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THE COMPARABILITY OF THE KOBAYASHI APPROACH REGION AND THE ADMISSIBLE APPROACH REGION

BY

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1. Introduction

Given a domain $\Omega \subseteq \mathbf{C}^n$, we denote by $F_K^\Omega(z, \xi)$ the infinitesimal form of the Kobayashi metric for Ω at z in the direction of the vector ξ. In [1] we have estimated the boundary behavior of the metric when ξ is fixed and z is allowed to approach a strongly pseudoconvex point P in the boundary of Ω. As a consequence of the work done in [1] we obtained the following estimate:

$$(*) \qquad F_K^\Omega(z, \xi) \approx c\frac{|\xi_{N_P}|}{\delta_\Omega(z)} + c\frac{|\xi_{T_P}|}{\sqrt{\delta_\Omega(z)}} \quad \text{for all } z \in U \cap \Omega$$

where U is a neighborhood of P where the eigenvalues of the Levi form at P are bounded from zero, and for any $\xi \in \mathbf{C}^n$, ξ_{N_P} is the complex normal component of ξ at P and ξ_{T_P} is the complex tangential component of ξ at P, and $\delta_\Omega(z)$ is the distance from z to the boundary.

N. Siboney in [10] has proven the inequality

$$F_K^\Omega(z, \xi) \geq c\frac{|\xi_{N_P}|}{\delta_\Omega(z)} + c\frac{|\xi_{T_P}|}{\sqrt{\delta_\Omega(z)}}$$

for the Kobayashi metric on strongly pseudoconvex (and other) domains, but it is not the precise asymptotic formula which is found in [1].

By means of the estimate $(*)$, it is possible to solve the following problem:

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[1]The work in this paper is contained in the author's Ph.D. thesis at the Pennsylvania State University, directed by Professor Steven G. Krantz.

Let $\Omega \subseteq \mathbf{C}^n$ be a pseudoconvex domain and $P \in \partial\Omega$ be strongly pseudoconvex point. For $\alpha > 1$, define the admissible approach region of Stein to be

$$\mathfrak{A}_\alpha(P) = \{z \in \Omega : |z - P|^2 < \alpha\delta_P(z); |\langle z - P, \nu_P\rangle| < \alpha\delta_P(z)\}$$

where $\delta_P(z) = \min\{\delta_\Omega(z); \operatorname{dist}(z, T_P(\partial\Omega))\}$ and $T_P(\partial\Omega)$ is the tangent space to $\partial\Omega$ at P.

Also, define the Kobayashi approach region to be

$$\mathscr{K}_\beta(P) = \{z \in \Omega : K_\Omega(z, -\nu_P) < \beta\} \quad \text{with } \beta > 0$$

where K_Ω represents the Kobayashi distance from z to $-\nu_P$.

Then our main result is:

THEOREM 1. *Under the above conditions, given $\alpha > 1$ there are two constants $B = B(\alpha)$ and $C = C(\alpha)$ which depend on Ω and the eigenvalues of the Levi form at P, and are functions of α, and there exists an open neighborhood U of P such that*

$$U \cap \mathscr{K}_{B(\alpha)}P \subseteq U \cap \mathfrak{A}_\alpha(P) \subseteq U \cap \mathscr{K}_{C(\alpha)}(P).$$

While our result is local, in the case that Ω is strongly pseudoconvex domain then $B(\alpha)$ and $C(\alpha)$ are uniform constants for all $P \in \partial\Omega$.

The theorem allows us give an invariant form of Fatou's Theorem [11].

By Fefferman's Theorem [6], biholomorphic maps of smooth strongly pseudoconvex domain extend smoothly, hence in particular C^1, to the boundary. Theorem 1 then yields immediately that Kobayashi approach regions are a biholomorphically invariant concept, hence so are admissible approach regions. An invariant metric approach to boundary behavior of holomorphic functions is explored in great detail in [7].

In the second part of this paper we want to discuss the following problem:

Given a pseudoconvex domain Ω of finite type in $\mathbf{C}^n$, Nagel, Stein and Wainger [8] introduced a family of balls on the boundary of Ω which is intimately linked to the complex geometry of Ω with respect to $\mathbf{C}^n$. They define approach regions in terms of these balls. The approach regions are denoted by $\mathscr{A}_\sigma$. By means of some estimates obtained by Catlin [3] for the Kobayashi metric on domains of finite type in $\mathbf{C}^2$, it is possible to show that the approach regions $\mathscr{A}_\sigma$ are comparable to Kobayashi approach regions $\mathscr{K}_\beta$.

Again we get an invariant form of Fatou's Theorem for pseudoconvex domains of finite type in $\mathbf{C}^2$.

I would like to thank Steven G. Krantz for all his help and good advice.

1. Notations and definitions

DEFINITION 1.1. If $e_1 = (1 + 0i, 0, \ldots, 0)$ then the infinitesimal form of the *Kobayashi metric* for Ω at z in the direction of ξ is

$$F_K^{\Omega}(z, \xi) = \inf\left\{ \frac{|\xi|}{|(f_*(0))(e_1)|} : f\colon B \to \Omega \text{ is holomorphic, } f(0) = z, \right.$$
$$\left. \text{and } (f_*(0))(e_1) \text{ is a constant multiple of } \xi \right\}.$$

DEFINITION 1.2. The *Kobayashi distance* between the points z, $w \in \Omega$ can be defined as

$$K_\Omega(z, w) = \inf_\gamma \int_0^1 F_K^\Omega(\gamma(t), \gamma'(t))\, dt$$

where the infimum is taken over C^1 curves $\gamma\colon [0,1] \to \Omega$ such that $\gamma(0) = z$ and $\gamma(1) = w$.

Remark 1.3. Royden [9] has shown that the infimum can be taken over all piece-wise differentiable curves.

For details about the metric and pseudoconvex domains see [6].

The following theorem has been proven in [1] and is a basic tool for our future calculation.

THEOREM 1.4. *Let $\Omega \subset\subset \mathbf{C}^n$ be a pseudoconvex domain with C^{n+1} boundary. Suppose $P \in \partial\Omega$ is a strongly pseudoconvex point and W is a neighborhood of P on which the eigenvalues of Levi form are bounded from zero by some number $\varepsilon > 0$. Let us assume without lost of generality that z_1 is the normal complex direction at P. Let ρ be a defining function for Ω such that $|\nabla_Z \rho(w)| = 1$ for all $w \in \partial\Omega$. Let Q be a unitary operator which diagonalized the Levi form at P, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the Levi form at P where the corresponding eigenvectors have respectively the directions $z_1, \ldots, z_n$. Let $z \in \Omega$ and S be the projection of z into $\partial\Omega$. Given $\xi \in \mathbf{C}^n$, let ξ_{N_S} be the complex normal component of ξ at S and ξ_{T_S} the complex tangential component of ξ at S. Define*

$$\eta(z) = \sqrt{2}\,\delta_\Omega(z)\xi_{N_S} + \sqrt{\delta_\Omega(z)}\,H(Q\xi_{T_S})$$

where H is diagonal matrix with entries $\lambda_i^{-1/2}$. *Then*

$$\lim_{\Omega \ni z \to P} F_K^{\Omega}(z, \eta(z)) = \frac{1}{\sqrt{2}} |\xi|.$$

As a consequence of the theorem we obtain the estimate

$$F_K^{\Omega}(z, \xi) \approx c \frac{|\xi_{N_P}|}{\delta_\Omega(z)} + c \frac{|\xi_{T_P}|}{\sqrt{\delta_\Omega(z)}} \quad \text{for all } z \in U \cap \Omega$$

where P is a strongly pseudoconvex point in the boundary of a pseudoconvex domain Ω, U is a neighborhood of P where the eigenvalues of the Levi form at P are bounded from zero and for any $\xi \in \mathbf{C}^n$, ξ_{N_P} is the complex normal component of ξ at P and ξ_{T_P} is the complex tangential component of ξ at P.

