ILLINOIS JOURNAL OF MATHEMATICS
Volume 33, Number 2, Summer 1989

INFINITESIMAL RIGIDITY OF PRODUCTS OF SYMMETRIC SPACES

BY

JACQUES GASQUI AND HUBERT GOLDSCHMIDT

Let (X, g) be a compact symmetric space. We say that a 1-form or a symmetric 2-form on X satisfies the zero-energy condition if all its integrals over the closed geodesics of X vanish; an exact 1-form and the Lie derivative of the metric g along a vector field on X always satisfy the zero-energy condition. The space (X, g) is infinitesimally rigid if the only symmetric 2-forms on X satisfying the zero-energy condition are the Lie derivatives of the metric g.

In this paper, which is a sequel to [6], we investigate the infinitesimal rigidity of a product $X = Y \times Z$ of compact symmetric spaces Y and Z and generalize the results of [6] concerning the product $S^1 \times \mathbf{RP}^n$. We give a criterion for the infinitesimal rigidity of $Y \times Z$ mainly in terms of properties of Y and Z (Theorem 2.1) from which we deduce the infinitesimal rigidity of an arbitrary product $X_1 \times \cdots \times X_r$, where each X_j is either a projective space, different from a sphere, or a flat torus, or a complex quadric of dimension ≥ 5. This englobes all the previously known infinitesimal rigidity results (see [8]) and gives the first known examples of non-flat infinitesimally rigid symmetric spaces of arbitrary rank.

One of the main ingredients of our proofs is the characterization of exact 1-forms on these spaces in terms of closed geodesics. In [14] and [7], it is shown that the 1-forms on a projective space, which is not a sphere, satisfying the zero-energy condition are exact (see also [8]); the corresponding fact for flat tori is given by [13], and for complex quadrics of dimension ≥ 4 by [3].

We consider the product $X = Y \times Z$ and assume that Y and Z are infinitesimally rigid. We also suppose that the 1-forms on Y and Z which satisfy the zero-energy condition are exact. Let h be a symmetric 2-form on X satisfying the zero-energy condition. To prove that h is a Lie derivative of the metric, most of the methods and computations introduced in [6] to treat the case of $S^1 \times \mathbf{RP}^n$, with $n \geq 2$, are used here. Several important new features occur, especially because the dimensions of Y and Z may both be greater than one. We first wish to show that h is locally a Lie derivative of the metric by proving that it lies in the kernel of the differential operator Q_g of order 3 of [4], which is the compatibility condition for the Killing operator. The in-

---

Received April 23, 1987.

finitesimal rigidity of Y and Z implies that we may assume that

$$h(\zeta_1, \zeta_2) = 0,$$

whenever the vectors ζ_1, ζ_2 are tangent to the same factor. We require a crucial additional assumption on h, which always holds if Y is either a projective space, a flat torus or a complex quadric (Lemma 1.9), namely: "averaging h along the closed geodesics of Y" is a C^∞-process which gives rise to another 2-form of the same type. This condition on h is used in verifying the identity (1.15) when Y and Z are both of dimension greater than one. In our proof that $Q_g h = 0$ and our computation of $L^h R$ (see Proposition 1.1), we do not require as in [6] exact formulas for the curvatures of Y and Z.

If the universal covering space of Y or of Z does not admit a Euclidean factor, we give a Künneth type decomposition for the harmonic space of symmetric 2-forms on the product $X = Y \times Z$ (Proposition 2.1), which enables us to conclude that a harmonic 2-form on X satisfying the zero-energy condition vanishes. Standard Hodge theory now gives us the infinitesimal rigidity of X (Theorem 2.1). The infinitesimal rigidity of the flat 2-torus $S^1 \times S^1$ is used in several instances during the course of our proof.

1. The zero-energy condition and local results

Let (X, g) be a Riemannian manifold of dimension n. We shall denote by $T = T_X$ the tangent bundle of X and by $T^* = T_X^*$ the cotangent bundle of X. By $\otimes^k T^*$, $S^k T^*$, we shall mean the k-th tensor product and the k-th symmetric product of T^*, respectively. Let $\nabla = \nabla^X$ be the Levi-Civita connection of g. Throughout this paper, we shall use the results and notations of §1 of [6]. In particular, we denote by $g_1 = g_1^X$ the symbol of the Killing equation of (X, g); it is the sub-bundle of $T^* \otimes T$ whose fiber at $x \in X$ is the Lie algebra of the orthogonal group of the Euclidean vector space $(T_x, g(x))$ (cf. [4, §3]). If X is locally symmetric, the space of Killing vector fields on a connected and simply connected open subset U of X is isomorphic to the space $R_{3,x} = R_{3,x}^X$ of jets of order 3 of Killing vector fields at $x \in U$ (see [4, Theorem 7.1]); moreover, we say that X does not admit a Euclidean factor at $x \in X$ if there exists a neighborhood of x isometric to an open subset of a product $M_+ \times M_-$, where M_+ and M_- are Riemannian globally symmetric spaces of the compact and non-compact type, respectively. If X is a compact symmetric space and $x \in X$, the set $C_{X,x}$ of vectors $\zeta \in T_x - \{0\}$, for which $\mathrm{Exp}_x \mathbf{R}\zeta$ is a closed geodesic of X, is a dense subset of T_x (see [10, Chapter IX, §5]).

Let (Y, g_Y) and (Z, g_Z) be two Riemannian manifolds and suppose that (X, g) is the Riemannian product of (Y, g_Y) and (Z, g_Z); we shall use the notations and conventions, introduced in §2 of [6], concerning the product $Y \times Z$. We shall identify a tensor on Y or Z with the one it determines on X. The musical isomorphisms $T \to T^*$, $T^* \to T$, sending $\xi \in T$ onto $\xi^\flat$ and $\alpha \in T^*$ onto $\alpha^\#$, associated to the metric g, induce isomorphisms

$$T_Y \longrightarrow T_Y^*, \quad T_Y^* \longrightarrow T_Y,$$

$$T_Z \longrightarrow T_Z^*, \quad T_Z^* \longrightarrow T_Z,$$

which are in fact the musical isomorphisms associated to g_Y and g_Z. We also denote by g_1^Y and g_1^Z the sub-bundles $\mathrm{pr}_Y^{-1} g_1^Y$ and $\mathrm{pr}_Z^{-1} g_1^Z$ of $T^* \otimes T$. We consider the isomorphism $\natural \colon T^* \otimes T \to T^* \otimes T$ of vector bundles defined as follows: if $u = \beta \otimes \xi$, with $\beta \in T^*$, $\xi \in T$, then $u^\natural = \xi^\flat \otimes \beta^\#$. The sub-bundle

$$g_1^{Y,Z} = \{ u - u^\natural \mid u \in T_Y^* \otimes T_Z \}$$

of $T^* \otimes T$ is isomorphic to $T_Y^* \otimes T_Z$; moreover, it is clear that $g_1^{Y,Z} \subset g_1$ and that:

LEMMA 1.1. *We have the direct sum*

$$g_1 = g_1^Y \oplus g_1^Z \oplus g_1^{Y,Z}.$$

We now suppose that (Y, g_Y) and (Z, g_Z) are connected and locally symmetric. If $\tilde{G}^Y$ (resp. $\tilde{G}^Z$) is the infinitesimal orbit of the curvature R_Y of (Y, g_Y) (resp. R_Z of (Z, g_Z)) of type $(0,4)$, we identify $\mathrm{pr}_Y^{-1} \tilde{G}_Y$ (resp. $\mathrm{pr}_Z^{-1} \tilde{G}_Z$) with a sub-bundle of G which we also denote by $\tilde{G}_Y$ (resp. $\tilde{G}_Z$). The curvature R of type $(0,4)$ of X is given by the relation $R = R_Y + R_Z$. If we set

$$\tilde{G}^{Y,Z} = \rho(g_1^{Y,Z}) R,$$

we have the surjective mapping

$$T_Y^* \otimes T_Z \to \tilde{G}^{Y,Z}, \tag{1.1}$$

sending u into $\rho(u - u^\natural)R$. Let G_1 denote the sub-bundle of G consisting of the elements ω of G for which $\omega(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = 0$, with $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in T$, whenever all the vectors ζ_i are tangent to the same factor or whenever two of the ζ_i are tangent to Y and the other two to Z. It is easily verified that

$$(1.2)\quad \tilde{G}^{Y,Z} = \left\{ \omega \in G_1 \;\middle|\; \begin{array}{l} \text{there exists } u \in T_Y^* \otimes T_Z \text{ such that} \\ \quad \omega(\xi_1, \eta_1, \eta_2, \eta_3) = R_Z(u(\xi_1), \eta_1, \eta_2, \eta_3), \\ \quad \omega(\eta_1, \xi_1, \xi_2, \xi_3) = -R_Y(u^{\natural}(\eta_1), \xi_1, \xi_2, \xi_3), \\ \text{for all } \xi_1, \xi_2, \xi_3, \in T_Y, \eta_1, \eta_2, \eta_3, \in T_Z \end{array} \right\}.$$

LEMMA 1.2. *Suppose that Y and Z are connected and locally symmetric. Then we have the direct sum*

$$(1.3)\qquad \tilde{G} = \tilde{G}^Y \oplus \tilde{G}^Z \oplus \tilde{G}^{Y,Z}.$$

Let $x = (y, z) \in X$; *if* Y (*or* Z) *does not admit a Euclidean factor at* y (*or* z), *the mapping* (1.1) *is an isomorphism at* x.

