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VARIATIONAL PROBLEMS AND THE EXTERIOR DIFFERENTIAL SYSTEMS

BY

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Introduction

In [8], by utilizing the theories and techniques of exterior differential systems, Griffiths has developed a new and systematic approach of solving general variational problems of one independent variable. Here by a general variational problem of one independent variable we mean a variational problem whose functional has as its domain of definition the integral curves of an exterior differential system on a manifold. This treatment of the calculus of variations is, while in greater generality than customary, particularly useful in variational problems arising from geometry, and it sheds new lights on even the classical Lagrange problems.

In this paper, some of the results in [8] are generalized to the case of several independent variables, that is, to general variational problems for functionals whose domain of definition consists of the integral manifolds of an exterior differential system (in particular, this includes the case of constrained variational problems). Of course, the first obvious difficulty in studying such general problems is the question of whether there exist even formally "enough" admissible integral manifolds. For this, one should at least assume the somewhat subtle condition of involutivity. Moreover, even if it is involutive, since there is no general C^∞ existence theorem, we must restrict ourselves to the real analytic case where the Cartan-Kähler theorem applies. However, since these are not our main concern, we shall just ignore these and proceed heuristically as in [8] assuming throughout the paper the existence of enough admissible variations to arrive at our Euler-Lagrange equations. For classical variational problems these coincide with the usual ones found in classical texts (e.g., [7]) as expected. Following the approach in [8], we then write the Euler-Lagrange equations as an involutive exterior differential system on an associated manifold, and a Noether's theorem is obtained.

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For a further useful application of the formalism in the present paper, one is referred to [5] in which by also adopting the ideas of "formal differential geometry" of Gelfand-Vinogradov-Manin, the author studies the Euler-Lagrange equations associated to general variational problems and obtains all the higher order conservation laws of such equations as the solution space of a linear differential operator with explicit formula. Meanwhile, in the same article a "higher order Noether's theorem" identifying intrinsically the higher order conservation laws with the "higher order Noether symmetries", where the latter is a generalization of the usual notion of "infinitesimal Noether symmetries", is also obtained with explicit formula.

Throughout the paper, the summation convention will be adopted. Manifolds will be denoted by X, Y, Z, etc. The algebra of smooth functions, the Lie algebra of smooth vector fields, and the de Rham complex on X are denoted by $C^\infty(X)$, ΓTX and $\Omega^*(X)$ respectively. If $\phi \in T^*X$ and $Y \subset X$ is a submanifold, we shall write ϕ_Y for $\phi|Y$ and $\phi \equiv 0 \bmod Y$ for $\phi_Y = 0$. If I is a closed differential ideal in $\Omega^*(X)$, then $\phi \equiv 0 \bmod I$ shall mean $\phi \in I$.

1. Definitions and basic terminologies

Let X be a manifold. On X an exterior differential system ([2], [3], [4], [9]) with independence condition (or simply an exterior differential system in short) (I, ω) is given by a closed differential ideal $I \subset \Omega^*(X)$ together with an n-form ω called the independence condition. Here n is called the number of independent variables. An integral element of (I, ω) is a pair (x, E) with $x \in X$ and E an n-plane in $T_x(X)$ satisfying the conditions $\theta|E = 0$, $\forall \theta \in I$, and $\omega|E \neq 0$. An integral manifold of (I, ω) is given by a smooth map $\phi\colon N \to X$ where N is a connected manifold of dimension n such that $(\phi(x), \phi_*(T_xN))$ is an integral element of (I, ω), $\forall x \in N$. The sets of all integral elements and integral manifolds of (I, ω) will be denoted by $V(I, \omega)$ and $\mathscr{I}(I, \omega)$ respectively.

Now we restrict our attention to the case where I is a Pfaffian system, i.e., I is locally generated by smooth sections of a subbundle $W^* \subset T^*(X)$. Since as long as integral manifolds are concerned, the independence n-form ω is only well-defined modulo I, what we will be concerned with is thus the filtration of sub-bundles

$$W^* \subset L^* \subset T^*(X), \tag{1.1}$$

where W^* generates I and L^*/W^* is a vector bundle of rank n such that $\omega \in C^\infty(X, \Lambda^n L^*)$ induces a non-zero cross-section of $\Lambda^n(L^*/W^*)$.

(1.2) *Example.* Let M be an m-dimensional manifold with local coordinates

$$(y^\alpha\colon \alpha = 1, \dots, m)$$

and consider $X = J^r(\mathbf{R}^n, M)$, the manifold of r-jets of locally defined maps from $\mathbf{R}^n$ to M. If (x^i, y_J^α) $(1 \leqq i \leqq n, 1 \leqq \alpha \leqq m,\ J \in \mathbf{N}^n,\ |J| \leqq r)$ is the local coordinate system (will be referred to as the standard local coordinate system of X) induced by (y^α), then the pair (I, ω) given by

$$I = \left\{ \theta_J^\alpha \colon = dy_J^\alpha - y_{J+(i)}^\alpha\, dx^i,\ 1 \leqq \alpha \leqq m,\ J \in \mathbf{N}^n,\ |J| < r \right\}$$

$$\omega = dx = dx^1 \wedge \ldots \wedge dx^n$$

is called the canonical Pfaffian system on X.

2. The setup of a general variational problem

On a manifold X we consider a Pfaffian system (I, ω) whose independence condition ω is locally a decomposable n-form in $\Omega^*(X)/I$. In terms of an adapted basis $\{\theta^\alpha, \omega^i\}$ $(1 \leqq \alpha \leqq m, 1 \leqq i \leqq n)$ of the corresponding filtration $W^* \subset L^*$ (cf. (1.1)), (I, ω) is then given locally by

$$\theta^\alpha = 0, \quad \omega = \omega^1 \wedge \ldots \wedge \omega^n \neq 0.$$

Now suppose we are given a Lagrangian n-form φ on X; for every $N \in \mathscr{I}(I, \omega)$, we set

$$\Phi(N) = \int_N \varphi. \tag{2.1}$$

Here we agree to consider only those integral manifolds N for which the integral exists, i.e., N may be non-compact but the improper integral should converge. We may view the assignment $N \to \int_N \varphi$ as a functional (perhaps not everywhere defined) $\Phi\colon \mathscr{I}(I, \omega) \to \mathbf{R}$ and we denote by $(X; I, \omega; \varphi)$ the variational problem associated to the functional (2.1).