DEFINITION 1.5. If $\Omega \subset\subset \mathbf{C}^n$ with C^2 boundary, $P \in \partial\Omega$, $z \in \Omega$, define

$$\delta_P(z) = \min\{\delta_\Omega(z), \operatorname{dist}(z, T_P(\partial\Omega))\}.$$

Then, for $\alpha > 1$, let the *admissible approach region* at P with aperture α be

$$\mathfrak{A}_\alpha(P) = \{z \in \Omega\colon |z - P|^2 < \alpha\delta_P(z);\ |\langle z - P, \nu_P\rangle| < \alpha\delta_P(z)\}.$$

$\mathfrak{A}_\alpha(P)$ is like a cone in the complex normal direction and like a paraboloid in the tangential direction. Notice that if Ω is convex, then $\delta_\Omega(z) = \delta_P(z)$. But $\delta_P(z)$ is used because near non-convex boundary points we still want $\mathfrak{A}_\alpha(P)$ to have the same shape.

DEFINITION 1.6. Let $\Omega \subset\subset \mathbf{C}^n$ with C^2 boundary, $P \in \partial\Omega$ and $\beta > 1$. The $\mathscr{K}$-admissible (for Kobayashi admissible) approach region of aperture β at P is

$$\mathscr{K}_\beta(P) = \{z \in \Omega\colon K_\Omega(z, -\nu_P) < \beta\}$$

where ν_P denotes the unit outward normal and

$$K_\Omega(z, -\nu_P) = \inf\{K_\Omega(z, w)\colon w \in -\nu_P\}.$$

2. Proof of Theorem 1

Through our work we will use the symbol c to denote constants whose values change from line to line, but independent of the relevant parameter.

THEOREM 1. *Let $\Omega \subset\subset \mathbf{C}^n$ be a pseudoconvex domain with C^{n+1} boundary. Let $P \in \partial\Omega$ be a strongly pseudoconvex point. Then, given $\alpha > 1$ there are two constants $B = B(\alpha)$ and $C = C(\alpha)$ which depend on Ω and the eigenvalues of the Levi form at P and are functions of α, and there exists an open neighborhood U of P such that*

$$U \cap \mathscr{K}_{B(\alpha)}(P) \subseteq U \cap \mathfrak{A}_\alpha(P) \subseteq U \cap \mathscr{K}_{C(\alpha)}(P).$$

Proof. Let U be a neighborhood of P such that

$$F_K^\Omega(z,\xi) \approx c\frac{|\xi_{N_P}|}{\delta_\Omega(z)} + c\frac{|\xi_{T_P}|}{\sqrt{\delta_\Omega(z)}} \quad \text{for all } z \in U \cap \Omega.$$

Part 1. Assume $z \in U \cap \mathfrak{A}_\alpha(P)$, we want to prove that $z \in U \cap \mathscr{K}_{C(\alpha)}(P)$. If $z \in U \cap \mathfrak{A}_\alpha(P)$ then

$$|(z-P)_{N_P}| < \alpha\delta_\Omega(z),$$

where $(z-P)_{N_P}$ is the projection of $(z-P)$ into $\mathscr{N}_P$ (the complex normal space to $\partial\Omega$ at P) and

$$|(z-P)_{T_P}| < \sqrt{\alpha\delta_\Omega(z)},$$

where $(z-P)_{T_P}$ is the projection of $(z-P)$ into $\mathscr{T}_P$ (the complex tangent space to $\partial\Omega$ at P). Let z^* be the projection of z into $\mathscr{N}_P$ and z' the projection of z into $-\nu_P$.

We have three possibilities:

(i) zz' is in $\mathscr{T}_P$;
(ii) zz' is in $\mathscr{N}_P$;
(iii) neither of the above.

Case (i). Here we have $|(z-P)_{T_P}| = |z-z'|$. Consider the curve $\delta_1(t) = (1-t)z + tz'$, $0 \le t \le 1$. Then

$$L_K^\Omega(\gamma_1) = \int_0^1 F_K^\Omega(\gamma_1(t);\gamma_1'(t))\,dt$$

where $\gamma_1'(t) = (\gamma_1'(t))_{T_P} = z - z' \in \mathscr{T}_P$. Since $\gamma_1(t) \in U$ it turns out that

$$F_K^\Omega(\gamma_1(t);\gamma_1'(t))\,dt \approx \frac{c|(\gamma_1')_{T_P}(t)|}{\sqrt{\delta_\Omega(\gamma_1(t))}} = \frac{c|z-z'|}{\sqrt{\gamma_\Omega(\gamma_1(t))}},$$

but $\sqrt{\alpha\delta_\Omega(z)} > |(z-P)_{T_P}| \geq |z-z'|$ and $\delta_\Omega(z) \leq \delta_\Omega(\gamma_1(t))$ so

$$F_K^\Omega(\gamma_1(t);\gamma_1'(t)) \leq \frac{c|z-z'|}{\sqrt{\delta_\Omega(z)}} \leq \frac{c\sqrt{\alpha\delta_\Omega(z)}}{\sqrt{\delta_\Omega(z)}} = c\sqrt{\alpha}\,.$$

Hence

$$K_\Omega(z,-\nu_P) \leq K_\Omega(z,z') \leq L_K^\Omega(\gamma_1(t)) \leq \int_0^1 c\sqrt{\alpha}\,dt = c\sqrt{\alpha}\,.$$

Case (ii) We have $zz' \in \mathcal{N}_P$. Hence

$$|z-z'| = |(z-z')_{N_P}| < \alpha\delta_\Omega(z).$$

Consider the curve $\gamma_2(t) = (1-t)z + tz'$, $0 \leq t \leq 1$. Then

$$L_K^\Omega(\gamma_2) = \int_0^1 F_K^\Omega(\gamma_2(t);\gamma_2'(t))\,dt$$

and

$$\gamma_2'(t) = (\gamma_2')_{N_P}(t) = z - z' \in \mathcal{N}_P.$$

Since $\gamma_2(t) \in U$ we have

$$F_K^\Omega(\gamma_2(t);\gamma_2'(t))\,dt \approx \frac{c|(\gamma_2')_{N_P}(t)|}{\delta_\Omega(\gamma_2(t))} \approx \frac{c|z-z'|}{\delta_\Omega(\gamma_2(t))}.$$

Then

$$F_K^\Omega(\gamma_2(t);\gamma_2'(t)) \leq \frac{c\alpha\delta_\Omega(z)}{\delta_\Omega(z)} = c\alpha,$$

so

$$K_\Omega(z,-\nu_P) \leq K_\Omega(z,z') \leq L^\Omega(\gamma_2(t)) \leq \int_0^1 c\alpha\,dt = c\alpha.$$

Case (iii) $zz' = (zz')_{T_P} + (zz')_{N_P} = zz^* + zz'$. Consider the curve

$$\gamma_3(t) = \begin{cases} \gamma_1(t), & 0 \leq t \leq t_0, \\ \gamma_2(t), & t_0 \leq t \leq 1, \end{cases}$$

where $\gamma_1(t)$ is the segment connecting z with z^* and $\gamma_2(t)$ is the segment connecting z^* with z'.

