ILLINOIS JOURNAL OF MATHEMATICS
Volume 33, Number 1, Spring 1989

CHARACTERIZATION OF BANACH SPACES OF CONTINUOUS VECTOR VALUED FUNCTIONS WITH THE WEAK BANACH-SAKS PROPERTY

BY

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Introduction

A Banach space E is said to have the Banach-Saks property (resp. weak Banach-Saks property) if for every bounded sequence (resp. weakly convergent sequence) (x_n) in E, you can choose a subsequence (x_n') of (x_n) such that the sequence

$$(y_n) = \left(\frac{x_1' + \cdots + x_n'}{n}\right)$$

converges in the E-norm.

We shall refer to these properties as the B.S.P. and the W.B.S.P.

It is known that a Banach space E with the B.S.P. is reflexive. So, it is clear that a $C(K)$ space (being $C(K)$, the Banach space of the continuous functions from K to $\mathbf{R}$, and being K, a compact Hausdorff space) has the B.S.P. iff K is finite.

Much more interesting in this context of $C(K)$ spaces is the W.B.S.P. The following characterization of $C(K)$ spaces with the W.B.S.P. is due essentially to N. Farnum (see [2]).

THEOREM 1. *Let K be a compact Hausdorff space. Then $C(K)$ possesses the W.B.S.P. if and only if*

$$K^{(\omega)} = \bigcap_{n=1}^{\infty} K^{(n)} = \emptyset$$

where $K^{(0)} = K$ and $K^{(n)}$ is the set of all accumulation points of $K^{(n-1)}$ for $n \in \mathbf{N}$.

Received January 6, 1987.

[1]Supported in part by a CAICYT grant.

The author wishes to thank Professor F. Bombal and the referee for their advice.

The target of this note is to characterize when $C(K, E)$, the Banach space of all continuous functions defined on a compact Hausdorff space K with values in a Banach space E, endowed with the supremum norm, has the W.B.S.P. Later, in Section 2, we'll show a Banach space E and a compact K such that

(a) $C(K)$ and E have the W.B.S.P.

(b) $C(K, E)$ has not the W.B.S.P.

Finally, in Section 3, we'll talk a little about two other properties that a $C(K)$ space may enjoy or not: the hereditary Dunford-Pettis and the alternate Banach-Saks properties.

The notations and terminology used and not explained here can be found in [2]. We only want to recall the definition of the spaces $E = (\Sigma \oplus E_n)_p$. If $(E_n, \|\cdot\|_n)$ is a Banach space, and p, $1 \leq p < \infty$ (resp. $p = 0$) we define E as the Banach space of all sequences (x_n), with $x_n \in E_n$, $(\|x_n\|_n) \to 0$ and such that

$$\left(\sum_n \|x_n\|_n^p\right)^{1/p} < \infty \quad (\textit{resp.}\ \sup\{\|x_n\|_n \colon n = 1, \dots\} < \infty)$$

being these expressions the norm of E, for p, $1 \leq p < \infty$ and $p = 0$ respectively.

1. When a $C(K, E)$ space has the weak Banach-Saks property

If K is a finite compact Hausdorff space, then it is immediate that $C(K, E)$ possesses the W.B.S.P. if and only if E does it. If K is infinite, we have the following result. First of all, we recall that $c_0(E)$ is the Banach space of all null sequences in E, endowed with the supremum norm.

THEOREM 2. *Let K be an infinite compact Hausdorff space. Then $C(K, E)$ has the W.B.S.P. if and only if $C(K)$ and $c_0(E)$ have the W.B.S.P.*

Proof. It is identical to the proof of Theorem 3 of [3], so we omit it.

Now the question is: when $c_0(E)$ has the W.B.S.P.? The following theorem gives us the answer.