Given a variational problem $(X; I, \omega; \varphi)$, we consider the following:

(2.2) *Problem.* Determine the variational equations and the "Euler-Lagrange equations" of (2.1)

Of course the present situation is much more complicated than the case $n = 1$ treated in [8], primarily because of the following two reasons:

(i) The Pfaffian system (I, ω) must satisfy the somewhat subtle condition of involutivity in order that there exist even formally enough integral manifolds for the variation.

(ii) Even when (I, ω) is involutive, there is no general C^∞ existence theorem for local integral manifolds, so we must restrict to the real analytic case where the Cartan-Kähler theorem applies.

However, we shall ignore these, and by assuming that there exist enough admissible variations we proceed heuristically to arrive at a set of equations that should express the condition that the functional (2.1) has a critical value at N, and we define them to be the Euler-Lagrange equations associated to $(X; I, \omega; \varphi)$.

(2.3) *Example.* We continue with Example (1.2). Take for (I, ω) the canonical Pfaffian system on X. Let L be a function on X and set $\varphi = L\omega$. Then $(X; I, \omega; \varphi)$ is called a classical r^{th} order variational problem. Integral manifolds $N \in \mathscr{I}(I, \omega)$ are locally r-jets of the maps (y^α): $\mathbf{R}^n \to M$ and so the functional L is given by

$$L = L\left(x^i, y_J^\alpha\right) = L\left(x^i, \partial^J y^\alpha(x)\right), \quad |J| \leqq r.$$

If $Y \subset X$ is a submanifold defined locally by equations

$$g^\sigma\left(x^i, y_J^\alpha\right) = 0, \quad 1 \leqq \sigma \leqq k,$$

such that $\omega | TY \neq 0$ and L is a function on Y (or in practice the restriction to Y of some function on X). Denote again by (I, ω) the restriction to Y of the canonical Pfaffian system on X and set $\varphi = L\omega$, then $(Y; I, \omega; \varphi)$ is called a classical r^{th} order variational problem with constraints. In this case the domain of $\Phi = \int \varphi$ consists of r-jets of mappings (y^α): $\mathbf{R}^n \to M$ satisfying

$$g^\sigma\left(x^i, \frac{\partial^{|J|} y^\alpha}{\partial x^J}(x)\right) = 0 \quad \text{for } \sigma = 1, \ldots, k.$$

(2.4) *Assumption.* Since practically all problems we are interested in have the form $\varphi = f\omega$ for some function f on X, we will simply assume that this is always the case.

3. Variational equations for integral manifolds for a Pfaffian system (I, ω)

To attack problem (2.2) we begin by deriving the variational equations for an integral manifold (N, ϕ) of (I, ω). Roughly speaking, we shall compute $T_{(N,\phi)}(\mathscr{I}(I, \omega))$, the tangent space of $\mathscr{I}(I, \omega)$ at the point (N, ϕ).

Now a variation of

$$\phi\colon N \to X \tag{3.1}$$

is given by Φ: $N \times [0, \varepsilon] \to X$ such that if we let

$$\phi_t\colon N \to X \tag{3.2}$$

be the restriction of Φ to $N \times \{t\} \cong N$, then $\phi_0 = \phi$. The associated infinitesimal variation $v \in C^\infty(\phi(N), T(X))$ is given by

$$v(\phi(s)) = \Phi|_*\left(\frac{\partial}{\partial t}\right),$$

where

$$\Phi|_*: T_{(s,0)}(N \times [0, \varepsilon]) \to T_{\phi(s)}(X)$$

denotes the differential of Φ at the point $(s, 0) \in N \times [0, \varepsilon]$.

(3.3) *Remarks*. (i) In all cases of concern to us, ϕ will be an immersion and v induces a section $[v] \in C^\infty(\phi(N), E)$, where $E =$ the normal bundle $T(X)/\phi_* T(N)$. It will turn out that our variational equations will only depend on the normal component $[v]$ of v.

(ii) Since our consideration is local, we may further assume that ϕ is an embedding. Let $V \in C^\infty(X, T(X))$ be any extension of v, then it is easy to see that for any $\theta \in T^*(X)$, $\phi^*(V \lrcorner d\theta + d(V \lrcorner \theta))$ is independent of the extension and we shall write

$$\phi^*(v \lrcorner d\theta + d(v \lrcorner \theta)) = \Phi^*(V \lrcorner d\theta + d(V \lrcorner \theta))|N.$$

(3.4) LEMMA. *If* (3.1) *is an immersion, then for every* $\theta \in T^*(X)$,

$$\mathscr{L}_{\partial/\partial t}(\Phi^*\theta)|N = \phi^*(v \lrcorner d\theta + d(v \lrcorner \theta)).$$

Proof. Assume that ϕ is an embedding and let V be an extension of v such that

$$V(\Phi(s, t)) = \Phi_*\left(\frac{\partial}{\partial t}\right) \in T_{\Phi(s,t)}(X).$$

By the H. Cartan formula, we have

$$\mathscr{L}_{\partial/\partial t}(\Phi^*\theta) = \Phi^*(d(V \lrcorner \theta) + V \lrcorner d\theta), \quad \forall \theta \in T^*(X),$$

and the lemma follows by restricting both sides to $N \cong N \times \{0\}$. Q.E.D.