Since γ_3 is a piece-wise differentiable curve joining z to z', according to Remark 1.3, we have

$$K_\Omega(z, -\nu_P) \le K_\Omega(z, z') \le L_K^\Omega(\gamma_3)$$

But

$$L_K^\Omega(\gamma_3) = \int_0^1 F_K^\Omega(\gamma_1(t); \gamma_1'(t))\, dt + \int_{t_0}^1 F_K^\Omega(\gamma_2(t); \gamma_2'(t))\, dt,$$

so by the previous two cases we have

$$L_K^\Omega(\gamma_3) \le c\sqrt{\alpha} + c\alpha = C(\alpha).$$

Therefore

$$K_\Omega(z, -\nu_P) \le C(\alpha).$$

Part 2. Assume $z \in U \cap \mathcal{K}_{B(\alpha)}(P)$, let us prove $z \in U \cap \mathfrak{A}_\alpha(P)$.

Let us prove the contrapositive. Take α very large. Suppose $z \notin \mathfrak{A}_\alpha(P)$; we want to show that $z \notin \mathcal{K}_{B(\alpha)}(P)$. We need to prove $K^\Omega(z, -\nu_P) \ge B(\alpha)$.

Let γ be a curve parametrized with respect to Euclidean arc length which connects z to $-\nu_P$, and let t_0 be the Euclidean length of γ. Fix two constants $D(\alpha) > 0$ and $M(\alpha) > 0$ such that $D(\alpha)$ is a small number and $M(\alpha)$ is a large number, to be selected. We have three possibilities:

(i) $\delta_\Omega(\gamma(t)) < D(\alpha)\delta_\Omega(z')$ for some t;
(ii) $\delta_\Omega(\gamma(t)) > M(\alpha)\delta_\Omega(z')$ for some t;
(iii) $D(\alpha)\delta_\Omega(z') \le \delta_\Omega(\gamma(t)) \le M(\alpha)\delta_\Omega(z')$ for all t.

Case (i) We have

$$L_K^\Omega(\gamma) \approx c\int_0^{t_0} \frac{|\gamma_{N_P}'(t)|}{\delta_\Omega(\gamma(t))}\, dt + c\int_0^{t_0} \frac{|\gamma_{T_P}'(t)|}{\sqrt{\delta_\Omega(\gamma(t))}}\, dt.$$

Define the curve

$$\mu(t) = z' + \int_0^t \gamma_{\hat{N}_P}'(s)\, ds$$

where $\gamma_{\hat{N}_P}'$ is the projection of $\gamma_{N_P}'(s)$ onto the real normal at P. We have

$$\mu'(t) = \gamma_{\hat{N}_P}'(t) \quad \text{for all } t.$$

Let $\gamma(t_1)$, $t_1 \in [0, t_0]$, be such that $\delta_\Omega(\gamma(t_1)) < D(\alpha)\delta_\Omega(z')$ and let w be the projection of $\gamma(t_1)$ into the real normal. Now

$$L_K^\Omega(\gamma) \approx c\int_0^{t_0} \frac{|\mu'(t)|}{\delta_\Omega(\mu(t))}\, dt \ge c\int_0^{t_0} \frac{|\hat{\mu}'(t)|}{\delta_\Omega(\hat{\mu}(t))}\, dt.$$

where the curve $\hat{\mu}$ is gotten from μ by discarding overlaps. Then

$$\begin{aligned} L_K^\Omega(\gamma) &\geq c\int_0^{t_1} \frac{|\hat{\mu}'(t)|}{\delta_\Omega(\hat{\mu}(t))} \\ &\geq cL_K^\Omega(\text{segment connecting } z' \text{ to } w) \\ &\geq c\int_0^{\delta_\Omega(z')-\delta_\Omega(w)} \frac{dt}{\delta_\Omega(w)+1} \geq c\ln\{\delta_\Omega(w)+t\}\Big|_0^{\delta_\Omega(z')-\delta_\Omega(w)} \\ &\geq c\ln\frac{\delta_\Omega(z')}{\delta_\Omega(w)}. \end{aligned}$$

But $\delta_\Omega(w) \leq D(\alpha)\delta_\Omega(z')$, hence

$$L_K^\Omega(\delta) \geq c\ln\frac{\delta_\Omega(z')}{D(\alpha)\delta_\Omega(z')} \geq c\ln\frac{1}{D(\alpha)}.$$

Case (ii) Again

$$L_K^\Omega(\gamma) \geq c\int_0^{t_0} \frac{|\gamma'_{N_P}(t)|}{\delta_\Omega(\gamma(t))}\,dt.$$

As in case (i) we define the curve

$$\mu(t) = z' + \int_0^t \gamma'_{N_P}(s)\,ds.$$

Following the same argument as above, we get

$$L_K^\Omega(\gamma) \geq cL_K^\Omega\ (\text{segment connecting } z' \text{ to } w)$$

where w is the projection of $\gamma(t_2)$, $t_2 \in [0; t_0]$ onto the real normal and

$$\delta_\Omega(\gamma(t_2)) > M(\alpha)\delta_\Omega(z').$$

Then

$$\begin{aligned} L^\Omega(\gamma) &\geq c\int_0^{\delta_\Omega(w)-\delta_\Omega(z')} \frac{dt}{\delta_\Omega(z')+t} \geq c\ln\{\delta_\Omega(w)+t\}\Big|_0^{\delta_\Omega(w)-\delta_\Omega(z')} \\ &= c\ln\frac{\delta_\Omega(w)}{\delta_\Omega(z')} \geq c\ln\frac{M(\alpha)\delta_\Omega(z')}{\delta_\Omega(z')} = c\ln M(\alpha) \end{aligned}$$

Case (iii) We have to divide this case into two subcases:
(a) $|z - P| \geq \sqrt{\alpha\delta_\Omega(z)}$;
(b) $|(z - P)_{N_P}| \geq \alpha\delta_\Omega(z)$.

Case (iii.a) We claim that if $|z - P| \geq \sqrt{\alpha\delta_\Omega(z)}$ then

$$|z - z^*| \geq T(\alpha)\sqrt{\delta_\Omega(z')}.$$

Since

$$|z - P|^2 = |z - z^*|^2 + |z^* - P|^2$$

we have

$$|z - z^*| \geq \sqrt{\alpha\delta_\Omega(z)} - |z^* - P|$$

but

$$|z^* - P| \leq k\sqrt{\delta_\Omega(z')}, \qquad 0 < k < 1 \text{ and } \delta_\Omega(z') \approx \delta_\Omega(z)$$

so

$$|z - z^*| \geq \sqrt{\alpha c\delta_\Omega(z')} - k\sqrt{\delta_\Omega(z')} = T(\alpha)\sqrt{\delta_\Omega(z')}$$

and $T(\alpha) > 0$ since we assume α very large.