THEOREM 3. *$c_0(E)$ has the* W.B.S.P. *if and only if E has the uniform W.B.S.P. That is to say, there exists a sequence $(a(n))$ of positive real numbers converging to 0 such that, for every sequence*

$$(x_n) \subset B(E), \quad (x_n) \xrightarrow{\omega} 0,$$

and for every $m \in \mathbf{N}$, *we can choose* $n(1) < \cdots < n(m)$, *these numbers depending on* m, *satisfying*

$$\left\| \frac{x_{n(1)} + \cdots + x_{n(m)}}{m} \right\| < a(m).$$

Proof. Suppose E has not the uniform W.B.S.P. Then, there exist a strictly increasing sequence of integers $(i(m))$, an $\varepsilon > 0$, and sequences

$$\left(x_n^{(1)}\right), \dots, \left(x_n^{(m)}\right), \dots$$

weakly convergent to 0, $(x_n^{(m)}) \subset B(E)$, such that

$$\left\| \frac{x_{n(1)}^{(m)} + \cdots + x_{n(i(m))}^{(m)}}{i(m)} \right\| > \varepsilon$$

for every $n(1) < \cdots < n(i(m))$. We can suppose $i(m) \geq 2^m$ without problem.

Let $(f_n) \subset B(c_0(E))$ be the sequence defined as follows:

$$f_1 = \left(x_1^{(1)}, 0, \dots\right),$$

$$f_n = \left(x_n^{(1)}, x_{n-1}^{(2)}, \dots, x_1^{(n)}, 0, \dots\right).$$

That is, $f_n(m) = 0$ if $n < m$; $f_n(m) = x_{n-m+1}^{(m)}$ if $n \geq m$. It is clear that, for every m fixed

$$(f_n(m) \colon n = 1, \dots) \xrightarrow{\omega} 0$$

and we can deduce that

$$(f_n) \xrightarrow{\omega} 0$$

(for instance, see [5]).

Let (f_n') be any subsequence of (f_n). It is clear that the sequence

$$(g_n) = ((f_1' + \cdots + f_n')/n)$$

does not converge in the $c_0(E)$-norm. In fact, if (g_n) converges to anything, it must be to 0. But

$$\begin{aligned}\|g_{i(m)+m}\|_\infty &= \left\|\frac{f_1' + \cdots + f'_{i(m)+m}}{i(m)+m}\right\|_\infty \\ &\geq \left\|\frac{f_1'(m) + \cdots + f'_{i(m)+m}(m)}{i(m)+m}\right\|_E \\ &= \left\|\frac{x_{n(1)}^{(m)} + \cdots + x_{n(j(m))}^{(m)}}{i(m)+m}\right\|_E\end{aligned}$$

It is easy to see that $i(m) \leq j(m) \leq i(m) + m$. And now, the inequality continues with

$$\begin{aligned}&\geq \left\|\frac{x_{n(1)}^{(m)} + \cdots + x_{n(i(m))}^{(m)}}{i(m)+m}\right\|_E - m/(i(m)+m) \\ &> \frac{\varepsilon i(m)}{i(m)+m} - \frac{m}{i(m)+m} \\ &> \frac{\varepsilon}{2} - \frac{m}{2^m}\end{aligned}$$

So we deduce that $c_0(E)$ does not have the W.B.S.P.

Let's suppose now that E has the uniform W.B.S.P. and let's see that $c_0(E)$ has the W.B.S.P. First of all, we need the following technical result.

LEMMA 4. *Let E be a Banach space with the uniform weak Banach-Saks property. Then, there exists a sequence (δ_m) of positive real numbers such that $(\delta_m) \to 0$ for which given any sequence (x_n) in $B(E)$ with $(x_n) \xrightarrow{\omega} 0$, there is a subsequence (y_n) of (x_n) such that for every subsequence (y_n') of (y_n), we have*

$$\left\|\sum_{j\leq m} y_j'/m\right\| \leq \delta_m$$

We leave the proof of this lemma to the end, and first finish the proof of our theorem. Let

$$(f_n) \xrightarrow{\omega} 0, \quad (f_n) \subset B(c_0(E)).$$

We need a subsequence (f_n') of (f_n) such that

$$(*) \qquad ((f_1' + \cdots + f_n')/n) \to 0$$

in the $c_0(E)$-norm. We suppose first that

$$(f_n) \subset c_{00}(E) = \{ f \in c_0(E) \colon \max\{k \colon f(k) \neq 0\} = N_f \in \mathbf{N} \}.$$

Since $c_{00}(E)$ is dense on $c_0(E)$, if we prove $(*)$ for a sequence $(f_n) \subset c_{00}(E)$, we also have $(*)$ for any sequence $(g_n) \subset c_0(E)$, $(g_n) \xrightarrow{\omega} 0$.