Now consider the variation (3.2) of the integral manifold given in (3.1). The conditions that (3.2) is an integral manifold are $\phi_t^*\theta^\alpha = 0$, $\forall \alpha$, i.e., $\Phi^*\theta^\alpha = g^\alpha(s, t)\, dt$, $\forall \alpha$, where the g^α's are some functions on $N \times [0, \varepsilon]$. By Lemma (3.4) these imply that

$$\phi^*(v \lrcorner d\theta^\alpha + d(v \lrcorner \theta^\alpha)) = g_t^\alpha(s, 0)\, dt|N = 0, \quad \forall \alpha.$$

Now we extend v again to a vector field (still denoted by v) on X and by dropping the reference to ϕ, we have

(3.5) PROPOSITION. *The variational equations for an integral manifold* (3.1) *of* (I, ω) *are*

$$v \lrcorner\, d\theta^\alpha + d(v \lrcorner\, \theta^\alpha) \equiv 0 \bmod N, \quad \forall \alpha. \tag{3.6}$$

(3.7) *Remarks*. (i) As noted in (ii) of (3.3), equations (3.6) depend only on the infinitesimal variation $v \in C^\infty(N, T(X))$ and not on its extension to $C^\infty(X, T(X))$.

(ii) Equations (3.6) depend only on the normal vector field

$$[v] \in C^\infty(N, T(X)/T(N))$$

induced by v. In fact, if $v \in \Gamma T(N)$, let v be an extension to $\Gamma T(X)$. Then the functions $v \lrcorner\, \theta^\alpha$ vanish on N and so $d(v \lrcorner\, \theta^\alpha) \equiv 0 \bmod N$. On the other hand, we have trivially $v \lrcorner\, d\theta^\alpha \equiv 0 \bmod N$.

(iii) It is also clear that (3.6) are independent of the choice of basis $\{\theta^\alpha\}$ of W^*. So the variational equations should really read

$$v \lrcorner\, d\theta + d(v \lrcorner\, \theta) \equiv 0 \bmod N, \quad \forall \theta \in \Gamma(W^*),$$

or, more generally, it is easy to see that they are equivalent to

$$v \lrcorner\, d\mu + d(v \lrcorner\, \mu) \equiv 0 \bmod N, \quad \forall \mu \in I. \tag{3.8}$$

Intrinsically equations (3.6) have the following meaning. Let $\{w_\alpha\} \in C^\infty(X, W)$ be a dual basis to the basis $\{\theta^\alpha\} \in C^\infty(X, W^*)$ and $E = T(X)/T(N)$. Define the first order linear differential operator

$$L\colon C^\infty(N, E) \to C^\infty(N, W \otimes T^*(N)) \tag{3.9}$$

by $L[v] = w_\alpha \otimes L^\alpha[v]$ with $L^\alpha[v] = v \lrcorner\, d\theta^\alpha + d(v \lrcorner\, \theta^\alpha)|N$. Then the variational equations (3.6) are just $L[v] = 0$. In view of this, we may refer to the "tangent space" $T_N(\mathscr{I}(I, \omega))$ as being the solution space of the linear PDE $L[v] = 0$.

4. The Euler-Lagrange equations associated to $(X; I, \omega; \varphi)$

In this section we shall derive the Euler-Lagrange equations associated to a variational problem $(X; I, \omega; \varphi)$ assuming the existence of enough solutions v

to the variational equations (3.6) satisfying the boundary conditions

$$v = 0 \quad \text{on } \partial N. \tag{4.1}$$

So we have the functional

$$\Phi(N) = \int_N \varphi, \tag{4.2}$$

and for a 1-parameter family $N_t \subset X$ of integral manifolds of (I, ω) with $N_0 = N$, we want to find the condition for

$$\frac{d}{dt}\int_{N_t} \varphi\bigg|_{t=0} = 0. \tag{4.3}$$

In general, of course, the boundary conditions (4.1) are much too stringent. However, we shall neglect that and formally proceed to derive heuristically, assuming the existence of enough admissible variations, a set of equations on N that must hold if (4.3) is to be true. This turns out to be a neat and beautiful system of equations in its own right, and we shall define them as the Euler-Lagrange equations associated to $(X; I, \omega; \varphi)$. Later in this section we shall loosen up the boundary conditions (4.1) and justify calling these equations obtained on N the Euler-Lagrange equations.

By Lemma (3.4), we have

$$\begin{aligned} \frac{d}{dt}\int_{N_t} \varphi\bigg|_{t=0} &= \int_N v \lrcorner\, d\varphi + d(v \lrcorner\, \varphi) \\ &= \int_N v \lrcorner\, d\varphi \end{aligned}$$

by Stokes' Theorem and (4.1). Recall the equations $L^\alpha[v] = 0$ that define $T_N(\mathscr{I}(I, \omega))$, the condition that (4.3) holds may be expressed as

$$L^\alpha[v] = 0, \forall \alpha,\ v = 0 \text{ on } \partial N \quad \Rightarrow \quad \int_N v \lrcorner\, d\varphi = 0.$$

By Stokes' Theorem the vanishing of $\int_N v \lrcorner\, d\varphi$ for sufficiently many v's should mean that $v \lrcorner\, d\varphi \equiv d\eta \bmod N$ where $\eta_{\partial N} = 0$ (this is one place where the "heuristic" comes in), or, η should depend linearly on v (so $\eta_{\partial N} = 0$ whenever $v_{\partial N} = 0$). Hence (4.3) means

$$L^\alpha[v] = 0, \forall \alpha,\ v = 0 \text{ on } \partial N \quad \Rightarrow \quad v \lrcorner\, d\phi \equiv d\eta(v) \bmod N.$$

Intuitively, this means that for any $v \in C^\infty(N, TX)$ satisfying $v_{\partial N} = 0$ the n-form $v \lrcorner\, d\varphi$ should be a linear combination of $L^\alpha[v]$ plus $d\eta(v)$ (this is the other place where the "heuristic" comes in), so (4.3) should mean

$$v \lrcorner\, d\varphi \equiv \lambda_\alpha \wedge L^\alpha[v] + d\eta(v) \bmod N, \forall v \in C^\infty(N, T(X)). \tag{4.4}$$

Here it is understood that $v_{\partial N} = 0$. The λ_α's are $(n-1)$ forms and $\eta(v)$ an $(n-1)$ form depending linearly on v.