Now

$$L_K^\Omega(\gamma) \geq c\int_0^{t_0} \frac{|\gamma'_{T_P}(t)|}{\sqrt{\delta_\Omega(\gamma(t))}}\, dt.$$

We define the curve

$$\mu_2(t) = z + \int_0^t \gamma'_{T_P}(s)\, ds.$$

We have $\mu_2'(t) = \gamma'_{T_P}(t)$ for all t. Then

$$\begin{aligned} L_K^\Omega(\gamma) &\geq c\int_0^{t_0} \frac{|\mu_2'(t)|}{\sqrt{\delta_\Omega(\mu_2(t))}}\, dt \geq cL^\Omega(\mu_2) \\ &\geq \frac{c|z - z^*|}{\sqrt{D(\alpha)\delta_\Omega(z')}} \geq \frac{cT(\alpha)\sqrt{\delta_\Omega(z')}}{\sqrt{D(\alpha)\delta_\Omega(z')}} = \frac{cT(\alpha)}{\sqrt{D(\alpha)}}. \end{aligned}$$

Case (iii.b) We claim that if $|(z - P)_{N_P}| > \alpha\delta_\Omega(z)$ then

$$|z^* - z'| > S(\alpha)\delta_\Omega(z).$$

We have $|(z - P)_{N_P}| = |z^* - P|$ and $|z^* - P|^2 = |z^* - z'|^2 + |z' - P|^2$ so

$$|z^* - z'| > \alpha\delta_\Omega(z) - |z' - P|.$$

But

$$|z' - P| = \delta_\Omega(z') \quad \text{and} \quad \delta_\Omega(z') \approx \delta_\Omega(z)$$

so

$$|z^* - z'| > \alpha c \delta_\Omega(z') - \delta_\Omega(z') = S(\alpha)\delta_\Omega(z).$$

But since we assume α very large then $S(\alpha) > 0$.

Now

$$L_K^\Omega(\gamma) \geq c\int_0^{t_0} \frac{|\gamma'_{N_P}(t)|}{\delta_\Omega(\gamma(t))}\,dt$$

$$\approx c\int_0^{t_0} \frac{|\gamma'_{N_P}(t)|}{D(\alpha)\delta_\Omega(z')}\,dt \geq \frac{c|z^* - z'|}{D(\alpha)\delta_\Omega(z')} \geq \frac{cS(\alpha)\delta_\Omega(z')}{D(\alpha)\delta_\Omega(z')} = \frac{cS(\alpha)}{D(\alpha)}.$$

Then

$$B(\alpha) = \sup\left\{ c \ln M(\alpha);\ c \ln \frac{1}{D(\alpha)};\ \frac{cT(\alpha)}{\sqrt{D(\alpha)}};\ \frac{cS(\alpha)}{D(\alpha)} \right\}$$

Then we have proved that if $z \notin \mathfrak{A}_\alpha(P)$ then $K(z, \nu_P) > B(\alpha)$, as desired. □

Remark 2.1. The constants $B(\alpha)$ and $C(\alpha)$ hold for all $P' \in \partial\Omega$ which are sufficiently near to P.

Remark 2.2. An alternative way to prove Theorem 1 would be the following:

If Ω is the unit ball in $\mathbf{C}^n$, then we can exploit the transitivity of the automorphism group to get a quick proof of the result. Now if Ω is strongly pseudoconvex domain then we can use the approximation ideas of Graham [3a] to get the full result.

3. Fatou's theorem on strongly pseudoconvex domains

As an application of Theorem 1 we can give a new statement of Fatou's theorem on strongly pseudoconvex domains in $\mathbf{C}^n$.

Let us recall the classical Fatou's theorem.

DEFINITION 3.1. Let $\Omega \subset\subset \mathbf{C}^n$ be a domain with defining function ρ. Let $\Omega_\varepsilon = \{z \in \mathbf{C}^n: \rho(z) < -\varepsilon\}$. For $0 < p < \infty$ we set

$$H^p(\Omega) = \left\{ f \text{ holomorphic in } \Omega: \sup_{\varepsilon>0} \int_{\partial\Omega_\varepsilon} |f(z)|^p \, d\mu_\varepsilon = \|f\|_{H^p}^p < \infty \right\}$$

$$H^\infty(\Omega) = \left\{ f \text{ holomorphic in } \Omega: \sup_{z\in\Omega} |f| < \infty \right\}$$

where $d\mu_\varepsilon$ is the area measure on $\partial\Omega_\varepsilon$.

We also define the *Nevalinna class* $N(\Omega)$ by

$$N(\Omega) = \left\{ f \text{ holomorphic in } \Omega: \sup_{\varepsilon>0} \int_{\partial\Omega_\varepsilon} \log^+ |f(z)| \, d\mu_\varepsilon < \infty \right\}$$

where $\log^+ u = \max\{0, \log u\}$.

FATOU'S THEOREM. *Let $0 < p \leq \infty$. Let $\alpha > 1$. If $\Omega \subset\subset \mathbf{C}^n$ has C^2 boundary and $f \in H^p(\Omega)$, then for almost every $P \in \partial\Omega$,*

$$\lim_{\mathfrak{A}_\alpha(P) \ni z \to P} f(z)$$

exists.

For details see [11].

The results in the unit ball $B \subseteq \mathbf{C}^n$ and on certain other classical domains were obtained by Koranyi [5] and all the principal ideas for arbitrary bounded domains in $\mathbf{C}^n$ with C^2 boundary are due to Stein [11].

Now, given $\Omega \subset\subset \mathbf{C}^n$ a strongly pseudoconvex domain and using the fact that for all $P \in \partial\Omega$, $\mathfrak{A}_\alpha(P) \approx \mathcal{K}_\beta(P)$, we obtain the following invariant form of Fatou's theorem:

THEOREM 3.1. *Let $\Omega \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain with C^2 boundary. Let $0 < p \leq \infty$ and $\beta > 1$. Let $f \in H^p(\Omega)$. Then for almost every $P \in \partial\Omega$,*

$$\lim_{\mathcal{K}_\beta(P) \ni z \to P} f(z)$$

exists.

Also, Stein [11] got the analogue of Fatou's theorem for the Nevanlinna class. Therefore we have the following result:

THEOREM 3.2. *Let* $\Omega \subset\subset \mathbf{C}^n$ *be a strongly pseudoconvex domain with* C^2 *boundary. Let* $0 < p \leq \infty$ *and* $\beta > 1$. *Let* $f \in N(\Omega)$. *Then for almost every* $P \in \partial\Omega$,

$$\lim_{\mathcal{K}_\beta(P) \ni z \to P} f(z)$$

exists.

4. Pseudoconvex domains of finite type in $\mathbf{C}^2$

DEFINITION 4.1. Let Ω be a smoothly bounded domain in $\mathbf{C}^2$ with defining function ρ. We define two holomorphic vector fields T_1 and T_2 by

$$T_1 = \frac{\partial \rho}{\partial z_2} \frac{\partial}{\partial z_1} - \frac{\partial \rho}{\partial z_1} \frac{\partial}{\partial z_2}$$

and

$$T_2 = \frac{\partial \rho}{\partial \bar{z}_1} \frac{\partial}{\partial z_1} + \frac{\partial \rho}{\partial \bar{z}_2} \frac{\partial}{\partial z_2}$$

Thus the vector fields T_1 and T_2 are respectively tangent and transverse to the boundary of Ω. For all $z \in \Omega$, we define the *Levi function* $\lambda(z)$ by

$$\lambda(z) = \langle \partial \rho; [T_1, \overline{T}_1] \rangle (z)$$

where $[T_1, \overline{T}_1] = T_1\overline{T}_1 - \overline{T}_1 T_1$ (Lie bracket).