So let's suppose $(f_n) \subset B(c_0(E))$, $(f_n) \subset c_{00}(E)$, $(f_n) \xrightarrow{\omega} 0$. For every $n \in \mathbf{N}$, we define

$$M(n) = \max\{k \colon f_n(k) \neq 0\}$$

If the sequence $(M(n))$ is bounded (for instance, by M), we can apply Lemma 4 to the sequences

$$(f_n(1)), \ldots, (f_n(M)) \subset E$$

and we can choose a subsequence (f_n') of (f_n) such that

$$\left\| \sum_{j \le m} f_j'(k)/m \right\| \le \delta_m$$

for $m \in \mathbf{N}$, and $1 \le k \le M$. So, due to the fact that $f_n(k) = 0$ if $k > M$, we have $\|\Sigma_{j \le m} f_j'/m\|_\infty \le \delta_m$ and we are finished.

If the sequence $(M(n))$ is not bounded, then we can assume (by passing to a subsequence if necessary) that $M(n))$ is strictly increasing.

Now, we build (f_n') subsequence of (f_n) by induction.

Case $n = 1$. Let $(f_n^{(1)}) = (f_n)$, $f_1' = f_1^{(1)}$. We also define $N(0) = 0$ and

$$N(1) = \max\{n \colon f_1'(n) \neq 0\}$$

Case $n = 2$. We consider the sequences

$$\left(f_n^{(1)}(1) \colon n = 1, \ldots\right), \ldots, \left(f_n^{(1)}(N(1)) \colon n = 1, \ldots\right)$$

If we apply Lemma 4 to these sequences, we have a subsequence $(f_n^{(2)})$ of $(f_n^{(1)})$ such that for every increasing sequence of integers $(n(j))$,

$$\left\| \sum_{j \le m} f_{n(j)}^{(2)}(k)/m \right\| \le \delta_m$$

for $m \in \mathbf{N}$ and $1 \le k \le N(1)$.

We choose $f_2' = f_2^{(2)}$ and define

$$N(2) = \max\{n: f_2'(n) \ne 0\}.$$

Case $n = r + 1$. Let's suppose that we have chosen the sequences $(f_n^{(t)}: n = 1, \dots)$, $t = 1, \dots, r$, the functions $f_1', \dots, f_r'$ and the numbers $N(0) < \cdots < N(r)$ satisfying:

(1) If $t > 1$, $(f_n^{(t)})$ is a subsequence of $(f_n^{(t-1)})$.

(2) $f_i' = f_i^{(i)}$, $1 \le i \le r$.

(3) $N(i) = \max\{n: f_i'(n) \ne 0\}$, $1 \le i \le r$.

(4) For every $t = 2, \dots, r$ and for every increasing sequence of integers $(n(j))$, we have

$$\left\| \sum_{j \le m} f_{n(j)}^{(t)}(k)/m \right\| \le \delta_m$$

for $m \in \mathbf{N}$ and $1 \le k \le N(t-1)$.

Now we consider the sequences

$$\left(f_n^{(r)}(N(r-1)+1): n = 1, \dots\right), \dots, \left(f_n^{(r)}(N(r)): n = 1, \dots\right)$$

If we apply Lemma 4 to these sequences, we have a subsequence $(f_n^{(r+1)})$ of $(f_n^{(r)})$ such that for every increasing sequence of integers $(n(j))$, we have

$$\left\| \sum_{j \le m} f_{n(j)}^{(r+1)}(k)/m \right\| \le \delta_m$$

for $m \in \mathbf{N}$ and $N(r-1) < k \le N(r)$. Note that the previous inequality is also true for every k, $1 \le k \le N(r-1)$, since $(f_n^{(r+1)})$ is a subsequence of $(f_n^{(r)})$. So, it is true for $1 \le k \le N(r)$. Now, we define $f_{r+1}' = f_{r+1}^{(r+1)}$, $N(r+1) = \max\{n: f_{r+1}'(n) \ne 0\}$ and the induction is finished.