Proof. Since

$$\rho(g_1^Y)R = \rho(g_1^Y)R_Y = \tilde{G}_Y,$$
$$\rho(g_1^Z)R = \rho(g_1^Z)R_Z = \tilde{G}_Z,$$

from Lemma 1.1 we obtain (1.3). If Y or Z satisfies the additional hypothesis at y or at z, by [10, Chapters V and VII] we see that

$$\dim R_{3,x} = \dim R^Y_{3,y} + \dim R^Z_{3,z}.$$

From the exactness of the sequence (5.4) of [4], it follows that

$$\dim \tilde{G}_x = \dim \tilde{G}^Y_y + \dim \tilde{G}^Z_z + \dim Y \cdot \dim Z;$$

we now deduce from this relation that (1.1) is an isomorphism at x.

We identify $T_Y^* \otimes T_Z^*$ with its image by the monomorphism of vector bundles $\iota\colon T_Y^* \otimes T_Z^* \to S^2T^*$ over X defined by

$$(\iota v)(\zeta_1, \zeta_2) = v(\zeta_1^Y, \zeta_2^Z) + v(\zeta_2^Y, \zeta_1^Z),$$

for $v \in T_Y^* \otimes T_Z^*, \zeta_1, \zeta_2 \in T$.

Assume that Y and Z are compact, connected locally symmetric spaces. Since the sequence (1.3) of [6] is the initial part of an elliptic complex, if Y (or Z) is infinitesimally rigid, then this property holds with parameters.

LEMMA 1.3. *Assume that Y and Z are infinitesimally rigid, and that Y or Z is a compact symmetric space. Let k be a symmetric 2-form on X satisfying the zero-energy condition and $x_0 \in X$. Then there exist a section h of $T_Y^* \otimes T_Z^*$ over X, with $h(x_0) = 0$, and a vector field ζ on X such that*

$$k = h + \mathscr{L}_\zeta g.$$

Proof. We write $k = k_1 + k_2 + k_3$, where k_1, k_2, k_3 are sections of $S^2T_Y^*$, $T_Y^* \otimes T_Z^*$ and $S^2T_Z^*$ respectively. For all $y \in Y$ and $z \in Z$, the restrictions of k_1 to $Y \times \{z\}$ and of k_2 to $\{y\} \times Z$ satisfy the zero-energy condition. Since Y and Z are infinitesimally rigid, there exist sections ξ of T_Y and η_1 of T_Z over X such that $\mathscr{L}_\xi g - k_1$ and $\mathscr{L}_{\eta_1} g - k_3$ are sections of $T_Y^* \otimes T_Z^*$. Then $\zeta_1 = \xi + \eta_1$ is a vector field on X and $h_1 = k - \mathscr{L}_{\zeta_1} g$ is a section of $T_Y^* \otimes T_Z^*$. We may assume without loss of generality that Z is a compact globally symmetric space. Let $\mathfrak{g}_Z$ denote the Lie algebra of Killing vector fields of Z and $C^\infty(Y, \mathfrak{g}_Z)$ the space of $\mathfrak{g}_Z$-valued functions on Y. We may also consider an element η of $C^\infty(Y, \mathfrak{g}_Z)$ as a section of T_Z over X; it is easily verified that $\mathscr{L}_\eta g$ is the section of $T_Y^* \otimes T_Z^*$ equal to the exterior derivative $d_Y \eta^\flat$ of the function $\eta^\flat$ on Y. Since Z is globally symmetric, for $z \in Z$ the mapping $\mathfrak{g}_Z \to T_{Z,z}$, sending η into $\eta(z)$, is surjective. Therefore, there exists a section η_2 of $C^\infty(Y, \mathfrak{g}_Z)$ such that

$$(\mathscr{L}_{\eta_2} g)(x_0) = (d_Y \eta_2^\flat)(x_0) = h_1(x_0).$$

Then $\zeta = \zeta_1 - \eta_2$ and $h = h_1 - \mathscr{L}_{\eta_2} g$ satisfy the desired conditions.

Let h be a section of $T_Y^* \otimes T_Z^*$. If $\zeta \in T$, we denote by h_ζ the element of T^* defined by the relation $h_\zeta(\zeta') = h(\zeta, \zeta')$, for $\zeta' \in T$; if $\zeta \in T_Y$ (resp. T_Z), then h_ζ belongs to T_Z^* (resp. T_Y^*).

For the remainder of this section, we consider a section h of $T_Y^* \otimes T_Z^*$. We have

$$\begin{aligned} \nu(h)(\xi_1, \xi_2, \xi_3, \xi_4) = \nu(h)(\eta_1, \eta_2, \eta_3, \eta_4) = 0, \\ \nu(h)(\xi_1, \eta_1, \zeta_1, \zeta_2) = 0, \end{aligned} \tag{1.4}$$

for $\xi_1, \xi_2, \xi_3, \xi_4 \in T_Y$, $\eta_1, \eta_2, \eta_3, \eta_4 \in T_Z$, $\zeta_1, \zeta_2 \in T$. We take this opportunity to point out that equation (3.1) of [6] is not correct and should be replaced by

$$\nu(h)(\xi_1, \xi_2, \xi_3, \xi_4) = \nu(h)(\partial_\theta, \xi_1, \partial_\theta, \xi_2) = \nu(h)(\partial_\theta, \xi_1, \xi_2, \xi_3) = 0,$$

for all $\xi_1, \xi_2, \xi_3, \xi_4 \in T_Z$, and that one must add the term $-\frac{1}{2}\nu(h)$ to the right-hand side of equation (3.2) of [6] and replace $1/(n+1)$ by $1/(n-1)$ there. By formulas (1.5) and (1.4) of [6], a computation similar to the one resulting in equation (3.2) of [6] yields the relations

$$(D_g h)(\xi_1, \xi_2, \xi_3, \xi_4) = (D_g h)(\eta_1, \eta_2, \eta_3, \eta_4) = 0, \tag{1.5}$$

$$\begin{aligned}(D_g h)(\xi_1, \eta_1, \xi_2, \eta_2) = -\tfrac{1}{2}\{&(\nabla^2 h)(\xi_1, \eta_1, \xi_2, \eta_2) \\ &+ (\nabla^2 h)(\xi_2, \eta_2, \xi_1, \eta_1)\},\end{aligned} \tag{1.6}$$

$$\begin{aligned}(D_g h)&(\xi, \eta_1, \eta_2, \eta_3) \\ &= \tfrac{1}{2}\{(\nabla^2 h)(\eta_1, \eta_3, \xi, \eta_2) - (\nabla^2 h)(\eta_1, \eta_2, \xi, \eta_3)\} \\ &= \tfrac{1}{2}\{(\nabla^2 h)(\eta_3, \eta_1, \xi, \eta_2) - (\nabla^2 h)(\eta_2, \eta_1, \xi, \eta_3) \\ &\qquad + R_Z(h^\#_\xi, \eta_1, \eta_2, \eta_3)\},\end{aligned} \tag{1.7}$$

for $\xi, \xi_1, \xi_2, \xi_3, \xi_4 \in T_Y$ and $\eta_1, \eta_2, \eta_3, \eta_4 \in T_Z$; similarly, we have

$$\begin{aligned}(D_g h)&(\eta, \xi_1, \xi_2, \xi_3) \\ &= \tfrac{1}{2}\{(\nabla^2 h)(\xi_3, \xi_1, \xi_2, \eta) - (\nabla^2 h)(\xi_2, \xi_1, \xi_3, \eta) \\ &\qquad + R_Y(h^\#_\eta, \xi_1, \xi_2, \xi_3)\},\end{aligned} \tag{1.8}$$

for $\xi_1, \xi_2, \xi_3 \in T_Y$, $\eta \in T_Z$.

For the remainder of this paper, we assume that Y and Z are compact symmetric spaces.

LEMMA 1.4. *Let k be a symmetric 2-form on X satisfying the zero-energy condition. Then we have*

$$(D_g k)(\xi, \eta_1, \xi, \eta_2) + (D_g k)(\xi, \eta_2, \xi, \eta_1) = 0,$$
$$(D_g k)(\xi_1, \eta, \xi_2, \eta) + (D_g k)(\xi_2, \eta, \xi_1, \eta) = 0,$$

for $\xi, \xi_1, \xi_2 \in T_Y$, $\eta, \eta_1, \eta_2 \in T_Z$.