Let $\mathcal{L}_0$ be the module spanned by T_1 and $\overline{T}_1$ over the C^∞ functions and let $\mathcal{L}_{k+1}$ be the module spanned by elements of $\mathcal{L}_k$ and elements of the form $[F, T_1]$ or $[F, \overline{T}_1]$ with $F \in \mathcal{L}_k$.

DEFINITION 4.2. A point $P \in \partial\Omega$ is said to be of *type* m $(m \geq 1)$ if

$$\langle \partial \rho(P), F(P) \rangle = 0 \quad \text{for all } F \in \mathcal{L}_{m-1}$$

while

$$\langle \partial \rho(P), F(P) \rangle \neq 0 \quad \text{for some } F \in \mathcal{L}_m$$

DEFINITION 4.2. If $\Omega \subset\subset \mathbf{C}^2$ is a pseudoconvex domain and $P \in \partial\Omega$ is of type m, then we say that $\partial\Omega$ is *pseudoconvex of type* m at P.

Remark 4.3. If $P \in \partial\Omega$ is strongly pseudoconvex point, then P is of type 1.

DEFINITION 4.4. Let $(i_0, i_1, \dots, i_m)$ be an $(m+1)$-tuple of zeros and ones; we define the vector field $T_1^{(i_0,\dots,i_m)}$ inductively by

$$T_1^{(0)} = T_1, \quad T_1^{(1)} = \overline{T}_1$$

and

$$T_1^{(i_0,\dots,i_m)} = \left[T_1, T_1^{(i_0,\dots,i_{m-1})}\right].$$

Then

$$\lambda^{(i_0,\dots,i_m)}(P) = \langle \partial\rho; T_1^{(i_0,\dots,i_m)}\rangle(P)$$

Remark 4.5. It can be proved, see [4], that the type of a given point P must be an odd integer if the boundary of Ω is pseudoconvex near P.

Remark 4.6. Let us define the function $C(z) = (T_1\overline{T}_1)^{k-1}\lambda(z)$. It is possible to show that when Ω is pseudoconvex near a point P in the boundary, then P is of type $2m-1$ if and only if $C_m(z) \neq 0$ and $C_k(z) = 0$ for all k, $1 \leq k < m$; for details see [4].

DEFINITION 4.7. Let $X = a_1T_1 + a_2T_2$ be a tangent vector of type $(1,0)$ at a point z in Ω. Define $M(z; X)$ by

$$M_m(z; X) = |a_2|\,|\rho(z)|^{-1} + |a_1| \sum_{k=1}^{m} |C_k(z)|^{1/2k}|\rho(z)|^{-1/2k}$$

Now we can state a theorem due to Catlin which allows us to estimate the Kobayashi metric; for details see [3].

THEOREM 4.8. *Let Ω be smoothly bounded domain in $\mathbf{C}^2$. Let P be a given point in the boundary of Ω; assume that P is of type $2m-1$. Then there exist a neighborhood U about P and positive constants c and C such that for every tangent vector $X = a_1T_1 + a_2T_2$ at a point $z \in U \cap \Omega$,*

$$cM_m(z; X) \leq F_K^{\Omega}(z; K) \leq CM_m(z; X).$$

Following the ideas introduced by Nagel, Stein and Wainger [10], we can define balls on the boundary of a smoothly bounded domain in $\mathbf{C}^2$.

First, we consider a domain $\Omega \subseteq \mathbf{R}^4$ with smooth boundary and finite type m. Let U be a neighborhood in $\partial\Omega$. Let X_1 and X_2 be smooth real vector fields defined in U and T be a non-vanishing transverse vector field in U, so that X_1, X_2 and T span the tangent space of each point of U.

Following Kohn [4], we define

$$\Lambda_k(x) = \left(\sum \{\lambda^{i_0 \cdots i_n}(x)\}^2\right)^{1/2}$$

where the sum is over the set of generators of $\mathcal{Y}_k$, the ideal over $C^\infty(\Omega)$ generated by the functions $\lambda^{i_0 \cdots i_n}$ with $n \leq k$. And let

$$\Lambda_\delta(x) = \sum_{k=1}^{m} \delta^k \Lambda_K(x),$$

assuming that Ω is of finite type m.

DEFINITION 4.9. Let

$$C_\delta^4 = \Bigg\{ \varphi\colon [0,1] \to \partial\Omega/\varphi \text{ is Lipschitz},$$

$$\varphi'(t) = \sum_{j=1}^{2} a_j(t) X_j(\varphi(t)) + b(t) T(\varphi(t)),$$

$$|a_j(t)| \leq \delta, |b(t)| \leq \Lambda_\delta(\varphi(t)) \Bigg\}$$

$$C_\delta^5 = \Bigg\{ \varphi\colon [0,1] \to \partial\Omega/\varphi \text{ is Lipschitz},$$

$$\varphi'(t) = \sum_{j=1}^{2} a_j X_j(\varphi(t)) + b T(\varphi(t)),$$

$$a_j, b \in \mathbf{R}, |a_j| \leq \delta, |b(t)| \leq \Lambda_\delta(\varphi(0)) \Bigg\}.$$

In order to keep the notation in [8], we use C_δ^4, C_δ^5. The curves C_δ^1, C_δ^2 and C_δ^3 will not be used in this work.

We can define corresponding distance and balls as follows.

DEFINITION 4.10. Given $x_0, y_0 \in \partial\Omega$ we say $\rho_j(x_0, y_0) < \delta$, $j = 4, 5$, if there exists $\varphi_j \in C_\delta^j$ with $\varphi_j(0) = x_0$, $\varphi_j^{(1)} = y_0$.

Also we define $B_j(x_0, \delta) = \{y_0 \in \partial\Omega\colon \rho(x_0, y_0) < \delta\}$.

It was proven in [8] that the balls B_4 and B_5 are equivalent.

From R. O. Wells [12], we have the following:

If V is a real vector space equipped with a complex structure J then V can be made into a complex vector space V_J by introducing the complex scalar multiplication

$$(\alpha + i\beta)v = \alpha v + \beta Jv, \quad \alpha, \beta \in \mathbf{R},\ v \in V,\ i = \sqrt{-1}$$

Alternatively, $V \otimes_{\mathbf{R}} \mathbf{C}$ is a complex vector space and J can be defined on $V \otimes_{\mathbf{R}} \mathbf{C}$ by

$$J(v \otimes \alpha) = J(v) \otimes \alpha \quad \text{for } v \in V,\ \alpha \in \mathbf{C}$$

This extended J has eigenvalues $+i$ and $-i$, since $J^2 = -I$.
The $+i$ eigenspace is called $V^{1,0}$.
The $-i$ eigenspace is called $V^{0,1}$.
Observe that, in the setup where $V = \mathbf{R}^{2n}$, then $V^{1,0}$ corresponds to span

$$\left\{ \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right\}$$

and $V^{0,1}$ corresponds to span

$$\left\{ \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right\}.$$

It can be checked that the complex vector space obtained from V by means of the complex structure J, denoted by V_J is $\mathbf{C}$-linearly isomorphic to $V^{1,0}$. This means we can canonically associate to any element of the "real" vector space a holomorphic vector space. This way we do it in the Euclidean space is by

$$(a_1, b_1, \dots, a_n, b_n) \to (a_1 + ib_1)\frac{\partial}{\partial z_1} + \cdots + (a_n + ib_n)\frac{\partial}{\partial z_n}$$

Let $\Omega \subset\subset \mathbf{C}^2$ be a domain and U be a neighborhood in $\partial\Omega$. Let X_1 be any complex tangent vector field on U. Let $X_2 = JX_1$. Let N be the vector field of unit outward normal vectors to $\partial\Omega$ on U and $T = -JN$.