Now, let's prove $(*)$. For every $m \in \mathbf{N}$, if $k \le N(1)$ then

$$\begin{aligned} \|(f_1' + \cdots + f_m')(k)/m\| &= \left\| \left(f_1' + f_n^{(2)} + \cdots + f_{n(m-1)}^{(2)}\right)(k)/m \right\| \\ &\le 1/m + \left((m-1)/m\right) \cdot \delta_{m-1} \end{aligned}$$

where $n(1) = 2 < \cdots < n(m-1)$ are suitable numbers.

If k is such that $N(1) < k \leq N(2)$ then

$$\|(f_1' + f_m')(k)/m\| = \left\|\left(f_2' + f_{n(1)}^{(3)} + \cdots + f_{n(m-2)}^{(3)}\right)(k)/m\right\|$$
$$\leq 1/m + ((m-2)/m) \cdot \delta_{m-2}$$

where $n(1) = 3 < \cdots < n(m-2)$ are suitable numbers.

Continuing in this way, it is clear that

$$\|(f_1' + \cdots + f_m')(k)/m\| < 1/m + ((m-t)/m)\delta_{m-t}$$

if $N(t-1) < k \leq N(t)$ (with the convention that δ_0 is any real number), $1 \leq t \leq m$, and it is immediate that

$$\|(f_1' + \cdots f_m')(k)/m\| = 0 \quad \text{if } N(m) < k.$$

So it only remains to prove that if $(\delta_m) \to 0$, then

$$(\eta_m) = \left(\sup\{(m-t)/m \cdot \delta_{m-t}: 1 \leq t \leq m-1\}\right) \to 0$$

We leave it as an easy exercise to the reader.

Proof of Lemma 4. Let $(x_n) \xrightarrow{\omega} 0$ and $(x_n) \subset B(E)$. If (x_n) has a subsequence (x_n') such that $\|x_n'\| \to 0$, we have finished. If not, then (x_n) has a subsequence (x_n') with the following good property: For every $b_1, \ldots, b_r$, if $n(1) < \cdots < n(r)$ then the limit

$$\lim_{n(1)\to\infty} \left\| \sum_{j \leq r} b_j x_{n(j)}' \right\|$$

exists. We call that limit $L(\sum_{j \leq r} b_j e_j)$ for convenience. See [1], Chapter 1, for a proof.

Since E has the uniform W.B.S.P. (see the definition at the beginning of Theorem 3) we can deduce that if we take $s(r) \leq r$, $s(r) \in \mathbf{N}$ and $b_1 = \cdots = b_{s(r)} = 0$, and $b_{s(r+1)} = \cdots = b_r = 1/r$, we have

$$(+) \qquad \left| L\left(\sum_{j \leq r} b_j e_j \right) \right| \leq a(r - s(r)) \cdot (r - s(r))/r$$

This holds because

$$\begin{aligned}\lim_{n(1)\to\infty}\left\|\sum_{j\le r} b_j x'_{n(j)}\right\| &= \frac{r-s(r)}{r}\lim_{n(1)\to\infty}\left\|\sum_{j=s(r)+1}^{r}\frac{x'_{n(j)}}{r-s(r)}\right\| \\ &\le \frac{r-s(r)}{r}a(r-s(r)).\end{aligned}$$

The last inequality is due to the fact that that limit exists and the definition of $a(m)$, for every $m \in \mathbf{N}$.

Let $s(m)=[\sqrt{m}]$ (where $[\cdot]$ is the greatest integer function). Now, we consider the finite set

$$A(i) = \{m \in \mathbf{N}\colon s(m) = i\}$$

It is clear that for every i we have an integer $N(i)$ such that if $n(i) \ge N(i)$ then

$$(++) \qquad \left| L\left(\sum_{j=i+1}^{m}\frac{e_j}{m}\right) - \left\|\sum_{j=i+1}^{m}\frac{x'_{n(j)}}{m}\right\| \right| < \frac{1}{m}$$

for every m, $s(m) = i$. Then, if we define $M(0) = 1$ and

$$M(i) = \max(N(i), M(i-1)+1)$$

the sequence $(x'_{M(i)})$ satisfies Lemma 4. In fact, if $(x'_{n(j)})$ is a subsequence of $(x'_{M(j)})$, we have

$$\left\|\sum_{j=s(m)+1}^{m}\frac{x'_{n(j)}}{m}\right\| \le \left| L\left(\sum_{j=s(m)+1}^{m}\frac{e_j}{m}\right)\right| + \frac{1}{m}$$