Now since the left-hand-side of (4.4) does not depend on any derivatives of v, in order that no derivatives of v appear on the right hand side, we set $\eta(v) = (-1)^n (v \lrcorner \theta^\alpha)\lambda_\alpha$. Then $\eta_{\partial N} = 0$ whenever $v_{\partial N} = 0$, and (4.4) reads (by changing a sign of the λ_α's if necessary)

$$v \lrcorner\, d(\varphi + \theta) \equiv 0 \bmod N, \forall v \in C^\infty(N, T(X)), \text{where } \theta = \lambda_\alpha \wedge \theta^\alpha. \quad (4.5)$$

(4.6) *Remarks.* (i) Since equations (4.5) do not involve any derivatives of v, they are pointwise conditions along N.

(ii) To have intrinsic meaning one must allow θ in (4.5) to be any n-form in I. Thus one must allow $\theta = \lambda_\alpha \wedge \theta^\alpha + \mu_\alpha \wedge d\theta^\alpha$ for any $(n-2)$ forms μ_α. But then

$$\theta = (\lambda_\alpha \pm d\mu_\alpha) \wedge \theta^\alpha + (-1)^n d(\mu_\alpha \wedge \theta^\alpha)$$

and the last term is closed. Thus we may assume that $\theta = \lambda_\alpha \wedge \theta^\alpha$ as given in (4.5).

(iii) Set $\omega_i = (-1)^{i-1}\omega^1 \wedge \ldots \wedge \hat{\omega}^i \wedge \ldots \wedge \omega^n$; then $\{\omega_i\}$ locally spans $\Lambda^{n-1}T^*(N)$ and λ_α's may be written as

$$\lambda_\alpha = \lambda^i_\alpha \omega_i + \sigma_\alpha$$

where $\sigma_\alpha \equiv 0 \bmod N$ and $\lambda^i_\alpha \in C^\infty(X)$. But since obviously

$$d(\sigma_\alpha \wedge \theta^\alpha) \equiv 0 \quad \bmod N,$$

we may assume that, with a change of sign for θ if necessary,

$$\begin{aligned} \lambda_\alpha &= \lambda^i_\alpha \omega_i, \\ \theta &= \lambda^i_\alpha \theta^\alpha \wedge \omega_i, \\ \lambda &= \left\| \lambda^i_\alpha v_i \otimes w^*_\alpha \right\| \in W^* \otimes V \end{aligned}$$

where $\{v_i\}$ is the basis for V dual to the ω^i's and $\{w^*_\alpha\}$ the frame for W^* corresponding to $\{\theta^\alpha\}$. The last equation above gives the intrinsic meaning of the functions $\{\lambda^i_\alpha\}$.

We record the above discussions as follows.

(4.7) Let $N \in \mathscr{I}(I, \omega)$ and $\{N_t\}$ any admissible variation of N ($N_t \in \mathscr{I}(I, \omega)$, $\forall t$) satisfying the boundary conditions (4.1). Then in order that

$$\frac{d}{dt}\int_{N_t} \varphi \bigg|_{t=0} = 0,$$

the following conditions on N should hold:

$$v \lrcorner\, d\left(\varphi + \lambda^i_\alpha \theta^\alpha \wedge \omega_i\right) \equiv 0 \bmod N, \quad \forall v \in C^\infty(N, T(X)) \tag{4.8}$$

where the λ^i_α's are functions to be determined. (Notice that in any case we only need to determine the λ^i_α's on N.)

DEFINITION. We shall call the conditions (4.8) imposed on $N \in \mathscr{I}(I, \omega)$ the Euler-Lagrange equations associated to the variational problem $(X; I, \omega; \varphi)$.

(4.9) *Example.* We consider a classical r^{th} order variational problem using notations in (1.2) and (2.3). Since the Euler-Lagrange equations (4.8) are in coordinate-free terms, we may assume $M = \mathbf{R}^m$. For the sake of simplicity we shall only work on the case where $r = 1$, while the general case can be done in exactly the same manner. First of all, we find

$$\begin{aligned} d\left(\varphi + \lambda^i_\alpha \theta^\alpha \wedge \omega_i\right) = {} & L_{y^\alpha} \theta^\alpha \wedge dx + L_{y^\alpha_i} dy^\alpha_i \wedge dx \\ & + d\lambda^i_\alpha \wedge \theta^\alpha \wedge dx_i - \lambda^i_\alpha\, dy^\alpha_i \wedge dx. \end{aligned}$$

In general, to compute equations (4.8) it is sufficient to use a set of vectors that span $T_x(X)$ at every $x \in N$. More importantly, we may use "n less" vectors as shown by the following algebraic lemma whose proof is completely trivial.