Proof. Let $x = (y, z) \in X$ and $\xi \in C_{Y,y}$, $\eta \in C_{Z,z}$. Then

$$\Gamma = \operatorname{Exp}_x(\mathbf{R}\xi \oplus \mathbf{R}\eta)$$

is a flat 2-torus totally geodesic in X. If $i\colon \Gamma \to X$ is the natural imbedding, then i^*k satisfies the zero-energy condition on Γ. According to [13], there is a vector field ζ on Γ such that $i^*k = \mathcal{L}_\zeta(i^*g)$. Since the sequence (1.7) of [6] is a complex, we see that $D_{i^*g}(i^*k) = 0$; from formula (1.8) of [6], we deduce that

$$(1.9) \qquad \big(D_g k\big)(\xi, \eta, \xi, \eta) = 0.$$

Since $C_{Y,y}$ is dense in $T_{Y,y}$ and $C_{Z,z}$ is dense in $T_{Z,z}$, (1.9) holds for all $\xi \in T_{Y,y}$, $\eta \in T_{Z,z}$ and we thus obtain the desired result.

If h satisfies the zero-energy condition, according to (1.6) and Lemma 1.4, we see that

$$(1.10) \qquad \big(D_g h\big)(\xi, \eta_1, \xi, \eta_2) = \big(D_g h\big)(\xi_1, \eta, \xi_2, \eta) = 0,$$

for all $\xi, \xi_1, \xi_2 \in T_Y$, $\eta, \eta_1, \eta_2 \in T_Z$.

If $y \in Y$ and $\xi \in C_{Y,y}$, we define a 1-form ω_ξ on Z by

$$\omega_\xi(\eta) = \frac{1}{L}\int_0^L h\big(\dot\gamma(t), \eta\big)\, dt,$$

for $\eta \in T_Z$, where $\gamma(t) = \mathrm{Exp}_y t\xi$ and $\dot\gamma(t)$ is the tangent vector to the closed geodesic γ of period L. We have $\omega_{\lambda\xi} = \lambda\omega_\xi$, for $\lambda \in \mathbf{R}$, with $\lambda \neq 0$.

The proof of Lemma 3.2 of [6] gives us the following:

LEMMA 1.5. *Assume that h satisfies the zero-energy condition. If $y \in Y$ and $\xi \in C_{Y,y}$, the 1-form ω_ξ on Z satisfies the zero-energy condition.*

The following lemma is a consequence of Lemma 1.5; its proof is similar to that of identity (3.9) of [6] and shall be omitted.

LEMMA 1.6. *Assume that h satisfies the zero-energy condition, and that the 1-forms on Z which satisfy the zero-energy condition are closed. Then we have*

$$(1.11) \qquad \tfrac{1}{2}\{(\nabla h)(\eta_1, \xi, \eta_2) + (\nabla h)(\eta_2, \xi, \eta_1)\} = \big(\nabla^Z \omega_\xi\big)(\eta_1, \eta_2),$$

for all $y \in Y$, $\xi \in C_{Y,y}$, $\eta_1, \eta_2 \in T_Z$.

Under the hypotheses of Lemma 1.6, if there is a section h_1 of $T_Y^* \otimes T_Z^*$ such that $h_1(\xi, \eta) = \omega_\xi(\eta)$ for all $y \in Y$, $\xi \in C_{Y,y}$, $\eta \in T_Z$, then, for $\eta_1, \eta_2 \in T_Z$, $y \in Y$, by Lemma 1.6 we have

$$(1.12) \qquad \tfrac{1}{2}\{(\nabla h)(\eta_1, \xi, \eta_2) + (\nabla h)(\eta_2, \xi, \eta_1)\} = (\nabla h_1)(\eta_1, \xi, \eta_2),$$

for all $\xi \in C_{Y,y}$; since $C_{Y,y}$ is dense in $T_{Y,y}$, this identity is then valid for all $\xi \in T_{Y,y}$.

Similarly, if $z \in Z$ and $\eta \in C_{Z,z}$, we define a 1-form β_η on Y by

$$\beta_\eta(\xi) = \frac{1}{L}\int_0^L h(\xi, \dot\gamma(t))\,dt,$$

for $\xi \in T_Y$, where $\gamma(t) = \mathrm{Exp}_z t\eta$ and $\dot\gamma(t)$ is the tangent vector to the closed geodesic γ of period L. We have $\beta_{\lambda\eta} = \lambda\beta_\eta$, for $\lambda \in \mathbf{R}$, with $\lambda \neq 0$.

LEMMA 1.7. *Suppose that h satisfies the zero-energy condition. If $y \in Y$, $z \in Z$ and $\xi \in C_{Y,y}$, $\eta \in C_{Z,z}$, we have*

$$\omega_\xi(\eta) + \beta_\eta(\xi) = h(\xi, \eta).$$

Proof. We may assume without loss of generality that $\|\xi\| = \|\eta\| = 1$. Set $\gamma_1(t) = \mathrm{Exp}_y t\xi$, $\gamma_2(t) = \mathrm{Exp}_z t\eta$ and let L_1, L_2 be the lengths of the closed geodesics γ_1 and γ_2, respectively. Consider the flat 2-torus $\Gamma = S^1 \times S^1$, where the first factor has length L_1 and the second has length L_2, and the totally geodesic imbedding $i\colon \Gamma \to X$ sending (θ_1, θ_2) into $(\gamma_1(\theta_1), \gamma_2(\theta_2))$. We identify a tensor on Γ with the corresponding doubly periodic tensor on the (θ_1, θ_2)-plane. According to Michel [13], there exists a vector field

$$\zeta = A_1(\theta_1, \theta_2)\frac{\partial}{\partial\theta_1} + A_2(\theta_1, \theta_2)\frac{\partial}{\partial\theta_2}$$

on Γ such that

$$\mathscr{L}_\zeta i^*g = \frac{\partial A_1}{\partial\theta_1}\,d\theta_1^2 + \left(\frac{\partial A_1}{\partial\theta_2} + \frac{\partial A_2}{\partial\theta_1}\right) d\theta_1 \cdot d\theta_2 + \frac{\partial A_2}{\partial\theta_2}\,d\theta_2^2 = i^*h.$$

Thus we see that $A_1 = A_1(\theta_2)$, $A_2 = A_2(\theta_1)$ and that

$$h(\dot\gamma_1(\theta_1), \dot\gamma_2(\theta_2)) = \frac{dA_1}{d\theta_2} + \frac{dA_2}{d\theta_1},$$

where $\dot\gamma_1(\theta_1)$, $\dot\gamma_2(\theta_2)$ are the tangent vectors to the geodesics γ_1, γ_2. Therefore

$$h(\xi, \eta) = h(\dot\gamma_1(0), \dot\gamma_2(0)) = \frac{dA_1}{d\theta_2}(0) + \frac{dA_2}{d\theta_1}(0).$$

On the other hand, we have

$$\begin{aligned}\omega_\xi(\eta) &= \frac{1}{L_1}\int_0^{L_1} h(\dot\gamma_1(\theta_1), \dot\gamma_2(0))\,d\theta_1 \\ &= \frac{1}{L_1}\int_0^{L_1}\left(\frac{dA_1}{d\theta_2}(0) + \frac{dA_2}{d\theta_1}(\theta_1)\right) d\theta_1 \\ &= \frac{dA_1}{d\theta_2}(0);\end{aligned}$$

similarly, we obtain

$$\beta_\eta(\xi) = \frac{dA_2}{d\theta_1}(0),$$

and the desired equality.

LEMMA 1.8. *Suppose that h satisfies the zero-energy condition. Let $y \in Y$, $z \in Z$ and $\xi_1, \xi_2 \in C_{Y,y}$. If $\xi_1 + \xi_2 \in C_{Y,y}$, for $\eta \in T_{Z,z}$ we have*

$$\omega_{\xi_1}(\eta) + \omega_{\xi_2}(\eta) = \omega_{\xi_1+\xi_2}(\eta).$$

Proof. Since $C_{Z,z}$ is dense in $T_{Z,z}$, we may assume that $\eta \in C_{Z,z}$. Then by Lemma 1.7, we have

$$\begin{aligned}\omega_{\xi_1}(\eta) + \omega_{\xi_2}(\eta) &= h(\xi_1, \eta) - \beta_\eta(\xi_1) + h(\xi_2, \eta) - \beta_\eta(\xi_2) \\ &= h(\xi_1 + \xi_2, \eta) - \beta_\eta(\xi_1 + \xi_2) = \omega_{\xi_1+\xi_2}(\eta).\end{aligned}$$

LEMMA 1.9. *Suppose that h satisfies the zero-energy condition and that there exists a C^∞-section h_1 of $T_Y^* \otimes T_Z^*$ such that*

$$h_1(\xi, \eta) = \omega_\xi(\eta), \tag{1.13}$$

for all $y \in Y$, $\xi \in C_{Y,y}$ and $\eta \in T_Z$. Then there exists a unique C^∞-section h_2 of $T_Y^* \otimes T_Z^*$ such that

$$h_2(\xi, \eta) = \beta_\eta(\xi), \tag{1.14}$$

for all $\xi \in T_Y$, $z \in Z$ and $\eta \in C_{Z,z}$; moreover, $h = h_1 + h_2$.