This inequality follows from $(++)$ $n(i) \ge M(i) \ge N(i)$. Now, by $(+)$, we continue the inequality with

$$\le a(m-s(m))\frac{m-s(m)}{m} + \frac{1}{m}.$$

And finally, it is clear that

$$\left\| \sum_{j \leq m} \frac{x'_{n(j)}}{m} \right\| \leq \left\| \sum_{j \leq s(m)} \frac{x'_{n(j)}}{m} \right\| + \left\| \sum_{j=s(m)+1}^{m} \frac{x'_{n(j)}}{m} \right\|$$
$$\leq \frac{s(m)}{m} + a(m - s(m)) \frac{m - s(m)}{m} + \frac{1}{m}$$
$$= \delta_m$$

It is obvious that $(\delta_m) \to 0$. So the sequence $(y_i) = (x'_{M(i)})$ satisfies our lemma.

2. A Banach space E with the weak Banach-Saks property but not in the uniform sense

We begin this section with the following question: If a $C(K)$ space has the W.B.S.P., does $C(K)$ possess the uniform W.B.S.P.? The answer is yes, and we deduce it in this way:

(a) As we saw in Theorem 1, a $C(K)$ space has the W.B.S.P. if and only if

$$(+) \qquad K^{(\omega)} = \bigcap_n K^{(n)} = \emptyset.$$

(b) $C(K)$ has the uniform W.B.S.P. if and only if $c_0(C(K))$ has the W.B.S.P. (by Theorem 3).

(c) As $c_0(C(K))$ is isomorphic to $C(\mathbf{N}^* \times K)$, where $\mathbf{N}^*$ is the Alexandroff compactification of $\mathbf{N}$, $c_0(C(K))$ has the W.B.S.P. if and only if

$$(++) \qquad (\mathbf{N}^* \times K)^{(\omega)} = \bigcap_n (\mathbf{N}^* \times K)^{(n)} = \emptyset.$$

(d) Proposition 10 of [3] proves that $(+) \Rightarrow (++)$, so we have finished.

The problem that we want to solve now is the following: we have seen that for a $C(K)$ space, the properties uniform W.B.S. and W.B.S. are equivalent, but is that true for any Banach space E? The answer is no. First of all, we need the following result. Remember that a Banach space E has the W.B.S.P. if and only if for every sequence $(x_n) \xrightarrow{\omega} 0$ there exists a subsequence (x'_n) of (x_n) such that for every subsequence (x''_n) of (x'_n), we have

$$\left\| \sum_{j \leq m} x''_j / m \right\| \to 0 \quad as\ m \to \infty$$

See [1], Chapter 2, for a proof.

THEOREM 5. *If each Banach space E_n has the W.B.S. property then so does* $(\Sigma \oplus E_n)_1$.

Proof. Let $(x^m) \xrightarrow{\omega} 0$ in $(\Sigma \oplus E_n)_1$. Then, it is known that if

$$x^m = \left(x_1^m, \ldots, x_n^m, \ldots\right)$$

where $x_n^m \in E_n$ and $\| \cdot \|_n$ is the norm of the Banach space E_n we have

$$(+) \qquad \lim_{k \to 0} \sup\left\{ \sum_{j=k}^{\infty} \|x_j^m\|_j \colon m = 1, \ldots \right\} = 0$$

Consider the sequence $(x_1^m) \xrightarrow{\omega} 0$ in E_1. E_1 has the W.B.S. property, so we can choose a subsequence $({}^{(1)}x_1^m)$ of (x_1^m) so that

$$\left\|{}^{(1)}x_1^{m(1)} + \cdots + {}^{(1)}x_1^{m(r)}\right\|_1 / r \to 0 \quad \text{in } E_1 \text{ as } r \to \infty,$$

for every increasing sequence of natural numbers $(m(j))$. Now let's consider the sequence

$$\left({}^{(1)}x_2^m\right) \xrightarrow{\omega} 0 \quad \text{in } E_2.$$

As E_2 has the W.B.S. property, there exists $({}^{(2)}x_2^m)$ a subsequence of $({}^{(1)}x_2^m)$, such that

$$\left\|{}^{(2)}x_2^{m(1)} + \cdots + {}^{(2)}x_2^{m(r)}\right\|_2 / r \to 0 \quad \text{in } E_2 \text{ as } r \to \infty,$$

for every increasing sequence of natural numbers $(m(j))$.