(4.10) LEMMA. *Let T be a vector space and $\Psi \in \Lambda^{n+1}T^*$. Let $v_1, \dots, v_s$ be ectors that span an s-dimensional subspace V of T and $w_1, \dots, w_n$ vectors not in such that $\{v_\alpha, w_i\}$ form a basis of T. Then if*

$$\langle v_\alpha \lrcorner \Psi; w_1, \dots, w_n \rangle = 0, \quad \forall 1 \leqq \alpha \leqq s,$$

we have

$$\langle v \lrcorner \Psi; w_1, \dots, w_n \rangle = 0, \quad \forall v \in T.$$

Now return to Example (4.9). Let

$$\left\{ \frac{\partial}{\partial x^i}; \frac{\partial}{\partial \theta^\alpha}; \frac{\partial}{\partial y^\alpha_i} \right\}$$

be the dual basis of $\{dx^i; \theta^\alpha; dy_i^\alpha\}$; using Lemma (4.10) the Euler-Lagrange equations (4.8) are given by

$$\frac{\partial}{\partial y_i^\alpha} \lrcorner\, d\left(\varphi + \lambda^i_\alpha \theta^\alpha \wedge dx_i\right) = \left(L_{y_i^\alpha} - \lambda^i_\alpha\right) dx \equiv 0 \mod N$$

$$\frac{\partial}{\partial \theta^\alpha} \lrcorner\, d\left(\varphi + \lambda^i_\alpha \theta^\alpha \wedge dx_i\right) = L_{y^\alpha} - d\lambda^i_\alpha \wedge dx_i \equiv 0 \mod N.$$

Since $dx \neq 0$, $\lambda^i_\alpha = L_{y_i^\alpha}$ on N. Thinking of N as an 1-jet

$$(x^i) \mapsto \left(x^i, y^\alpha(x), \frac{\partial y^\alpha(x)}{\partial x^i}\right),$$

the Euler-Lagrange equations become

$$\frac{\partial}{\partial x^i} L_{y_i^\alpha} = L_{y^\alpha},$$

which are the classical Euler-Lagrange equations. Similarly, one finds that our Euler-Lagrange equations for classical r^{th} order variational problems coincide with the classical ones.

(4.11) *Remark.* If $(y^\alpha) = (y^\alpha(x))$ is a solution for the Euler-Lagrange equations and (y_t^α) a variation of (y^α), the natural boundary conditions are clearly $\partial^I y_t^\alpha | \partial N = \partial^I y^\alpha | \partial N$, $\forall I \in \mathbf{N}^n$, $|I| \leq r$.

Let us now go back to the derivation of the Euler-Lagrange equations (4.8). We see that the weaker conditions

$$v \lrcorner\, \omega^i = v \lrcorner\, \theta^\alpha = 0 \text{ on } \partial N, \forall \alpha, \forall i, \tag{4.12}$$

are all that was needed. Also, it is clear that they are exactly the corresponding conditions on the infinitesimal variation $v \in T_N(\mathscr{I}(I, \omega))$ of the natural boundary conditions of Example (4.9). Of course, in general (4.12) may still be too stringent. But we shall not get into the details of determining the correct boundary conditions individually because again that will involve the structure theory of (I, ω) which could be different for different problems. Instead, we shall just assume that (4.12) are the correct boundary conditions and proceed to justify calling (4.8) the Euler-Lagrange equations associated to $(X; I, \omega; \varphi)$.

(4.13) THEOREM. *If $N \in \mathscr{I}(I, \omega)$ satisfies the Euler-Lagrange equations (4.8), then*

$$\frac{d}{dt} \int_{N_t} \varphi \bigg|_{t=0} = 0$$

for any admissible variation of N_t of N that satisfies the boundary conditions (4.12).

Proof. Let v be the infinitesimal variation associated to $\{N_t\}$. By (3.4) we have

$$\begin{aligned}
\frac{d}{dt}\int_{N_t}\varphi\Big|_{t=0} &= \int_N v \lrcorner\, d\varphi + d(v \lrcorner\, \varphi) \\
&= \int_N v \lrcorner\, d\varphi \quad \text{by (4.12) and Stokes' theorem} \\
&= -\int_N v \lrcorner\, d\left(\lambda^i_\alpha \theta^\alpha \wedge \omega_i\right) \quad \text{by (4.8)} \\
&= \int_N d\left(v \lrcorner \lambda^i_\alpha \theta^\alpha \wedge \omega_i\right) \quad \text{by (3.8)} \\
&= 0 \quad \text{by (4.12) and Stokes' theorem.}
\end{aligned}$$

Q.E.D.

(4.14) *Example.* We shall derive, in our setting, the familiar condition that a submanifold $M^n \subset \mathbf{E}^{n+r}$ be minimal. For this we shall use moving frames in $\mathbf{E}^{n+r}$ (see [2], [4], [8]) and the following ranges of indices

$$\{1 \leqq i, j \leqq n;\ n+1 \leqq \alpha, \beta \leqq n+r;\ 1 \leqq a, b \leqq n+r\}.$$

Let X be the manifold consisting of pairs (x, T) where $x \in \mathbf{E}^{n+r}$ and $T \subset \mathbf{E}^{n+r}$ is an n-plane through x. Let $\mathscr{F}(\mathbf{E}^{n+r})$ be the orthonormal frame bundle over $\mathbf{E}^{n+r}$. There is a natural fibration

$$\pi\colon \mathscr{F}(\mathbf{E}^{n+r}) \to X \tag{4.15}$$

given by $\pi(x; e_1, \ldots, e_{n+r}) = (x; e_1 \wedge \ldots \wedge e_n)$, where $e_1 \wedge \ldots \wedge e_n$ means the translation by x of the n-plane through the origin spanned by the e_i's. Let $U \subset X$ be an open set. Choose a cross-section $(x; e_i, e_\alpha)$ of (4.15) and consider the Pfaffian system

$$I = \{\theta^\alpha := \omega^\alpha\}, \quad \omega = \omega^1 \wedge \ldots \wedge \omega^n \neq 0$$

on X. It is clear that (I, ω) is independent of the choice of the section and hence is well-defined.