Proof. We set $h_2 = h - h_1$; then by Lemma 1.7, if $y \in Y$, $z \in Z$, $\eta \in C_{Z,z}$, we have (1.14) for all $\xi \in C_{Y,y}$ and, since $C_{Y,y}$ is dense in $T_{Y,y}$, for all $\xi \in T_{Y,y}$.

We always consider the projective spaces endowed with their canonical metrics as in [1]. In particular, the metric on the complex projective space $\mathbf{CP}^n$ is the Fubini-Study metric with constant holomorphic curvature 4. We also consider the complex quadric Q_n, which is the hypersurface of $\mathbf{CP}^{n+1}$, with $n \geq 3$, defined by the equation

$$\zeta_0^2 + \zeta_1^2 + \cdots + \zeta_{n+1}^2 = 0$$

in terms of the homogeneous coordinates $\zeta_0, \zeta_1, \ldots, \zeta_{n+1}$; the metric on Q_n is that induced by the Fubini-Study metric of $\mathbf{CP}^{n+1}$. If $Y = Q_n$, a field ν of unit tangent vectors of the hypersurface Y of $\mathbf{CP}^{n+1}$, normal to Y and defined on an open subset U of Y, determines an involution K of $T_{Y|U}$ and a decomposition

$$T_{Y|U} = T^+ \oplus T^-,$$

where T^+, T^- are the sub-bundles of $T_{Y|U}$ consisting of the eigenvectors of K corresponding to the eigenvalues $+1$ and -1, respectively (see [8]). According to [3], if $y \in U$ and F is the subspace of $T_{Y,y}$ generated by an orthonormal set $\{\xi, \eta\}$ of vectors of T_y^+ or of T_y^-, then $\mathrm{Exp}_y F$ is a closed totally geodesic surface of Y isometric to the sphere S^2 of constant curvature 2. It follows that, if ξ is a non-zero vector of T_y^+ or of T_y^-, then $\mathrm{Exp}_y \mathbf{R}\xi$ is a closed geodesic of Y of length $\pi\sqrt{2}$.

LEMMA 1.10. *Assume that Y is either a projective space, different from a sphere, or a flat torus, or a complex quadric Q_n, with $n \geq 3$. If h satisfies the zero-energy condition, there exists a unique C^∞-section h_1 of $T_Y^* \otimes T_Z^*$ satisfying the relation* (1.13).

Proof. If Y is a projective space, different from a sphere, the geodesic flow φ_s of Y is periodic of period π. In this case, we define a C^∞-function h_1 on $(T_Y - \{0\}) \times T_Z$ by

$$h_1(\xi, \eta) = \frac{1}{\pi}\int_0^\pi h(\varphi_s \xi, \eta)\, ds,$$

for $\xi \in T_Y - \{0\}$, $\eta \in T_Z$; clearly (1.13) holds, since $C_{Y,y} = T_{Y,y} - \{0\}$, for $y \in Y$. We set $h_1(\xi, \eta) = 0$, for $\xi \in T_Y$, $\eta \in T_Z$, whenever ξ vanishes. If $y \in Y$ and $\xi_1, \xi_2 \in T_{Y,y} - \{0\}$, with $\xi_1 + \xi_2 \neq 0$, by Lemma 1.8 we have

$$h_1(\xi_1, \eta) + h_1(\xi_2, \eta) = h_1(\xi_1 + \xi_2, \eta),$$

for all $\eta \in T_Z$. Therefore, since $h_1(\lambda\xi_1, \eta) = \lambda h_1(\xi_1, \eta)$, for all $\lambda \in \mathbf{R}$, $\eta \in T_{Z,z}$, we see that h_1 is a C^∞-section of $T_Y^* \otimes T_Z^*$. If Y is a flat torus $\mathbf{R}^q/\Gamma$, where Γ is a lattice of maximal rank in $\mathbf{R}^q$, choose a basis $e_1, \ldots, e_q$ of $\mathbf{R}^q$ generating Γ and let $\{\theta_1, \ldots, \theta_q\}$ be the corresponding coordinate system. Then the vector fields $\partial_i = \partial/\partial\theta_i$ and the 1-forms $d\theta_i$ on $\mathbf{R}^q$ induce tensors on Y which we denote in the same way. We define a C^∞-section h_1 of $T_Y^* \otimes T_Z^*$ over X by

$$h_1(\xi, \eta) = \sum_{i=1}^{q} a_i \omega_{\partial_i}(\eta),$$

where $\xi = \sum_{i=1}^q a_i \partial_i$ is an element of T_Y and η of T_Z; since $\{\partial_1, \ldots, \partial_q\}$ is a global frame for Y, we see that h_1 is differentiable. If $y \in Y$ and $\xi = \sum_{i=1}^q p_i \partial_i$, where $p_1, \dots, p_q \in \mathbf{Z}$, then $\xi \in C_{Y,y}$ and by Lemma 1.8 we see that

$$\omega_\xi(\eta) = \sum_{i=1}^{q} p_i \omega_{\partial_i}(\eta),$$

for all $\eta \in T_Z$. From this relation, we deduce that (1.13) holds. Finally, suppose that Y is the complex quadric Q_n, with $n \geq 3$. Let $y \in Y$ and ν be a field of unit tangent vectors on the hypersurface Y of $\mathbf{CP}^{n+1}$, normal to Y and defined on a neighborhood U of y. Consider the sub-bundles T^+ and T^- of $T_{Y|U}$ determined by ν. If φ_s is the geodesic flow of Y, we define C^∞-functions h_1^+ on $(T^+ - \{0\}) \times T_Z$ and h_1^- on $(T^- - \{0\}) \times T_Z$ by

$$h_1^+(\xi, \eta) = \frac{1}{L}\int_0^L h(\varphi_s\xi, \eta),$$

$$h_1^-(\zeta, \eta) = \frac{1}{L}\int_0^L h(\varphi_s\zeta, \eta)\, ds,$$

for $\xi \in T^+ - \{0\}$, $\zeta \in T^- - \{0\}$ and $\eta \in T_Z$, where $L = \pi\sqrt{2}$. According to the remarks preceding the lemma, for all $a \in U$, the non-zero vectors of T_a^+ and T_a^- belong to $C_{Y,a}$ and

$$h_1^+(\xi, \eta) = \omega_\xi(\eta), \quad h_1^-(\zeta, \eta) = \omega_\zeta(\eta),$$

for all $\xi \in T^+ - \{0\}$, $\zeta \in T^- - \{0\}$ and $\eta \in T_Z$. We set $h_1^+(\xi, \eta) = 0$ and $h_1^-(\zeta, \eta) = 0$, for $\xi \in T^+$, $\zeta \in T^-$ and $\eta \in T_Z$, whenever ξ and ζ vanish. By Lemma 1.8, we have

$$h_1^+(\xi_1, \eta) + h_1^+(\xi_2, \eta) = h_1^+(\xi_1 + \xi_2, \eta),$$
$$h_1^-(\zeta_1, \eta) + h_1^-(\zeta_2, \eta) = h_1^-(\zeta_1 + \zeta_2, \eta),$$

for all $\xi_1, \xi_2 \in T^+ - \{0\}$, $\zeta_1, \zeta_2 \in T^- - \{0\}$, whenever $\xi_1 + \xi_2 \neq 0$ and $\zeta_1 + \zeta_2 \neq 0$. Therefore, since

$$h_1^+(\lambda\xi, \eta) = \lambda h_1^+(\xi, \eta), \quad h_1^-(\lambda\zeta, \eta) = \lambda h_1^-(\zeta, \eta),$$

for all $\lambda \in \mathbf{R}$, $\xi \in T^+$, $\zeta \in T^-$ and $\eta \in T_Z$, the function h_1 on $T_{Y|U} \times T_Z$, defined by

$$h_1(\xi, \eta) = h_1^+(\xi^+, \eta) + h_1^-(\xi^-, \eta),$$

for $\xi \in T_{Y|U}$, $\eta \in T_Z$, where $\xi = \xi^+ + \xi^-$ is the decomposition of ξ, with $\xi^+ \in T^+$ and $\xi^- \in T^-$, is a C^∞-section of $T_Y^* \otimes T_Z^*$ over $U \times Z$. Now let ξ be an element of $C_{Y,a}$, with $a \in U$; we write $\xi = \xi^+ + \xi^-$, where $\xi^+ \in T^+$ and $\xi^- \in T^-$. If ξ^+ or ξ^- vanishes, then we know that (1.13) holds for all $\eta \in T_Z$.

