l_p$ contains no infinite dimensional closed subspace of l_∞. Suppose $S \subset \bigcup_{n=1}^{\infty} l_n$ and S is closed in l_∞. Now $S = \bigcup_{n=1}^{\infty} l_n \cap S$, so $S = l_n \cap S$ for some n, i.e. $S \subset l_n$. But S is closed in c_0, hence in l_n, and c_0 and l_n have no isomorphic infinite dimensional closed subspaces. Thus, $\dim S < \infty$. ■

Note that it is possible but awkward to consider an MKZ property which depends on the particular universe H. For instance, let $Z = l_\infty$. If W is a BZ space, then $W \subset Z$, so certainly $W \cap Z$ is closed in W. Thus, $Z \to Z$ has an MKZ property relative to $H = Z$ but certainly not relative to $H = \omega$. In the context of 4.3 one can show without difficulty that the MKZ property with respect to a universe H implies the MKZ property, if no infinite dimensional w^*-closed subspace of Y^* is closed in H.

Finally, the known duality between strictly cosingular and w^* strictly singular operators yields the following. Theorem 2 in [10] is a very special case.

4.7 Theorem. *Let T be a map from the separable Banach space Y into Z with dense range. The following conditions are equivalent*:

(i) *T is strictly cosingular.*

(ii) *$TY < Z$.*

(iii) *T^* is w^* strictly singular.*

(iv) *$T^*Z^* \to Y^*$ has MKZ.*

(v) *T^* is strictly singular.*

Proof. The equivalence of (i) and (ii) is 3.2, and (i) is known to be equivalent to (iii). Since T^* is an injection, the equivalence of (iii), (iv), and (v) follows from 4.3. ■

The assumption that T has dense range (and hence that Z is separable) is required for the implication (iii) $\to$ (iv). For instance, let T be the injection of c_0 into l_∞. Then T is strictly cosingular but T^* is a surjection. A counterexample with Z separable may be found in Orlicz sequence spaces.

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