Now for every submanifold $M^n \to \mathbf{E}^{n+r}$ there is a Gauss map $\gamma\colon M \to X$ given by $\gamma(x) = (x, T_x(M))$ for each $x \in M$. Notice that $\gamma(M) \in \mathscr{I}(I, \omega)$ and every $N \in \mathscr{I}(I, \omega)$ arises from a sub-manifold $M^n \subset \mathbf{E}^{n+r}$ in this manner. (Note that a cross-section of (4.15) over $\gamma(M)$ is just a field of Darboux frames over M.) Hence the problem of minimal submanifolds $M^n \subset \mathbf{E}^{n+r}$ is equivalent to the variational problem $(X; I, \omega; \varphi)$ with $\varphi = \omega$, the volume form.

Using the structure equations (e.g., [8], [11]) for $\mathbf{E}^{n+r}$, we get

$$\begin{aligned} &d(\varphi + \lambda^i_\alpha \theta^\alpha \wedge \omega_i) \\ &\quad = \theta^\alpha \wedge \omega^i_\alpha \wedge \omega_i + d\lambda^i_\alpha \wedge \theta^\alpha \wedge \omega_i - \lambda^i_\alpha \omega^\alpha_i \wedge \omega - \lambda^i_\alpha \theta^\alpha \wedge d\omega_i \\ &\quad\quad + \lambda^i_\alpha \theta^\beta \wedge \omega^\alpha_\beta \wedge \omega_i. \end{aligned}$$

Note that the 1-forms $\{\theta^\alpha; \omega^i; \omega^i_\alpha\}$ are horizontal for the fibration (4.15) and they form a coframe for X. Contracting $d(\varphi + \lambda^i_\alpha \theta^\alpha \wedge \omega_i)$ with $\partial/\partial \omega^i_\alpha$ and $\partial/\partial \theta^\alpha$, we get

(i) $\lambda^i_\alpha \omega \equiv 0 \bmod M$,

(ii) $\omega^i_\alpha \wedge \omega_i - d\lambda^i_\alpha \wedge \omega_i - \lambda^i_\alpha \, d\omega_i \equiv 0 \bmod M$.

Since $\omega \neq 0$, these combine and give

$$\omega^i_\alpha \wedge \omega_i \equiv 0 \mod M. \tag{4.16}$$

But since $d\theta^\alpha \equiv \omega^\alpha_i \wedge \omega^i \equiv 0 \bmod M$, by Cartan's lemma these imply that there are functions $H^\alpha_{ij} = H^\alpha_{ji}$ such that

$$\omega^\alpha_i = H^\alpha_{ij} \omega^j, \tag{4.17}$$

where $H = H^\alpha_{ij} e_\alpha \otimes \omega^i \omega^j$ is just the 2nd fundamental form of M in $\mathbf{E}^{n+r}$. Substituting (4.17) into (4.16) we get

$$H^\alpha_{ii} \equiv 0 \mod M,$$

which are just the familiar equations of a minimal submanifold.

(4.18) *Example.* We compute the Euler-Lagrange equations for the Willmore functional (cf. [1] and the references cited there). For this we let $M \subset \mathbf{E}^3$ be a 2-dimensional submanifold with Gaussian and mean curvatures K and H respectively. Then the Willmore functional is given by

$$\Phi(M) = \int_M H^2 \, dA.$$

Let X be the manifold defined in example (4.14) with $n = 2$ and $r = 1$ and consider the fibration (cf. (4.15))

$$\pi\colon \mathscr{F}(\mathbf{E}^3) \to X.$$

Let $\tilde{X} = X \times \mathbf{R}^3_{(a,b,c)}$. On $\tilde{X}$ we consider the Pfaffian system

$$I = \begin{cases} \theta^1 = \omega^3 \\ \theta^2 = \omega_1^3 - a_\omega^1 - b\omega^2 \\ \theta^3 = \omega_2^3 - b_\omega^1 - c\omega^2 \end{cases}$$

$$\omega = \omega^1 \wedge \omega^2 \neq 0.$$

Notice that every 2-dimensional submanifold $M \subset \mathbf{E}^3$ determines a (unique up to orientations) lift (given by the Darboux frames) $\tilde{M} \in \mathscr{I}(I, \omega)$ and conversely, every $\tilde{M} \in \mathscr{I}(I, \omega)$ arises in this way. So the Willmore problem of minimizing the Willmore functional is equivalent to the variational problem $(\tilde{X}; I, \omega; \varphi)$ with $\varphi = H^2\omega = \frac{1}{4}(a+c)^2\omega$.

Following our prescription and using the structure equations of $\mathbf{E}^3$ (for example, see [8], [11]), it is not hard to find the Euler-Lagrange equations of $(\tilde{X}; I, \omega; \varphi)$ to be (see [6] for more details)

$$\Delta H + 2H(H^2 - K) = 0$$

which is quite well-known.

5. The Euler-Lagrange differential system; momentum space

Given a variational problem $(X; I, \omega; \varphi)$ we want to write the associated Euler-Lagrange equations as an involutive exterior differential system (J, ω) on an associated manifold Y.

Given a variational problem $(X; I, \omega; \varphi)$, we notice that the Euler-Lagrange equations (4.8) are actually lying in the enlarged manifold $Z = X \times \mathbf{R}^{nm}$, where the $\mathbf{R}^{nm}$ has coordinates $\{\lambda^i_\alpha\}$. On Z, define

$$\psi = \varphi + \lambda^i_\alpha \theta^\alpha \wedge \omega_i, \quad \Psi = d\psi,$$

and the Cartan system $\mathscr{C}(\Psi)$ as the differential ideal generated by the differential forms $\{v \lrcorner \Psi \colon v \in \Gamma T(Z)\}$.

(5.1) Proposition. *The solutions to the Euler-Lagrange equations* (4.8) *associated to the variational problem* $(X; I, \omega; \varphi)$ *are in a natural one-to-one correspondence with* $\mathscr{I}(\mathscr{C}(\Psi), \omega)$ *in* Z.