If ξ^+ and ξ^- are both non-zero, by Lemma 1.8, we see that

$$\begin{aligned} h_1(\xi, \eta) &= h_1^+(\xi^+, \eta) + h_1^-(\xi^-, \eta) \\ &= \omega_{\xi^+}(\eta) + \omega_{\xi^-}(\eta) \\ &= \omega_\xi(\eta), \end{aligned}$$

for all $\eta \in T_Z$. As $C_{Y,a}$ is dense in $T_{Y,a}$, these relations give us the uniqueness of h_1 on $U \times Z$, and thus there exists a global section h_1 of $T_Y^* \otimes T_Z^*$ over X satisfying (1.13).

PROPOSITION 1.1. *Assume that the* 1*-forms on Y and Z satisfying the zero-energy condition are closed. Suppose that h satisfies the zero-energy condition and that there exists a C^∞-section h_1 of $T_Y^* \otimes T_Z^*$ satisfying the relation* (1.13). *Then we have*

$$(D_g h)(\xi_1, \eta_1, \xi_2, \eta_2) = 0, \tag{1.15}$$

$$(D_g h)(\xi, \eta_1, \eta_2, \eta_3) = R_Z\big(h^\sharp_{2,\xi}, \eta_1, \eta_2, \eta_3\big), \tag{1.16}$$

$$(D_g h)(\eta, \xi_1, \xi_2, \xi_3) = R_Y\big(h^\sharp_{1,\eta}, \xi_1, \xi_2, \xi_3\big), \tag{1.17}$$

for all $\xi, \xi_1, \xi_2, \xi_3 \in T_Y$, $\eta, \eta_1, \eta_2, \eta_3 \in T_Z$, and

$$D_2 h = 0.$$

Moreover, if h vanishes at x_0, then

$$(D_1 h)(x_0) = 0.$$

Proof. Because of (1.13) and our hypothesis on Z, by Lemma 1.6 we know that (1.12) holds. Hence by (1.10) and (1.6), we have

$$\begin{aligned} 0 &= (D_g h)(\xi, \eta_1, \xi, \eta_2) \\ &= -\tfrac{1}{2}\{(\nabla^2 h)(\xi, \eta_1, \xi, \eta_2) + (\nabla^2 h)(\xi, \eta_2, \xi, \eta_1)\} \\ &= -(\nabla^2 h_1)(\xi, \eta_1, \xi, \eta_2), \end{aligned}$$

for $\xi \in T_Y$, $\eta_1, \eta_2 \in T_Z$. By our hypothesis on Y, by Lemma 1.6 the analogue of (1.12) holds for h_2; namely, we have

$$\tfrac{1}{2}\{(\nabla h)(\xi_1, \xi_2, \eta) + (\nabla h)(\xi_2, \xi_1, \eta)\} = (\nabla h_2)(\xi_1, \xi_2, \eta), \tag{1.18}$$

for $\xi_1, \xi_2 \in T_Y$, $\eta \in T_Z$. Therefore by (1.10) and (1.6), we also have

$$\left(\nabla^2 h_2\right)(\xi_1, \eta, \xi_2, \eta) = 0,$$

for $\xi_1, \xi_2 \in T_Y$, $\eta \in T_Z$. Thus by (1.12), (1.18) and the above relations, for $\xi_1, \xi_2 \in T_Y$, $\eta_1, \eta_2 \in T_Z$, we see that

$$\left(\nabla^2 h_1\right)(\xi_1, \eta_1, \xi_2, \eta_2)$$

is symmetric in η_1, η_2 and skew-symmetric in ξ_1, ξ_2, while

$$\left(\nabla^2 h_2\right)(\xi_1, \eta_1, \xi_2, \eta_2)$$

is symmetric in ξ_1, ξ_2 and skew-symmetric in η_1, η_2. Hence since $h = h_1 + h_2$, by (1.6) we have

$$\begin{aligned}(D_g h)(\xi_1, \eta_1, \xi_2, \eta_2) = -\tfrac{1}{2}\Big\{&\left(\nabla^2 h_1\right)(\xi_1, \eta_1, \xi_2, \eta_2) + \left(\nabla^2 h_2\right)(\xi_1, \eta_1, \xi_2, \eta_2)\\ &+\left(\nabla^2 h_1\right)(\xi_2, \eta_2, \xi_1, \eta_1) + \left(\nabla^2 h_2\right)(\xi_2, \eta_2, \xi_1, \eta_1)\Big\}\\ = 0.\end{aligned}$$

By (1.7) and (1.12), we obtain

$$\begin{aligned}(D_g h)(\xi, \eta_1, \eta_2, \eta_3) &= \tfrac{1}{2}\Big\{(\nabla^2 h)(\eta_2, \eta_3, \xi, \eta_1) - (\nabla^2 h)(\eta_3, \eta_2, \xi, \eta_1)\\ &\qquad +R_Z\left(h^\sharp_\xi, \eta_1, \eta_2, \eta_3\right)\Big\}\\ &\qquad +\left(\nabla^2 h_1\right)(\eta_3, \eta_1, \xi, \eta_2) - \left(\nabla^2 h_1\right)(\eta_2, \eta_1, \xi, \eta_3)\\ &= R_Z\left(h^\sharp_\xi - h^\sharp_{1,\xi}, \eta_1, \eta_2, \eta_3\right)\\ &= R_Z\left(h^\sharp_{2,\xi}, \eta_1, \eta_2, \eta_3\right),\end{aligned}$$

for all $\xi \in T_Y$, $\eta_1, \eta_2, \eta_3 \in T_Z$; similarly, from (1.8) and (1.18), we deduce (1.17). We now compute $L^h R$. Let $\eta_1, \eta_2, \eta_3 \in T_Z$; we set $\eta = \tilde{R}_Z(\eta_2, \eta_3)\eta_1$. For $\zeta \in T$, $\xi \in T_Y$, by formula (4.8) of [4], we have

$$\begin{aligned}(L^h R)(\zeta, \xi, \eta_1, \eta_2, \eta_3) &= -R\left(L^h_\zeta \xi, \eta_1, \eta_2, \eta_3\right)\\ &= \tfrac{1}{2}\{(\nabla h)(\zeta, \xi, \eta) + (\nabla h)(\xi, \zeta, \eta) - (\nabla h)(\eta, \xi, \zeta)\}.\end{aligned}$$

If $\zeta \in T_Y$, then by (1.18) we see that

$$\begin{aligned}(L^h R)(\zeta, \xi, \eta_1, \eta_2, \eta_3) &= \tfrac{1}{2}\{(\nabla h)(\zeta, \xi, \eta) + (\nabla h)(\xi, \zeta, \eta)\}\\ &= (\nabla h_2)(\zeta, \xi, \eta);\end{aligned}$$

on the other hand, if $\zeta \in T_Z$, then by (1.12) we have

$$\begin{aligned}(L^hR)(\zeta, \xi, \eta_1, \eta_2, \eta_3) &= \tfrac{1}{2}\{(\nabla h)(\zeta, \xi, \eta) - (\nabla h)(\eta, \xi, \zeta)\} \\ &= (\nabla h)(\zeta, \xi, \eta) - (\nabla h_1)(\zeta, \xi, \eta) \\ &= (\nabla h_2)(\zeta, \xi, \eta).\end{aligned}$$

Since $\nabla R_Z = 0$, from the above relations we deduce that

$$(L^hR)(\zeta, \xi, \eta_1, \eta_2, \eta_3) = -R_Z\big((\nabla h_2)^{\#}_{\zeta,\xi}, \eta_1, \eta_2, \eta_3\big),$$

for all $\zeta \in T$, $\xi \in T_Y$, where $(\nabla h_2)_{\zeta,\xi}$ is the element of T_Z^* defined by

$$(\nabla h_2)_{\zeta,\xi}(\eta') = (\nabla h_2)(\zeta, \xi, \eta'),$$

for $\eta' \in T_Z$. If $\zeta_1, \zeta_2 \in T_Z$, by formula (4.8) of [4], we have

$$\begin{aligned}R\big(L^h_{\zeta_1}\zeta_2, \eta_1, \eta_2, \eta_3\big) = -\tfrac{1}{2}\{&(\nabla h)(\zeta_1, \zeta_2, \eta) + (\nabla h)(\zeta_2, \eta, \zeta_1) \\ &-(\nabla h)(\eta, \zeta_1, \zeta_2)\},\end{aligned}$$

and so we obtain

$$(L^hR)(\zeta_1, \zeta_2, \eta_1, \eta_2, \eta_3) = 0.$$

Similarly, we have

$$\begin{aligned}(L^hR)(\zeta, \eta, \xi_1, \xi_2, \xi_3) &= -R_Y\big((\nabla h_1)^{\#}_{\zeta,\eta}, \xi_1, \xi_2, \xi_3\big), \\ (L^hR)(\zeta_1, \zeta_2, \xi_1, \xi_2, \xi_3) &= 0,\end{aligned}$$