In the same way, for every k there exists $({}^{(k)}x^m \colon m = 1, \ldots)$, a subsequence of $({}^{(k-1)}x^m \colon m = 1, \ldots)$, such that

$$\left\|{}^{(k)}x_k^{m(1)} + \cdots + {}^{(k)}x_k^{m(r)}\right\|_k / r \to 0 \quad \text{in } E_k \text{ as } r \to \infty,$$

for every increasing sequence of natural numbers $(m(j))$.

Define the subsequence (y^m) of (x^m) by

$$y^m = {}^{(m)}x^m$$

We will show that $\|y^1 + \ldots y^m\|/m \to 0$. Let $\varepsilon > 0$. Then, by $(+)$, there exists $k(\varepsilon)$ such that

$$\sum_{j=k(\varepsilon)+1}^{\infty} \|y_j^m\|_j < \varepsilon/2 \quad \text{for every } m.$$

For each $j = 1, \dots, k(\varepsilon)$, we have

$$\{y_j^m \colon m = j, \dots,\} = \{{}^{(j)}x_j^{k(m)} \colon m = j, \dots\}$$

where $k(m)$ is an increasing sequence of integers with $k(j) = j$. So, it is clear that, for every $j = 1, \dots, k(\varepsilon)$, there exist i_j such that for every $i > i_j$ we have

$$\|y_j^1 + \cdots + y_j^i\|_j / i < \varepsilon / 2k(\varepsilon)$$

Now, taking $i_0 = \max\{i_1, \dots, i_{k(\varepsilon)}\}$, if $i > i_0$ we obviously have

$$\begin{aligned} \left\|\sum_{s=1}^{i} y^s\right\| \Big/ i &= \sum_{j=1}^{\infty} \left\|\sum_{s=1}^{i} y_j^s\right\|_j \Big/ i \\ &= \sum_{j=1}^{k(\varepsilon)} \left\|\sum_{s=1}^{i} y_j^s\right\|_j \Big/ i + \sum_{j=k(\varepsilon)+1}^{\infty} \left\|\sum_{s=1}^{i} y_j^s\right\|_j \Big/ i \\ &< \sum_{j=1}^{k(\varepsilon)} \varepsilon/2k(\varepsilon) + \sum_{s=1}^{i}\left(\sum_{j=k(\varepsilon)+1}^{\infty} \|y_j^s\|_j\right) \Big/ i \\ &< \varepsilon \end{aligned}$$

So the Banach space $(\Sigma \oplus E_n)_1$ has the W.B.S. property.

Now we can establish the main result of this section.

COROLLARY 6. *Let* $\mathbf{N}^*$ *be the Alexandroff compactification of* $\mathbf{N}$. *There exists a Banach space* E *with the weak Banach-Saks property such that* $c(\mathbf{N}^*, E)$ *does not have the weak Banach-Saks property despite the fact that* $c(\mathbf{N}^*)$ *has the* W.B.S.P.

Proof. Take $E_n = c_0(\omega^n)$. It is known that E_n is isomorphic to c_0 (see [6]), so every E_n has the W.B.S. property. Define $E = (\Sigma \oplus E_n)_1$. As we have seen before, E has the W.B.S. property. But $c_0(E)$ does not (and so neither does $c(\mathbf{N}^*, E)$). To prove my point, we consider the subspace of $c_0(E)$,

$$Z = \{f \colon \mathbf{N} \to E,\ f \in c_0(E)/f(n) \in E_n\}.$$

It is immediately seen that this subspace is isometric to

$$(\Sigma \oplus E_n)_0 = (\Sigma \oplus c_0(\omega^n))_0$$

which, in fact, is isometric to $c_0(\omega^\omega)$, a very well known example of a Banach space which does not have the W.B.S. property (see [2] for a proof). So $c_0(E)$ has not this property, either.