Proof. Note first that on Z,

$$\Psi = d\lambda^i_\alpha \wedge \theta^\alpha \wedge \omega_i + \big(\text{terms not involving } d\lambda^i_\alpha\text{'s}\big),$$

and so $\mathscr{C}(\Psi)$ contains

$$\frac{\partial}{\partial \lambda^i_\alpha} \lrcorner \Psi = \theta^\alpha \wedge \omega_i. \tag{5.2}$$

Now if $N \in \mathscr{I}(I, \omega)$ satisfies (4.8) let $s = (s^i)$ be local coordinates on N. Then we may determine functions $\lambda^i_\alpha(s)$ such that for all $v \in C^\infty(N, T(X))$,

$$v \lrcorner d\left(\varphi + \lambda^i_\alpha \theta^\alpha \wedge \omega_i\right)|N = 0. \tag{5.3}$$

(Note: Strictly speaking we must extend the functions λ^i_α from N to X and compute $d(\varphi + \lambda^i_\alpha \theta^\alpha \wedge \omega_i)$. But the point is that since $\theta^\alpha | N = 0$, the left hand side of (5.3) is independent of the extension of the λ^i_α's.)

Associated to N and $\{\lambda^i_\alpha(s)\}$ is a natural manifold

$$\tilde{N} = N \times \left(\lambda^i_\alpha\right) \subset Z.$$

It is now clear that $\tilde{N} \in \mathscr{I}(\mathscr{C}(\Psi), \omega)$.

Conversely, if $\tilde{N} \in \mathscr{I}(\mathscr{C}(\Psi), \omega)$, let $N = \pi(\tilde{N}) \subset X$, where π is the natural projection $Z \to X$. By (5.2) we see that $N \in \mathscr{I}(I, \omega)$, and the condition $v \lrcorner \Psi | \tilde{N} = 0$ for all $v \in \Gamma T(X) \subset \Gamma T(Z)$ shows that N satisfies (4.8). Q.E.D.

In general, $(\mathscr{C}(\Psi), \omega)$ on Z is not involutive, so we have to apply the Cartan-Kuranishi theorem [10] to construct an involutive system (J, ω) on an associated manifold Y from $(\mathscr{C}(\Psi), \omega)$ on Z. To do this, let $G_n(Z)$ be the prolongation of the pair (Z, n), i.e., π: $G_n(Z) \to Z$ is the Grassmann bundle over Z whose fibers $\pi^{-1}(z) = G_n(T_z(Z))$ are Grassmann manifolds of n-planes in the tangent space $T_z(Z)$. Notice that $V(\mathscr{C}(\Psi), \omega) \subset G_n(Z)$. Denote by $Z_1 \subset Z$ the image of $V(\mathscr{C}(\Psi), \omega)$ under π and assume that Z_1 is a manifold. (This is not strictly necessary: in the real analytic case Z_1 will be an analytic subvariety and we may just consider the open dense subset of smooth points of Z_1 instead.) Set

$$\mathscr{C}_1(\Psi) = \mathscr{C}(\Psi)|Z_1, \quad \omega = \omega|Z_1, \quad \Psi_1 = \Psi|Z_1.$$

Obviously we have $\mathscr{C}(\Psi_1) \subset \mathscr{C}_1(\Psi)$ but the equality may not hold. The point is that an integral element of $(\mathscr{C}(\Psi), \omega)$ may not be tangent to Z_1. However, it is not hard to see that we always have $\mathscr{I}(\mathscr{C}(\Psi), \omega) = \mathscr{I}(\mathscr{C}_1(\Psi), \omega)$.

Replacing $(Z; \mathscr{C}(\Psi), \omega)$ by $(Z_1; \mathscr{C}_1(\Psi), \omega)$ and repeating the above constructions inductively, we get a sequence of exterior differential systems $(\mathscr{C}_k(\Psi), \omega)$ on manifolds Z_k with $Z_k \supset Z_{k+1}$ and $\mathscr{C}_{k+1}(\Psi) = \mathscr{C}_k(\Psi)|Z_{k+1}$. By the Cartan-Kuranishi theorem [10] this construction terminates with an involutive differential system (J, ω) on a manifold Y after a finite number of steps. (Here of course, in order to apply the Cartan-Kuranishi theorem real analyticity or some suitable constant rank assumptions are necessary, and we always assume these to hold.) By these constructions we have the next result.

(5.4) PROPOSITION. *Y and (J, ω) are characterized as follows*:
(i) *The projection of π: $V(J, \omega) \to Y$ is surjective, and*
(ii) $\mathcal{I}(J, \omega) = \mathcal{I}(\mathcal{C}(\Psi), \omega)$.

(5.5) COROLLARY. *The solutions to the Euler-Lagrange equations* (4.8) *associated to the variational problem $(X; I, \omega; \varphi)$ are in a natural one-to-one correspondence with $\mathcal{I}(J, \omega)$ in Y.*

DEFINITION. We shall call Y the momentum space and (J, ω) on Y the Euler-Lagrange differential system associated to the variational problem $(X; I, \omega; \varphi)$.

(5.6) *Remark.* From equations (5.2) we see that $I \subset \mathcal{C}(\Psi)$ and hence $I \subset J$, and in general, J is not a Pfaffian system.