for all $\zeta \in T$, $\xi_1, \xi_2, \xi_3, \zeta_1, \zeta_2 \in T_Y$, $\eta \in T_Z$, where $(\nabla h_1)_{\zeta,\eta}$ is the element of T_Y^* defined by

$$(\nabla h_1)_{\zeta,\eta}(\xi) = (\nabla h_1)(\zeta, \xi, \eta),$$

for $\xi \in T_Y$. Moreover, since $R = R_Y + R_Z$, for $\zeta \in T$, $\xi_1, \xi_2 \in T_Y$, $\eta_1, \eta_2 \in T_Z$, we easily see that

$$(L^hR)(\zeta, \xi_1, \eta_1, \xi_2, \eta_2) = 0.$$

Since $\nabla R = 0$, from (1.16) and (1.17), we deduce that

$$\begin{aligned}(\nabla D_g h)(\zeta, \xi, \eta_1, \eta_2, \eta_3) &= R_Z\big((\nabla h_2)^{\#}_{\zeta,\xi}, \eta_1, \eta_2, \eta_3\big), \\ (\nabla D_g h)(\zeta, \eta, \xi_1, \xi_2, \xi_3) &= R_Y\big((\nabla h_1)^{\#}_{\zeta,\eta}, \xi_1, \xi_2, \xi_3\big),\end{aligned}$$

for all $\zeta \in T$, $\xi, \xi_1, \xi_2, \xi_3 \in T_Y$ and $\eta, \eta_1, \eta_2, \eta_3 \in T_Z$. From all these relations involving $\nabla D_g h$ and $L^h R$ and from (1.4), (1.5) and (1.15), by formula (1.9) of [6] we obtain

$$(D_2h)(\xi, \xi_1, \xi_2, \xi_3, \xi_4) = (D_2h)(\eta, \eta_1, \eta_2, \eta_3, \eta_4) = 0,$$
$$(D_2h)(\zeta, \xi_1, \eta_1, \xi_2, \eta_2) = 0,$$
$$(D_2h)(\zeta, \xi, \eta_1, \eta_2, \eta_3) = (D_2h)(\zeta, \eta, \xi_1, \xi_2, \xi_3) = 0,$$

for all $\xi, \xi_1, \xi_2, \xi_3, \xi_4 \in T_Y$, $\eta, \eta_1, \eta_2, \eta_3, \eta_4 \in T_Z$ and $\zeta \in T$. Since D_2h is a section of H, these relations imply that $D_2h = 0$. If $h(x_0) = 0$, we define elements $u \in (T_Y^* \otimes T_Z)_{x_0}$, $v \in (T_Z^* \otimes T_Y)_{x_0}$ by

$$u(\xi) = h^\sharp_{2,\xi}, \qquad v(\eta) = h^\sharp_{1,\eta},$$

for $\xi \in T_{Y, x_0}$, $\eta \in T_{Z, x_0}$; then by (1.16) and (1.17), we have

$$(D_gh)(\xi, \eta_1, \eta_2, \eta_3) = R_Z(u(\xi), \eta_1, \eta_2, \eta_3),$$
$$(D_gh)(\eta, \xi_1, \xi_2, \xi_3) = R_Y(v(\eta), \xi_1, \xi_2, \xi_3),$$

for all $\xi, \xi_1, \xi_2, \xi_3 \in T_{Y, x_0}$, $\eta, \eta_1, \eta_2, \eta_3 \in T_{Z, x_0}$. As $h(x_0) = 0$, we know that $(D_gh)(x_0) \in G$, and that $(D_1h)(x_0) = 0$ if and only if $(D_gh)(x_0) \in \tilde{G}$. According to Lemma 1.2, (1.2), (1.5) and (1.15), this last condition holds if $v = -u^\natural$; this equality is true, since

$$\begin{aligned} g(u(\xi), \eta) + g(\xi, v(\eta)) &= g\left(h^\sharp_{2,\xi}, \eta\right) + g\left(\xi, h^\sharp_{1,\eta}\right) \\ &= h_2(\xi, \eta) + h_1(\xi, \eta) \\ &= h(\xi, \eta) \\ &= 0, \end{aligned}$$

for $\xi \in T_{Y, x_0}$, $\eta \in T_{Z, x_0}$. Thus $(D_1h)(x_0) = 0$.

PROPOSITION 1.2. *Assume that Y and Z are infinitesimally rigid and that the 1-forms on Y and Z satisfying the zero-energy condition are closed. Suppose moreover that the conclusion of Lemma* 1.10 *holds for every section h of $T_Y^* \otimes T_Z^*$ satisfying the zero-energy condition. If k is a symmetric 2-form on X satisfying the zero-energy condition, then*

$$Q_gk = 0.$$

Proof. Let k be a symmetric 2-form on X satisfying the zero-energy condition and $x_0 \in X$. By Lemma 1.3, we may write $k = \mathscr{L}_\xi g + h$, where ξ is a vector field on X and h is a section of $T_Y^* \otimes T_Z^*$, with $h(x_0) = 0$, satisfying the zero-energy condition. By Proposition 1.1, we see that

$$(D_1 k)(x_0) = (D_1 h)(x_0) = 0 \quad \text{and} \quad D_2 k = D_2 h = 0.$$

PROPOSITION 1.3. *Assume that Y is either a projective space, different from a sphere, or a flat torus or a complex quadric Q_n, with $n \geq 5$. Assume that Z is infinitesimally rigid and that the 1-forms on Z satisfying the zero-energy condition are closed. If k is a symmetric 2-form on X satisfying the zero-energy condition, then*

$$Q_g k = 0.$$

Proof. The 1-forms on Y satisfying the zero-energy condition are exact and Y is infinitesimally rigid, according to [14], [7], [12] and [15] (see also [1], [5], [8] and [9]) in the case of a projective space, or to [13] in the case of a torus, or to [3] and [9] in the case of a complex quadric. The conclusion follows from Lemma 1.10 and Proposition 1.2.

2. Harmonic infinitesimal deformations

We continue to assume that Y and Z are compact symmetric spaces and that $X = Y \times Z$. We denote by $\tilde{Y}$ and $\tilde{Z}$ the universal covering spaces of Y and Z. We say that $\tilde{Y}$ (resp. $\tilde{Z}$) does not admit a Euclidean factor if it is isometric to a product $M_+ \times M_-$, where M_+ and M_- are symmetric spaces of compact and non-compact type, respectively.

LEMMA 2.1. *If $\tilde{Y}$ does not admit a Euclidean factor, then every parallel vector field on Y vanishes.*

Proof. According to a result of H.C. Wang (see [11, Theorem 4.6, Chapter VI]), a parallel vector field ξ on Y is invariant under the identity component of the group of isometries of Y. Thus by passing to the universal covering space of Y if necessary, we easily see that it suffices to consider the case of an irreducible symmetric space (of compact or non-compact type) and a vector field which is invariant under the identity component of the group of isometries; such a vector field must necessarily vanish.

Let Θ, Θ_Y and Θ_Z be the sheaves of Killing vector fields on X, Y and Z, respectively. We consider the harmonic spaces

$$\boldsymbol{H}^1 = \left\{ h \in C^\infty(S^2T^*) \,|\, D_0^* h = 0,\, Q_g h = 0 \right\}$$

on X and the analogous harmonic spaces $\mathbf{H}^1_Y$ and $\mathbf{H}^1_Z$ on Y and Z, respectively. According to Theorem 1.1 of [6], we have isomorphisms

$$(2.1)\qquad H^1(X,\Theta) \approx \mathbf{H}^1, \qquad H^1(Y,\Theta_Y) \approx \mathbf{H}^1_Y, \qquad H^1(Z,\Theta_Z) \approx \mathbf{H}^1_Z.$$

We denote by $\mathbf{H}^1_{Y,Z}$ (resp. $\mathbf{H}^1_{Z,Y}$) the subspace of $C^\infty(S^2T^*)$ generated by the elements $\alpha \cdot \xi^\flat$, where α is a harmonic 1-form on Z (resp. Y) and ξ is a Killing vector field on Y (resp. Z).