Then, to the vector field $a_1X_1 + a_2X_2 + bT$ on U, where $a_1, a_2, b \in \mathbf{R}$, corresponds the holomorphic vector field

$$a_1T_1 + ia_2T_2 + bT_2 = (a_1 + ia_2)T_1 + bT_2 \quad \text{on } U.$$

Then we can define the curves C_δ^4 and C_δ^5 in terms of holomorphic vector fields by

$$\begin{aligned} C_\delta^4 = \{\varphi\colon [0,1] \to \partial\Omega/\varphi \text{ is Lipschitz};\\ \varphi'(t) = a_1(t)T_1(\varphi(t)) + a_2(t)T_2(\varphi(t));\\ |a_1(t)| < \delta,\ |a_2(t)| < \Lambda_\delta(\varphi(t))\}.\\ C_\delta^5 = \{\varphi\colon [0,1] \to \partial\Omega/\varphi \text{ is Lipschitz};\\ \varphi'(t) = a_1T_1(\varphi(t)) + a_2T_2(\varphi(t));\\ a_1, a_2 \in \mathbf{C},\ |a_1| < \delta,\ |a_2| < \Lambda_\delta(\varphi(0))\}. \end{aligned}$$

So we have equivalent notations of distances and balls.

We can define approach regions in $\Omega \subset\subset \mathbf{C}^2$ in terms of the families of ball on $\partial\Omega$. By B we mean any of the equivalent balls.

DEFINITION 4.11. Let $\tilde{\Omega} = \overline{\Omega} \cap$ (small neighborhood of $P \in \partial\Omega$). Let π be any smooth projection from Ω to $\partial\Omega$. For $z \in \Omega$ set

$$D(z) = \inf_{1 \le k \le m-1} \left\{ \frac{\delta_\Omega(z)}{\Lambda_k(\pi(z))} \right\}^{1/k}$$

DEFINITION 4.12. Given $\sigma > 0$, $P \in \partial\Omega$, then

$$\begin{aligned} \mathscr{A}_\sigma(P) &= \{z \in \tilde{\Omega}\colon \pi(z) \in B(P, \sigma D(z))\}\\ &= \left\{ z \in \tilde{\Omega}\colon \rho(\pi(z), P) < \sigma \inf_{1 \le k \le m-1} \left\{ \frac{\delta_\Omega(z)}{\Lambda_k(\pi(z))} \right\}^{1/k} \right\}, \end{aligned}$$

where ρ denotes any of the equivalent metrics ρ_4 or ρ_5 and B any of the equivalent balls B_4 or B_5.

5. Comparability of the Kobayashi approach region and the approach region $\mathscr{A}_\sigma(P)$

THEOREM 5.1. *Let $\Omega \subset\subset \mathbf{C}^2$ be a pseudoconvex domain of finite type. Let P be a given point in the boundaray of Ω, and assume that P is of type $2m - 1$.*

Then given $\sigma > 1$ there are two positive constants, $B = B(\sigma)$ and $C = C(\sigma)$, which depend on Ω and are functions of σ, and an open neighborhood U of P such that

$$U \cap \mathscr{K}_{C(\sigma)}(P) \subseteq U \cap \mathscr{A}_\sigma(P) \subseteq U \cap \mathscr{K}_{B(\sigma)}(P).$$

Proof. Let U be a neighborhood of P where Catlin's estimates hold.

Part 1. Assume $z_0 \in U \cap \mathscr{A}_\sigma(P)$; we want to prove $z_0 \in U \cap \mathscr{K}_{B(\sigma)}(P)$.

If $z_0 \in U \cap \mathscr{A}_\sigma(P)$, then $\pi(z_0) \in B(P, \sigma D(z_0))$ and this implies there exists a curve β: $[0,1] \to \partial\Omega$, Lipschitz with $\beta(0) = P$, $\beta(1) = \pi(z_0)$ and

$$\beta'(t) = a_1 T_1(\beta(t)) + a_2 T_2(\beta(t))$$

where $|a_1| < \sigma D(z_0)$ and $|a_2| < \Lambda_{\sigma D(z_0)}(\beta(0))$.

Consider the curve in $\Omega \cap U$, defined by

$$\hat{\beta}(t) = \beta(t) - \delta_\Omega(z)\nu_{\beta(t)}.$$

Then, applying Catlin's estimates we have

$$\begin{aligned} K(z_0, -\nu_P) &\leq L_K^\Omega(\hat{\beta}(t)) \\ &= \int_0^1 F_K^\Omega(\hat{\beta}(t), \hat{\beta}'(t))\, dt \\ &\leq \int_0^1 CM_m(\hat{\beta}(t); a_1 T_1(\hat{\beta}(t)) + a_2 T_2(\hat{\beta}(t)))\, dt \\ &\leq C\int_0^1 \Bigg\{ |a_2|\,|\rho(\hat{\beta}(t))|^{-1} \\ &\qquad + |a_1| \sum_{k=1}^{m-1} |C_k(\hat{\beta}(t))|^{1/2k} |\rho(\hat{\beta}(t))|^{-1/2k} \Bigg\}\, dt. \end{aligned}$$

where ρ is a defining function for Ω.

Since Ω is a domain of finite type, let us assume $\pi(z_0)$ is of type $2s - 1$ with $s \leq m$. Then $\alpha = \Lambda_m(\pi(z_0)) \neq 0$. Therefore

$$\begin{aligned} |a_2| &\leq \sum_{k=1}^{m-1} (\sigma D(z_0))^k \Lambda_k(P) \leq \sum_{k=s}^{m-1} \left\{ \sigma \left[\frac{\delta_\Omega(z_0)}{\alpha} \right]^{1/s} \right\}^k \Lambda_k(P) \\ &\leq \sigma^m \Lambda_{m-1}(P) \sum_{k=s}^{m-1} \left[\frac{\delta_\Omega(z_0)}{\alpha} \right]^{k/s}, \\ |a_1| &\leq \sigma \left[\frac{\delta_\Omega(z_0)}{\Lambda_s(\pi(z_0))} \right]^{1/s} \leq \sigma \left[\frac{\delta_\Omega(z_0)}{\alpha} \right]^{1/s}. \end{aligned}$$

We also have $|\rho(\hat{\beta}(t))| \approx \delta_\Omega(z_0)$ for all t. Hence,

$$\begin{aligned} K(z_0, -\nu_P) &\le C\int_0^1 \left\{ \sigma^{m-1}\Lambda_{m-1}(P)\left[\sum_{k=s}^{m-1}\left[\frac{\delta_\Omega(z_0)}{\alpha}\right]^{k/s}\right]\delta_\Omega^{-1}(z_0) \right. \\ &\qquad \left. +\sigma\left[\frac{\delta_\Omega(z_0)}{\alpha}\right]^{1/s}\left[\sum_{k=s}^{m-1}|C_k(\hat{\beta}(t))|^{1/2k}\delta_\Omega^{-1/2k}(z_0)\right]\right\} dt \\ &\le C\frac{\sigma^m}{\alpha}\Lambda_{m-1}(P)\sum_{k=s}^{m-1}\delta_\Omega^{k/s-1}(z_0) \\ &\qquad + C\sigma\alpha^{-1/s}\delta_\Omega^{1/2s}(z_0)\int_0^1\sum_{k=s}^{m-1}|C_k(\hat{\beta}(t))|^{1/2k}\,dt \\ &\le C\frac{\sigma^m}{\alpha}\Lambda_{m-1}(P) + C\sigma\alpha^{-1/s} = B(\sigma). \end{aligned}$$

Part 2. Assume $z_0 \in U \cap \mathscr{K}_{C(\sigma)}(P)$; we want to prove $z_0 \in U \cap \mathscr{A}_\sigma(P)$. Let us prove the contrapositive.