(5.7) *Example.* We continue example (4.9) with $r = 1$. Using the notations there we set $Z = X \times \mathbf{R}^{nm}$ and on Z we have

$$\Psi = L_{y^\alpha}\theta^\alpha \wedge dx + L_{y_i^\alpha}\, dy_i^\alpha \wedge dx + d\lambda^i_\alpha \wedge \theta^\alpha \wedge dx_i - \lambda^i_\alpha\, dy_i^\alpha \wedge dx,$$

and the Cartan system $\mathcal{C}(\Psi)$ is given by
(i) $(\partial/\partial\lambda^i_\alpha)\lrcorner\Psi = \theta^\alpha \wedge dx_i \equiv 0 \bmod N$,
(ii) $(\partial/\partial y_i^\alpha)\lrcorner\Psi = \left(L_{y_i^\alpha} - \lambda^i_\alpha\right) dx \equiv 0 \bmod N$,
(iii) $(\partial/\partial\theta^\alpha)\lrcorner\Psi = L_{y^\alpha}\, dx - d\lambda^i_\alpha \wedge dx_i \equiv 0 \bmod N$.
Equations (i) together with $\omega \neq 0$ give $I \subset \mathcal{C}(\Psi)$. Equations (ii) imply that integral elements of $(\mathcal{C}(\Psi), dx)$ lie over

$$Z_1 = \left\{\lambda^i_\alpha = L_{y_i^\alpha}\right\} \subset Z.$$

We will show that if for general $\xi = (\xi_i) \in \mathbf{R}^n$,

$$\det \|L_{y_i^\alpha y_j^\beta}\xi_i\xi_j\| \neq 0, \tag{5.8}$$

then the restriction $(\mathcal{C}_1(\Psi), dx)$ of $(\mathcal{C}(\Psi), dx)$ to $Z_1 \subset Z$ is involutive. That is, $Y = Z_1$, $J = \mathcal{C}_1(\Psi)$.

To see this, notice that $(\mathcal{C}_1(\Psi), dx)$ is the exterior differential system associated to the familiar Euler-Lagrange equations obtained in example (4.9) which may be rewritten as

$$L_{y_i^\alpha y_j^\beta}\frac{\partial^2}{\partial x^i\,\partial x^j}y^\beta(x) = g^\alpha\!\left(x; y(x); \frac{\partial y(x)}{\partial x}\right) \tag{5.9}$$

which is a 2^{nd} order PDE system whose symbol is a map

$$\sigma\colon T^*(\mathbf{R}^n) \otimes \mathbf{R}^m \to \mathbf{R}^m$$

given by

$$(\sigma(\xi \otimes \mu))^\alpha = \left(L_{y_i^\alpha y_j^\beta} \mu^\beta \xi_i \xi_j\right),$$

where $\xi = (\xi_i) \in \mathbf{R}^n$, $\mu = (\mu^\alpha) \in \mathbf{R}^m$. It is a well-known result that (5.8) (in the form $\det \sigma_\xi \not\equiv 0$) implies the involutivity of the PDE system (5.9).

6. Noether's theorem

DEFINITION. An (infinitesimal) Noether symmetry of a variational problem $(X; I, \omega; \varphi)$ is given by a vector field v on X satisfying

$$\mathscr{L}_v I \subset I, \quad \mathscr{L}_v \varphi \equiv 0 \bmod I. \tag{6.1}$$

(6.2) NOETHER'S THEOREM. *If v is a Noether symmetry of $(X; I, \omega; \varphi)$, then $v \lrcorner \psi$ is a closed $(n-1)$ form on each integral manifold N of (I, ω) which satisfies the Euler-Lagrange equations* (4.8).

Proof. Let $\theta = \lambda^i_\alpha \theta^\alpha \wedge \omega_i$ and v be the vector field on $Z = X \times \mathbf{R}^{nm}$ induced by v on X by the product structure. By H. Cartan's formula and (4.8),

$$\begin{aligned} d(v \lrcorner \psi) &\equiv d(v \lrcorner (\varphi + \theta)) + v \lrcorner\, d(\varphi + \theta) \mod N \\ &\equiv \mathscr{L}_v(\varphi + \theta) \mod N \\ &\equiv 0 \mod N \end{aligned}$$

for every $N \in \mathscr{I}(I, \omega)$ that satisfies the Euler-Lagrange equations. Q.E.D.

Now since the construction of Y from $(X; I, \omega; \varphi)$ is functorial, if $F_t\colon X \to X$ is the 1-parameter group induced by v, then the induced 1-parameter group $\tilde{F}_t\colon Y \to Y$ in turn induces a vector field v_Y on Y. We call v_Y the vector field on Y induced by v on X.

By the natural one-to-one correspondence between $\mathscr{I}(J, \omega)$ and the solutions of the Euler-Lagrange equations of $(X; I, \omega; \varphi)$, we have:

(6.3) COROLLARY. *If v is a Noether symmetry of $(X; I, \omega; \varphi)$, then $v_Y \lrcorner \psi_Y$ is a closed $(n-1)$-form on each integral manifold of (J, ω) on Y.*

REFERENCES

1. R. BRYANT, *A duality theorem for Willmore surfaces*, J. Diff. Geom., vol. 20 (1984), pp. 23–53.
2. R. BRYANT, S.S. CHERN and P. GRIFFITHS, *Exterior differential systems*, Proc. Beijing Symposium on Differential Geometry and Differential Equations, 1980.
3. R. BRYANT, S.S. CHERN, R. GARDNER and P. GRIFFITHS, *Essays on exterior differential systems*, in preparation.
4. E. CARTAN, *Les systèmes différentielles extérieurs et leurs applications géométriques*, Herman, Paris, 1945.
5. W.S. CHEUNG, *Higher order conservation laws and a higher order Noether's theorem*, Adv. Appl. Math., vol. 8 (1987), pp. 446–485.
6. ______, *On higher order conservation laws*, Thesis, Harvard Univ., 1985.
7. I. GELFAND and G. FOMIN, *Calculus of variations*, Prentice Hall, Englewood Cliffs, N.J., 1963.
8. P. GRIFFITHS, *Exterior differential systems and the calculus of variations*, Birkäuser, Boston, 1983.
9. R. HERMAN, *Differential geometry and the calculus of variations*, 2nd edition, Math. Sci. Press, Brookline, Mass., 1977.
10. M. KURANISHI, *On E. Cartan's prolongation theorem of exterior differential systems*, Amer. J. Math., vol. 79 (1957), pp. 1–47.
11. S. STERNBERG, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

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