PROPOSITION 2.1. *Assume that Y, Z are compact symmetric spaces. If $\tilde{Y}$ or $\tilde{Z}$ does not admit a Euclidean factor, then*

$$(2.2)\qquad \mathbf{H}^1 = \mathbf{H}^1_Y \oplus \mathbf{H}^1_Z \oplus \mathbf{H}^1_{Y,Z} \oplus \mathbf{H}^1_{Z,Y}.$$

Proof. If h is an element of $\mathbf{H}^1_Y$, then clearly $D_0^*h = 0$ on X; since h can be written locally as a Lie derivative of the metric g_Y on Y, we see that $Q_g h = 0$. Thus $\mathbf{H}^1_Y$ and $\mathbf{H}^1_Z$ are subspaces of $\mathbf{H}^1$. Next, let ξ be a Killing vector field on Y and α be a harmonic 1-form on Z. If U is a simply connected open subset of Z, we may write $\alpha = df$, for some real-valued function f on U, and then we have

$$\mathscr{L}_{f\xi} g = df \cdot \xi^\flat + f\mathscr{L}_\xi g = \alpha \cdot \xi^\flat$$

on $Y \times U$. On the other hand, if δ is the formal adjoint of d and if $\operatorname{Tr} h$ denotes the trace of symmetric 2-form h on X, we have

$$D_0^*(\alpha \cdot \xi^\flat) = -\delta\alpha \cdot \xi^\flat + 2\operatorname{Tr}(\mathscr{L}_\xi g) \cdot \alpha = 0.$$

Thus $\mathbf{H}^1_{Y,Z}$ and $\mathbf{H}^1_{Z,Y}$ are also subspaces of $\mathbf{H}^1$. If $\tilde{Y}$ or $\tilde{Z}$ does not admit a Euclidean factor, we now show that $\mathbf{H}^1_{Y,Z} \cap \mathbf{H}^1_{Z,Y} = 0$. Let $\alpha_1, \ldots, \alpha_p$ (resp. $\beta_1, \ldots, \beta_q$) be a basis of the space of harmonic 1-forms on Y (resp. Z). Suppose that there are Killing vector fields $\xi_1, \ldots, \xi_q$ on Y and $\eta_1, \ldots, \eta_p$ on Z such that

$$(2.3)\qquad \sum_{j=1}^{p} \alpha_j \cdot \eta_j^\flat + \sum_{k=1}^{q} \xi_k^\flat \cdot \beta_k = 0.$$

For $1 \leq j \leq p$, since η_j is a Killing vector field on Z, $\delta\eta_j^\flat = 0$; hence there exist a 2-form φ_j on Z and constants b_{jl} such that

$$\eta_j^\flat = \delta\varphi_j + \sum_{k=1}^{q} b_{jk}\beta_k.$$

Similarly, for $1 \leq k \leq q$, there exist a 2-form ω_k on Y and constants a_{kj} such that

$$\xi_k^\flat = \delta\omega_k + \sum_{j=1}^{p} a_{kj}\alpha_j.$$

From (2.3), it follows that

$$\text{(2.4)} \qquad \sum_{j=1}^{p} \alpha_j \cdot \delta\varphi_j + \sum_{k=1}^{q} \delta\omega_k \cdot \beta_k + \sum_{j=1}^{p}\sum_{k=1}^{q} \left(a_{kj} + b_{jk}\right)\alpha_j \cdot \beta_k = 0.$$

We denote by $(\ ,\ )$ the L^2-scalar product on $C^\infty(S^mT^*)$ induced by the metric g. As $(\delta\varphi_j, \beta_k) = 0$, we see that

$$\left(\alpha_j \cdot \delta\varphi_j, \alpha_l \cdot \beta_k\right) = 0,$$

for $1 \leq j,\ l \leq p$, $1 \leq k \leq q$; similarly, we have

$$\left(\delta\omega_k \cdot \beta_k, \alpha_j \cdot \beta_r\right) = \left(\alpha_j \cdot \delta\varphi_j, \delta\omega_k \cdot \beta_r\right) = 0,$$

for $1 \leq j \leq p$, $1 \leq k,\ r \leq q$. Hence from (2.4), we deduce that

$$\text{(2.5)} \qquad \sum_{j=1}^{p} \alpha_j \cdot \delta\varphi_j = 0, \quad \sum_{k=1}^{q} \delta\omega_k \cdot \beta_k = 0,$$

$$\sum_{j=1}^{p}\sum_{k=1}^{q} \left(a_{kj} + b_{jk}\right)\alpha_j \cdot \beta_k = 0.$$

Since $\alpha_1, \ldots, \alpha_p$ are linearly independent (over $\mathbf{R}$), if η is a vector field on Z, the first of equations (2.5) implies that $\langle \eta, \delta\varphi_j \rangle = 0$ and hence that $\delta\varphi_j = 0$, for $1 \leq j \leq p$. Similarly, we obtain

$$\delta\omega_k = 0, \quad a_{kj} + b_{jk} = 0,$$

for $1 \leq j \leq p$, $1 \leq k \leq q$. Thus

$$\xi_k^\flat = \sum_{j=1}^{p} a_{kj}\alpha_j, \quad \eta_j^\flat = \sum_{k=1}^{q} b_{jk}\beta_k.$$

Since $d\alpha_j = 0$, $d\beta_k = 0$ and ξ_k, η_j are Killing vector fields, we see that ξ_k and η_j are parallel vector fields. According to Lemma 2.1, the parallel vector fields on Y or Z vanish, and so $a_{kj} = b_{jk} = 0$ and $\xi_k = 0$, $\eta_j = 0$, for $1 \leq j \leq p$, $1 \leq k \leq q$. We have thus shown that the sum on the right-hand side of (2.2) is direct. Our hypothesis on $\tilde{Y}$ or on $\tilde{Z}$ implies that

$$\Theta = \mathrm{pr}_Y^{-1}\Theta_Y \oplus \mathrm{pr}_Z^{-1}\Theta_Z.$$

Künneth's formula [2, Theorem II, 18.2] tells us that

$$\begin{aligned} H^1(X,\Theta) = {} & \big(H^0(Y,\mathbf{R}) \otimes H^1(Z,\Theta_Z)\big) \oplus \big(H^1(Y,\mathbf{R}) \otimes H^0(Z,\Theta_Z)\big) \\ & \oplus \big(H^0(Y,\Theta_Y) \otimes H^1(Z,\mathbf{R})\big) \oplus \big(H^1(Y,\Theta_Y) \otimes H^0(Z,\mathbf{R})\big). \end{aligned}$$

Since Y and Z are connected, from the isomorphisms (2.1) we deduce the equality (2.2).

In fact, we have shown that (2.2) represents a "Künneth decomposition" of the harmonic space $\mathbf{H}^1$. If Z is of compact type, then $H^1(Z,\mathbf{R}) = 0$ and so $\mathbf{H}^1_{Y,Z} = 0$; in this case, the proof of Proposition 2.1 is considerably simpler.

LEMMA 2.2. *Assume that the* 1*-forms on* Y *and* Z *which satisfy the zero-energy condition are exact. Let* k *be a symmetric* 2*-form on* X *which can be written in the form*

$$k = \sum_{j=1}^{p} \alpha_j \cdot \beta_j, \tag{2.6}$$

where α_j *are* 1*-forms on* Y *and* β_j *are* 1*-forms on* Z *satisfying* $\delta\alpha_j = 0$, $\delta\beta_j = 0$. *If* k *satisfies the zero-energy condition, then it vanishes.*

Proof. Assume that k is non-zero and satisfies the zero-energy condition, and that p is the least integer for which we can write k in the form (2.6), where α_j are non-zero 1-forms on Y and β_j are non-zero 1-forms on Z satisfying $\delta\alpha_j = 0$, $\delta\beta_j = 0$. There exists a closed geodesic γ_1 of Y such that

$$\int_{\gamma_1} \alpha_1 = c_1 \neq 0. \tag{2.7}$$

Indeed, if this were false, α_1 would satisfy the zero-energy condition and, so by our hypothesis on Y, would be exact. Since Y is compact and $\delta\alpha_1 = 0$, we would have $\alpha_1 = 0$. If

$$c_j = \int_{\gamma_1} \alpha_j,$$

for $2 \le j \le p$, then

$$k = \alpha_1 \cdot \left(\beta_1 + \sum_{j=2}^{p} \frac{c_j}{c_1} \beta_j \right) + \sum_{j=2}^{p} \left(\alpha_j - \frac{c_j}{c_1} \alpha_1 \right) \cdot \beta_j,$$

where

$$\int_{\gamma_1} \left(\alpha_j - \frac{c_j}{c_1} \alpha_1 \right) = 0.$$

Thus we may assume without loss of generality that there exists a closed geodesic γ_1 of Y such that (2.7) holds and that

$$\int_{\gamma_1} \alpha_j = 0, \tag{2.8}$$

for $2 \le j \le p$. Let γ_2: $[0, L_2] \to Z$ be an arbitrary closed geodesic of Z parametrized by its arc-length. Let L_1 be the length of the closed geodesic γ_1 of Y. Consider the flat 2-torus $\Gamma = S^1 \times S^1$, where the first factor has length L_1 and the second has length L_2, and the totally geodesic imbedding i: $\Gamma \to X$ sending (θ_1, θ_2) into $(\gamma_1(\theta_1), \gamma_2(\theta_2))$. According to Michel [13] and the proof of Lemma 1.7, there exists a vector field

$$\zeta = A_1(\theta_2) \frac{\partial}{\partial \theta_1} + A_2(\theta_1) \frac{\partial}{\partial \theta_2}$$

on Γ such that $\mathscr{L}_\zeta i^*g = i^*k$. Then we see that

$$\sum_{j=1}^{p} \alpha_j(\dot{\gamma}_1(\theta_1)) \beta_j(\dot{\gamma}_2(\theta_2)) = \frac{dA_1}{d\theta_2}(\theta_2) + \frac{dA_2}{d\theta_1}(\theta_1);$$

from (2.7) and (2.8), it follows that

$$c_1 \beta_1(\dot{\gamma}_2(\theta_2)) = L_1 \frac{dA_1}{d\theta_2}(\theta_2)$$

and, since $c_1 \neq 0$, that

$$\int_{\gamma_2} \beta_1 = 0.$$

Our hypothesis on Z implies that β_1 is exact; since $\delta \beta_1 = 0$ and Z is compact, we see that $\beta_1 = 0$, which shows that p was not minimal.