Assume $z_0 \notin U \cap \mathscr{A}_\sigma(P)$; we will prove that $K(z_0, -\nu_P) > C(\sigma)$.

If $z_0 \notin U \cap \mathscr{A}_\sigma(P)$ then $\pi(z_0) \notin B(P, \sigma D(z_0))$. Therefore for any curve φ: $[0,1] \to \partial\Omega$, Lipschitz with $\varphi(0) = P$ and $\varphi(1) = \pi(z_0)$ such that $\varphi'(t) = a_1T_1(\varphi(t)) + a_2T_2(\varphi(t))$ we have

$$|a_1| > \sigma D(z_0) \quad \text{or} \quad |a_2| > \sum_{k=1}^{m-1}\sigma D(z_0)^k\Lambda_k(P).$$

Take a curve γ: $[0,1] \to \Omega$ such that the Euclidean length of γ is t_0 and it connects z_0 with $-\nu_P$. Then the curve

$$\Psi(t) = \gamma(t) + \delta_\Omega(\gamma(t))\nu_{\pi(\gamma(t))}$$

is a curve in $\partial\Omega$ such that $\Psi(0) = P$ and $\Psi(t_0) = \pi(z_0)$.

Fix two constants $N(\sigma) > 0$ and $M(\sigma) > 0$ such that $N(\sigma)$ is a small number and $M(\sigma)$ is a large number.

There are three possibilities:

(i) $\delta_\Omega(\gamma(t)) \approx \delta_\Omega(z_0)$ for all $t \in [0;1]$;

(ii) $\delta_\Omega(\gamma(t)) < N(\sigma)\delta_\Omega(z_0)$ for some t;

(iii) $\delta_\Omega(\gamma(t)) > M(\sigma)\delta_\Omega(z_0)$ for some t.

Case (i) Since γ is parametrized with respect to Euclidean arc length then $|\gamma'(t)| = 1$ for all t and

$$\Psi'(t) = \gamma'(t) + \delta'_\Omega(\gamma(t))\nu_{\pi(\gamma(t))} + \delta_\Omega(\gamma(t))\nu'_{\pi(\gamma(t))}.$$

Since $\delta_\Omega(\gamma(t)) \approx \delta_\Omega(z_0)$, the second and third terms of $\Psi'(t)$ are negligible, so

$$\begin{aligned} L_K^\Omega(\gamma(t)) &= \int_0^{t_0} F_K^\Omega(\gamma(t);\gamma'(t))\,dt \\ &\geq C\int_0^{t_0} M_{m-1}(\gamma(t);\gamma'(t))\,dt \\ &= C\int_0^{t_0}\Big\{|a_2|\,|\rho(\gamma(t)|^{-1} \\ &\qquad + |a_1| \sum_{k=1}^{m-1} |C_k(\delta(t))|^{1/2k} |\rho(\gamma(t))|^{-1/2k}\Big\}\,dt. \end{aligned}$$

Assume that $\pi(z_0)$ is a point of type $2s-1$ with $s \leq m$. We have

$$|a_2| > \sigma^s \frac{\delta_\Omega(z_0)}{\Lambda_s(\pi(z_0))}\Lambda_s(P)$$

or

$$|a_1| > \sigma\left[\frac{\delta_\Omega(z_0)}{\Lambda_s(\pi(z_0))}\right]^{1/s}$$

so

$$\begin{aligned} L_K^\Omega(\gamma(t)) &\geq C\int_0^{t_0} \sigma^s \frac{\delta_\Omega(z_0)}{\Lambda_s(\pi(z_0))}\delta_\Omega^{-1}(z_0)\Lambda_s(P)\,dt \\ &= C\sigma^s\Lambda_s(\Pi(z_0))t_0\Lambda_s^{-1}(P) \\ &= h(\sigma) \end{aligned}$$

or

$$\begin{aligned} L_K^\Omega(\gamma(t)) &\geq C\int_0^{t_0} \sigma\delta_\Omega^{1/s}(z_0)\Lambda_s^{-1/s}(\pi(z_0))|C_s(\gamma(t))|^{1/2s}\delta_\Omega^{-1/2s}(z_0)\,dt \\ &= C\sigma\delta_\Omega^{1/2s}(z_0)\int_0^{t_0}|C_s(\gamma(t))|^{1/2s}\,dt \\ &= f(\sigma). \end{aligned}$$

Case (ii) We have

$$\gamma'(t) = c_1T_1(\gamma(t)) + c_2T_2(\gamma(t))$$

where

$$c_2 = \langle \gamma'(t); T_2(\gamma(t)) \rangle \approx \gamma'_{N_P}(t),$$

$$L_K^\Omega(\gamma(t)) = \int_0^{t_0} F_K^\Omega(\gamma(t); \gamma'(t))\,dt \geq C\int_0^{t_0} M_{m-1}(\gamma(t); \gamma'(t))\,dt$$

$$= C\int_0^{t_0} |c_2|\,|\rho(\gamma(t))|^{-1}\,dt \geq C\int_0^{t_0} \frac{|\gamma_{N_P}(t)|}{\delta_\Omega(\gamma(t))}\,dt.$$

Define the curve

$$\mu(t) = z_0' + \int_0^t \gamma'_{\tilde{N}_P}(s)\,ds, \quad 0 \leq t \leq t_0,$$

where $\gamma'_{\tilde{N}_P}(s)$ is the projection of $\gamma'_{N_P}(s)$ onto the real normal at P and z_0 is the projection of z_0 onto $-\nu_P$. We have $\mu'(t) = \gamma'_{\tilde{N}_P}(t)$ for all t. Then

$$L^\Omega(\gamma(t)) \geq C\int_0^{t_0} \frac{|\mu'(t)|}{\delta_\Omega(\mu(t))}\,dt \geq \int_0^{t_0} \frac{|\hat{\mu}'(t)|}{\delta_\Omega(\hat{\mu}(t))}\,dt$$

where $\hat{\mu}$ is gotten from μ by discarding overlaps.

Let $\gamma(t_1)$ be such that $\delta_\Omega(\gamma(t_1)) < N(\sigma)\delta_\Omega(z_0)$ and m is the projection of $\gamma(t_1)$ onto the real normal.