The previous corollary is remarkable because it shows (using Theorem 3) that the properties W.B.S. and uniform W.B.S. are not equivalent. In fact we have a better result.

THEOREM 7. *There is a Banach space E with the Banach-Saks property such that E does not have the uniform weak Banach-Saks property.*

Proof. Using Lemma 5.2 of [1], it is very easy to prove that if each Banach space E_n has the Banach-Saks property, so does $E = (\Sigma \oplus E_n)_2$.

For any n, one can take

$$E_n = \{x\colon \mathbf{N} \to \mathbf{R} \text{ such that } (*) < +\infty\}$$

where

$$(*) = \sup\left\{\left(\sum_m \left(\sum_{k \in A_m} |x(k)|\right)^2\right)^{1/2} : \mathrm{Card}(A_m) = n,\ \bigcup_m A_m = \mathbf{N} \text{ and } A_m \cap A_{m'} = \emptyset \text{ if } m \neq m'\right\}$$

We take $\|\cdot\|_n = (*)$. It is clear that $(E_n, \|\cdot\|_n)$ is isomorphic to l^2, so it has the Banach-Saks property. By Lemma 5.2 of [1], E has the Banach-Saks property. But, for every $n \in \mathbf{N}$, if we take

$$x_k^{(n)} = \left(0, \ldots, {}^{n)}e_k, 0, \ldots\right)$$

it is clear that $(x_k^{(n)}) = \xrightarrow{\omega} 0$ and, for every $m(1) < \cdots < m(n)$, we have

$$\left\|\left(x_{m(1)}^{(n)} + \cdots + x_{m(n)}^{(n)}\right)/n\right\| = 1$$

So E does not have the uniform weak Banach-Saks property.

3. Alternate Banach-Saks and the hereditary Dunford-Pettis properties of $C(K,E)$ spaces

DEFINITION 8. (a) A Banach space E is said to have the alternate Banach-Saks property (A.B.S.P.) if for every bounded sequence (x_n) in E, we can choose a subsequence (x_n') of (x_n) such that the sequence

$$(y_n) = \left(\frac{(-1)x_1' + \cdots + (-1)^n x_n'}{n}\right)$$

converges in the E-norm.

(b) A Banach space E is said to have the hereditary Dunford-Pettis property (H.D.P.P.) if for every sequence weakly convergent to 0, not convergent in norm, there is a subsequence (x_n') of (x_n) which is equivalent to the unit vector basis of c_0 (see [3] for this definition).

Everything we have done in Section 1 with the W.B.S.P. we can do it with the A.B.S.P. For instance:

THEOREM 9. (a) *$C(K, E)$ has the A.B.S.P if and only if $C(K)$ and $c_0(E)$ have the A.B.S.P.*

(b) *$C(K)$ has the A.B.S.P if and only if $K^{(\omega)} = \bigcap_n K^{(n)} = \emptyset$*

(c) *$c_0(E)$ has the A.B.S.P. if and only if E has the uniform A.B.S.P That is there exists a sequence $(a(n))$ of positive real numbers converging to 0 such that, for every sequence $(x_n) \subset B(E)$, and for every $m \in \mathbf{N}$, we can choose $n(1) < \cdots < n(m)$, these numbers depending on m, such that*

$$\left\|\left((-1)x_{n(1)} + \cdots + (-1)^m x_{n(m)}\right)/m\right\| < a(m)$$

(d) *The space E of Theorem 7 has the A.B.S.P. but not the uniform A.B.S.P. Then the space $C(\mathbf{N}', E)$ does not have the A.B.S.P. although*

(i) *$c(\mathbf{N}')$ has the A.B.S.P. where $\mathbf{N}'$ is the Alexandroff compactification of* $\mathbf{N}$, *and*

(ii) *E has the A.B.S.P.*

The hereditary Dunford-Pettis property on $C(K, E)$ spaces was intensely studied in [3]. The uniform H.D.P.P. was defined there as follows:

$(*)$ There exists $M > 0$ such that every normalized weakly null sequence $(x_n) \subset E$, has a subsequence (x_n') that is equivalent to the unit vector basis of c_0 and satisfies

$$\left\|\sum_n a_n y_n\right\| \leq M \sup|a_n| \quad \text{for all } (a_n) \in c_0$$

and the problem "does every Banach space with the H.D.P.P. satisfy $(*)$" is still open.