THEOREM 2.1. *Assume that Y and Z are infinitesimally rigid compact symmetric spaces. Assume that the 1-forms on Y and Z which satisfy the zero-energy condition are exact, and that $\tilde{Y}$ or $\tilde{Z}$ does not admit a Euclidean factor. Let k be a symmetric 2-form on X. Then the following assertions are equivalent*:

(i) *k satisfies the zero-energy condition and $Q_g k = 0$;*

(ii) *there exists a vector field ξ on X such that $\mathcal{L}_\xi g = k$.*

If moreover the conclusion of Lemma 1.10 holds for every section h of $T_Y^* \otimes T_Z^*$ satisfying the zero-energy condition, then X is infinitesimally rigid.

Proof. By Proposition 1.2, it suffices to show that (i) $\Rightarrow$ (ii). Assume that (i) holds. By Theorem 1.1 of [6], we may write

$$k = \mathcal{L}_\xi g + k', \tag{2.9}$$

where ξ is a vector field on X and $k' \in \mathbf{H}^1$. The hypotheses of Proposition 2.1 are satisfied and so, by (2.2), we have $k' = k_1 + k_2 + k_3$, where $k_1 \in \mathbf{H}^1_Y$, $k_2 \in \mathbf{H}^1_Z$ and $k_3 \in \mathbf{H}^1_{Y,Z} \oplus \mathbf{H}^1_{Z,Y}$. By (2.9), k' satisfies the zero-energy condition; hence k_1 (resp. k_2) satisfies the zero-energy condition on Y (resp. Z). From the infinitesimal rigidity of Y and Z, we see that $k_1 = 0$, $k_2 = 0$, and hence that $k' = k_3$. Since a Killing vector field ζ on Y or Z satisfies $\delta\zeta^b = 0$, we see that k_3 satisfies all the hypotheses of Lemma 2.2. Thus $k_3 = 0$ and $k = \mathcal{L}_\xi g$.

Since projective spaces, different from spheres, flat tori and complex quadrics of dimension ≥ 5 are infinitesimally rigid and the 1-forms on these spaces satisfying the zero-energy condition are exact (see the proof of Proposition 1.3), the following theorem is a direct consequence of Proposition 1.3 and Theorem 2.1.

THEOREM 2.2. *Assume that Y is either a projective space, different from a sphere, or a flat torus, or a complex quadric Q_n, with $n \geq 5$. Assume that Z is an infinitesimally rigid compact symmetric space and that the 1-forms on Z satisfying the zero-energy condition are exact. If Y is a flat torus, suppose moreover that $\tilde{Z}$ does not admit a Euclidean factor. Then X is infinitesimally rigid.*

PROPOSITION 2.2. *Assume that Y and Z are compact symmetric spaces, and that the 1-forms on Y and Z which satisfy the zero-energy condition are exact. Then the 1-forms on X which satisfy the zero-energy condition are exact.*

Proof. Let α be a 1-form on X satisfying the zero-energy condition. Then by our hypothesis, for all $y \in Y$, $z \in Z$, the restrictions of α to $Y \times \{z\}$ and

$\{y\} \times Z$ are exact. Therefore

$$(d\alpha)(\xi_1, \xi_2) = 0, \quad (d\alpha)(\eta_1, \eta_2) = 0,$$

for all $\xi_1, \xi_2 \in T_Y$, $\eta_1, \eta_2 \in T_Z$. Let $x = (y, z) \in X$ and $\xi \in C_{Y,y}$, $\eta \in C_{Z,z}$. Then

$$\Gamma = \operatorname{Exp}_x(\mathbf{R}\xi \oplus \mathbf{R}\eta)$$

is a flat 2-torus totally geodesic in X. If i: $\Gamma \to X$ is the natural imbedding, then $i^*\alpha$ satisfies the zero-energy condition on Γ. According to [13], $i^*\alpha$ is exact; thus

$$(d\alpha)(\xi, \eta) = 0. \tag{2.10}$$

Since $C_{Y,y}$ is dense in $T_{Y,y}$ and $C_{Z,z}$ is dense in $T_{Z,z}$, (2.10) holds for all $\xi \in T_{Y,y}$, $\eta \in T_{Z,z}$. Hence α is closed. As Y and Z are connected, by the Künneth formula, we have

$$H^1(X, \mathbf{R}) \simeq H^1(Y, \mathbf{R}) \oplus H^1(Z, \mathbf{R});$$

hence by Hodge theory, we may write

$$\alpha = df + \beta_1 + \beta_2,$$

where f is a real-valued function on X, and β_1, β_2 are harmonic 1-forms on Y and Z respectively. Clearly β_1 and β_2 satisfy the zero-energy condition on Y and Z respectively, and therefore are exact. It follows that $\beta_1 = 0$, $\beta_2 = 0$ and $\alpha = df$.

The following theorem is a consequence of the fact that 1-forms on projective spaces, different from spheres, on flat tori, or on complex quadrics of dimension ≥ 5 satisfying the zero-energy condition are exact, and of Theorem 2.2 and Proposition 2.2.

THEOREM 2.3. *A product $X_1 \times \cdots \times X_r$ of Riemannian manifolds, where each X_j is either a projective space, different from a sphere, or a flat torus, or a complex quadric Q_n, with $n \geq 5$, is infinitesimally rigid.*

REFERENCES

1. A. BESSE, *Manifolds all of whose geodesics are closed*, Ergeb. Math. Grenzgeb., Band 93, Springer-Verlag, New York, 1978.
2. G. BREDON, *Sheaf theory*, McGraw-Hall, New York, 1967.
3. Y. DIENG, *Quelques résultats de rigidité infinitésimale pour les quadriques complexes*, C. R. Acad. Sci. Paris Sér. I Math., vol. 304 (1987), pp. 393–396.
4. J. GASQUI and H. GOLDSCHMIDT, *Déformations infinitésimales des espaces riemanniens localement symétriques*. I, Adv. in Math., vol. 48 (1983), pp. 205–285.

5. J. GASQUI and H. GOLDSCHMIDT, *Déformations infinitésimales des espaces riemanniens localement symétriques*. II. *La conjecture infinitésimale de Blaschke pour les espaces projectifs complexes*, Ann. Inst. Fourier (Grenoble), vol. 34 (1984), fasc. 2, pp. 191–226.
6. ______, *Infinitesimal rigidity of* $S^1 \times \mathbf{RP}^n$, Duke Math. J., vol. 51 (1984), pp. 675–690.
7. ______, *Une caractérisation des formes exactes de degré* 1 *sur les espaces projectifs*, Comment. Math. Helv., vol. 60 (1985), pp. 46–53.
8. ______, "Some rigidity results in the deformation theory of symmetric spaces" in *Deformation theory of algebras and structures and applications*, edited by M. Hazewinkel and M. Gerstenhaber, Kluwer, Boston, 1988, pp. 839–851.
9. ______, *Rigidité infinitésimale des espaces projectifs et des quadriques complexes*, J. Reine Angew. Math., to appear.
10. S. HELGASON, *Differential geometry and symmetric spaces*, Academic Press, New York, 1963.
11. S. KOBAYASHI and K. NOMIZU, *Foundations of differential geometry*, vol. I, Interscience, New York, 1963.
12. R. MICHEL, *Problèmes d'analyse géométrique liés à la conjecture de Blaschke*, Bull. Soc. Math. France, vol. 101 (1973), pp. 17–69.
13. ______, (a) *Un problème d'exactitude concernant les tenseurs symétriques et les géodésiques*, C. R. Acad. Sc. Paris Sér. A, vol. 284 (1977), pp. 183–186; (b) *Tenseurs symétriques et géodésiques*, C. R. Acad. Sc. Paris Sér. A, vol. 284 (1977), pp. 1065–1068.
14. ______, *Sur quelques problèmes de géométrie globale des géodésiques*, Bol. Soc. Brasil Mat., vol. 9 (1978), pp. 19–38.
15. C. TSUKAMOTO, *Infinitesimal Blaschke conjectures on projective spaces*, Ann. Sci. École Norm. Sup. (4), vol. 14 (1981), pp. 339–356.

UNIVERSITÉ SCIENTIFIQUE, TECHNOLOGIQUE, ET MÉDICALE DE GRENOBLE
GRENOBLE, FRANCE
COLUMBIA UNIVERSITY
NEW YORK