Then

$$L_K^\Omega(\gamma(t)) \geq CL_K^\Omega \text{ (segment connecting } m \text{ with } z_0')$$

$$\approx C\int_0^{\delta_\Omega(z_0')-\delta_\Omega(m)} \frac{dt}{\delta_\Omega(m)+t} \approx C\ln\{\delta_\Omega(m)+t\}\Big|_0^{\delta_\Omega(z_0')-\delta_\Omega(m)}$$

$$\approx C\ln\frac{\delta_\Omega(z_0')}{\delta_\Omega(m)}.$$

But $\delta_\Omega(m) \leq N(\sigma)\delta_\Omega(z_0) \leq N(\sigma)\delta_\Omega(z_0')$ since $\delta_\Omega(z_0) \leq \delta_\Omega(z_0')$. So

$$L_K^\Omega(\gamma) \geq C\ln\frac{\delta_\Omega(z_0')}{N(\sigma)\delta_\Omega(m)} \geq C\ln\frac{1}{N(\sigma)}.$$

Case (iii) Fixing the large constant $M(\sigma)$ such that $\delta_\Omega(\gamma(t)) \geq M(\sigma)\delta_\Omega(z_0)$ for some t, we follow the same argument applied in case (ii). Therefore

$$L_K^\Omega(\gamma(t)) = C\int_0^{t_0} \frac{|\gamma_{N_P}(t)|}{\delta_\Omega(\gamma(t))}\,dt.$$

and we can define the curve

$$\mu_1(t) = z_0' + \int_0^t \gamma_{\hat{N}_P}'(s)\,ds, \quad 0 \le t \le t_0,$$

where $\gamma_{\hat{N}_P}'(s)$ is the projection of $\gamma_{N_P}'(s)$ onto the real normal at P and z_0' is the projection of z_0 onto $-\nu_P$. We have $\mu'(t) = \gamma_{\hat{N}_P}'(t)$ for all t. Then

$$L^{\Omega}(\gamma(t)) \ge C\int_0^{t_0} \frac{|\mu'(t)|}{\delta_\Omega(\mu(t))}\,dt \ge C\int_0^{t_0} \frac{|\hat{\mu}'(t)|}{\delta_\Omega(\hat{\mu}(t))}\,dt$$

where μ is gotten from μ by discarding overlaps.

Let $\gamma(t_2)$ be such that $\delta_\Omega(\gamma(t_2)) > M(\sigma)\delta_\Omega(z_0)$ and let m be the projection of $\gamma(t_2)$ onto the real normal. Then

$$\begin{aligned}
L_K^{\Omega}(\gamma(t)) &\ge CL_K^{\Omega}\ (\text{segment connecting } z_0' \text{ with } m)\\
&\approx C\int_0^{\delta_\Omega(m)-\delta_\Omega(z_0')} \frac{dt}{\delta_\Omega(z_0')+t}\\
&\approx C\ln\{\delta_\Omega(z_0')+t\}\Big|_0^{\delta_\Omega(m)-\delta_\Omega(z_0')}\\
&\approx C\ln\frac{\delta_\Omega(m)}{\delta_\Omega(z_0')} \ge C\ln\frac{M(\sigma)\delta_\Omega(z_0)}{\delta_\Omega(z_0)} \ge C\ln M(\sigma)
\end{aligned}$$

If we let

$$C(\sigma) = \sup\{C\ln M(\sigma);\ C\ln 1/N(\sigma);\ f(\sigma)\}$$

then we have proven that if $z_0 \notin \mathscr{A}_\sigma(P)$ then $K(z_0, -\nu_P) > C(\sigma)$, as desired. □

6. Fatou's theorem on domains of finite type

As an application of theorem 5.1 we can give a new invariate form of Fatou's theorem for domains of finite type in $\mathbf{C}^2$.

Following the ideas in Section 6 of [8] we have the following:

DEFINITION 6.1. Let f be holomorphic on $\Omega \subseteq \mathbf{C}^2$, $P \in \partial\Omega$ and $\beta > 1$. We set

$$\mathscr{M}_\beta f(P) = \sup_{z \in \mathscr{K}_\beta(P)} |f(z)|$$

Then we have the following theorems.

THEOREM 6.2. *Let $\Omega \subseteq \mathbf{C}^2$ be a domain of finite type.*

(i) *For $0 \leq p < \infty$ if $f \in H^p(\Omega)$ then $\mathcal{M}_\beta f \in L^p(\partial\Omega)$ and $\|\mathcal{M}_\beta f\|_{L^p} \leq \|f\|_{H^p}$.*

(ii) *If $f \in N(\Omega)$, then $\mathcal{M}_\beta f$ is finite almost everywhere, and*

$$m\left\{\log^+ \mathcal{M}_\beta f > \lambda\right\} \leq c/\lambda.$$

The proof this theorem is similar to the proof of Theorem 9 in [8]. We have to use the fact that $\mathcal{A}_\sigma(P) \approx \mathcal{K}_\beta(P)$.

THEOREM 6.3. *Given f holomorphic in Ω, a domain of finite type in $\mathbf{C}^2$, the following two conditions are equivalent for almost every $P \in \partial\Omega$.*

(i) $\mathcal{M}_\beta f(P) < \infty$.

(ii) $\lim_{z \to P,\, z \in \mathcal{K}_\beta(P)} f(z)$ *exists.*

In the proof we use the ideas of Theorem 11 in [8].

REFERENCES

1. G. ALADRO, *The boundary behavior of the Carathéodory and Kobayashi metrics*, preprint.
2. ______, *Some consequences of the boundary behavior of the Carathéodory and Kobayashi metrics and applications to normal holomorphic functions*, Pennsylvania State University Ph.D. Thesis, 1985.
3. D.W. CATLIN, *Invariant metrics on pseudoconvex domains*, preprint.
3A. I. GRAHAM, *Boundary behavior of the Carathéodory and Kobayashi metrics on strongly pseudoconvex domains in $\mathbf{C}^n$ with smooth boundary*, Trans. Amer. Math. Soc., vol. 267 (1975), pp. 219–240.
4. J.J. KOHN, *Boundary behavior of ∂ on weakly pseudoconvex manifold of dimension two*, J. Differential Geometry, vol. 6 (1972), 523–542.
5. A. KORANYI, *Harmonic functions on Hermitian hyperbolic space*, Trans. Amer. Math. Soc. vol. 135 (1969), pp. 507–516.
6. S.G. KRANTZ, *Function theory of several complex variables*, Wiley-Interscience, New York, 1982.
7. ______, *Fatou theorems on domains in $\mathbf{C}^n$*, Bull. Amer. Math. Soc., vol. 16 (1987), pp. 93–96.
8. A. NAGEL, E. STEIN and S. WAINGER, *Boundary behavior of functions holomorphic in domains of finite type*, Proc. Nat. Acad. Sci. USA, vol. 78 (1981), pp. 6596–6599.
9. H.L. ROYDEN, "Remarks on the Kobayashi metric" in *Several complex variables, II*, Proc. International Conf., Univ. of Maryland, 1970, Lectures Notes in Mathematics, Vol. 185, Springer-Verlag, Berlin, 1975, pp. 125–137.
10. N. SIBONY, "A class of hyperbolic manifolds" in *Recent developments in several complex variables*, J.E. Fornaess ed., Princeton Univ. Press, Princeton, 181, pp. 357–372.
11. E.M. STEIN, *Boundary behavior of holomorphic functions of several complex variables*, Princeton University Press, Princeton, N.J., 1972.
12. R.O. WELLS, *Differential analysis on complex manifolds*, Springer-Verlag, New York, 1980.

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