We do not have the answer to this difficult question. Someone suggested that the space $E = (\Sigma \oplus E_n)_1$ with $E_n = c_0(\omega^n)$ could be the answer, and we are going to prove that it is not the case. Of course, we saw in Corollary 6 that $c_0(E)$ has a subspace isometric to $c_0(\omega^\omega)$, a well known example of a Banach space without the H.D.P.P. (see [2], for a proof), and so $c_0(E)$ does not have the H.D.P.P. The problem is that neither does E have this property. Let's prove this.

THEOREM 10. *The space $E = (\Sigma \oplus c_0(\omega^n))_1$ is not hereditarily Dunford-Pettis (although E has the weak Banach-Saks and the Dunford-Pettis properties).*

Proof. Remember that a Banach space E has the Dunford-Pettis property if for every $(x_n) \xrightarrow{\omega} 0$, $(x_n) \subset E$ and for every $(x'_n) \xrightarrow{\omega} 0$, $(x'_n) \subset E'$, the sequence of real numbers $(\langle x_n, x'_n \rangle) \to 0$.

To prove that E has the Dunford-Pettis property is very easy with the ideas of Theorem 5.

To prove that E does not have the H.D.P.P., we begin with the fact that the space $(\Sigma \oplus c_0(\omega^n))_0$ isometric to $c_0(\omega^\omega)$ does not have the H.D.P.P. Then, using the same technique as Cembranos in [3], there is no $M > 0$ such that $(*)$ is satisfied for every $c_0(\omega^n)$ with the same M. In other words, for every $M_k = 3^k$, there is an $n(k)$ (we take $n(k) > n(k-1)$) such that in the space $E_{n(k)} = c_0(\omega^{n(k)})$ there is a sequence

$$\left(x_j^{(k)}\colon j = 1, \ldots\right)$$

with the following properties (where $\|\cdot\|_k$ is the norm of $E_{n(k)}$):

(a) $(x_j^{(k)}) \xrightarrow{\omega} 0$ in $E_{n(k)}$, $\|x_j^{(k)}\|_k = 1$.

(b) For every subsequence $(x_{m(j)}^{(k)}\colon j = 1, \ldots)$ of $(x_j^{(k)}\colon j = 1, \ldots)$ there is a sequence $a = (a_1, \ldots, a_n, \ldots) \in c_0$ (depending on the subsequence) such that

$$3^k \sup|a_j| \le \left\| \sum a_j x_{m(j)}^{(k)} \right\|_k$$

Consider the following sequence in $(\Sigma \oplus E_{n(k)})_1$, a subspace of $(\Sigma \oplus c_0(\omega^n))_1$:

$$z_1 = \left(x_1^{(1)}/2, \ldots, x_1^{(k)}/2^k, \ldots\right),$$
$$z_m = \left(x_m^{(1)}/2, \ldots, x_m^{(k)}/2^k, \ldots\right)$$

It is very easy to prove that

(1) $z_m \in (\Sigma \oplus E_{n(k)})_1$,

(2) $(z_m) \xrightarrow{\omega} 0$ in $(\Sigma \oplus E_{n(k)})_1$.

But it is not difficult to see that, for every subsequence (z'_n) of (z_n), and every $k \in \mathbf{N}$, there is a sequence depending on (z'_m) and k such that

$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} a_j z'_j \right\| &= \sum_{k=1}^{\infty} \left\| \sum_{j=1}^{\infty} a_j z'_j(k) \right\|_k \\
&= \sum_{k=1}^{\infty} \left\| \sum_{j=1}^{\infty} \frac{a_j x_{m(j)}^{(k)}}{2^k} \right\|_k \\
&\ge \left\| \sum_{j=1}^{\infty} \frac{a_j x_{m(j)}^{(k)}}{2^k} \right\|_k \\
&\ge (3^k/2^k)\sup|a_j| \quad \text{due to the choice of a} = (a_1, \ldots)
\end{aligned}$$

So (z'_m) can not be equivalent to the canonic base of c_0, and so $(\Sigma \oplus c_0(\omega^n))_1$ does not have the H.D.P.P.

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