ILLINOIS JOURNAL OF MATHEMATICS
Volume 33, Number 2, Summer 1989

VECTOR-VALUED HARMONIC FUNCTIONS AND CONE ABSOLUTELY SUMMING OPERATORS

BY

OSCAR BLASCO

1. Introduction

Throughout this paper $\mathbf{R}^{n+1}_{+}$ denotes the half space

$$\{(x, y) : x \in \mathbf{R}^n,\ y > 0\}$$

and $(B, \|\ \ \|_B)$ denotes a real Banach space.

The objective of this paper is to extend to a vector-valued setting the problem of characterizing the boundary values of certain spaces of harmonic functions in the upper half space. In the scalar-valued case the result may be stated as follows [8]:

$$h^p(\mathbf{R}^{n+1}_{+}) = L^p(\mathbf{R}^n) \quad (1 < p \le \infty), \tag{1.1}$$

$$h^1(\mathbf{R}^{n+1}_{+}) = M(\mathbf{R}^n) \tag{1.2}$$

where these isometries are given by the Poisson integral. Let us define the space h^p for B-valued functions. $h^p_B(\mathbf{R}^{n+1}_{+})$ denotes the space of B-valued harmonic functions $u : \mathbf{R}^{n+1}_{+} \to B$ such that

$$|u|_p = \sup_{y>0}\left(\int \|u(x, y)\|_B^p\, dx\right)^{1/p} < +\infty \quad (1 \le p < \infty), \tag{1.3}$$

$$|u|_\infty = \sup_{(x, y) \in \mathbf{R}^{n+1}_{+}} \|u(x, y)\|_B$$

In [1] the author solved the same problem for the unit disc D, by using certain classes of operators. In that case $h^p_B(D)$ may be interpreted, for $1 < p < \infty$, as the Dinculeanu class of operators. In addition a class of operators from a Banach lattice into a Banach space was defined in [1], the so-called positive

Received April 7, 1987.

p-summing operators, which turns out to be the Dinculeanu class in the special case of L_p-spaces. Later on in [2] the author discovered that, for L_p-spaces, these classes also coincide with a class defined by Schaefer [7] and which are called cone absolutely summing operators (c.a.s.).

Here we shall present a proof in the case of the half-space which uses the c.a.s. operators and which is quite different from the original one [1]. Now we shall use the w^*-compactness of the unit ball in a dual space.

Let us introduce some notation here to establish the main result of the paper.

Denoting by X a Banach lattice, $\mathscr{L}(X, B)$, $\Lambda^1(X, B)$ and $\pi^1(X, B)$ will be the spaces of all bounded operators, the cone absolutely summing ones and the absolutely summing operators respectively (see definitions below).

With all this the main theorem states the following identifications:

$$h_B^\infty(\mathbf{R}_+^{n+1}) = \mathscr{L}(L^1(\mathbf{R}^n), B)$$

$$h_B^p(\mathbf{R}_+^{n+1}) = \Lambda^1(L^{p'}(\mathbf{R}^n), B), \quad 1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1,$$

$$h_B^1(\mathbf{R}_+^{n+1}) = \pi^1(C_0(\mathbf{R}^n), B)$$

We shall finally prove that (1.1) remains valid in the vector-valued setting if and only if B has the Radon-Nikodym property.

2. Preliminary results and definitions

Let us begin with the properties which still hold for functions with values in B.

Recall that $P(x, y)$ denotes the Poisson Kernel on the half-space; that is,

$$P(x, y) = c_n \frac{y}{(y^2 + |x|^2)^{\frac{n+1}{2}}}.$$

If f belongs to $L_B^p(\mathbf{R}^n)$ we still can define the B-valued harmonic function

$$u(x, y) = P(\cdot, y) * f(x) = \int f(t) P(x - t, y)\, dt. \tag{2.1}$$

PROPOSITION 1. *Let* $1 \leq p \leq \infty$. *If* $f \in L_B^p(\mathbf{R}^n)$ *then* $u \in h_B^p(\mathbf{R}_+^{n+1})$ *and* $\|f\|_p = |u|_p$.

Proof. From Minkowski's inequality follows $|u|_p \leq \|f\|_p$ and since $P(\cdot, y) * f \to f$ as y goes to zero, Fatou's lemma implies the converse inequality. ■

So the Poisson integral embeds $L^p_B(\mathbf{R}^n)$ into $h^p_B(\mathbf{R}^{n+1}_+)$.

Remark 1. The Poisson integral is not surjective in general. The reader can easily check that if $B = L^1(\mathbf{R}^n)$ and $u(x, y)(t) = P(x - t, y)$ then $u \in h^\infty_B(\mathbf{R}^{n+1}_+)$ but u is not the Poisson integral of any function in $L^\infty_B(\mathbf{R}^n)$. So in order to find the corresponding space of boundary values for $h^p_B(\mathbf{R}^{n+1}_+)$ we should look for a larger space than $L^p_B(\mathbf{R}^n)$ but which contain it. The key point is to look at functions as operators. In order to unify results let $X_p = L^p(\mathbf{R}^n)$ for $1 \le p < \infty$ and let $X_\infty = C_0(\mathbf{R}^n)$ (continuous functions which converge to zero at infinite), and with this notation we can write

$$L^p_B(\mathbf{R}^n) \subseteq \mathscr{L}(X_{p'}, B), \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

The identification is given as follows: f in $L^p_B(\mathbf{R}^n)$ defines the operator T_f,

$$T_f(\varphi) = \int \varphi(t) f(t)\, dt \quad \text{for all } \varphi \text{ in } X_{p'}. \tag{2.2}$$

From Hölder's inequality $\|T_f\| \le \|f\|_p$. Notice that for $p = \infty$ we have an isometry ($\|f\|_\infty = \|T_f\|$). To embed $L^p_B(\mathbf{R}^n)$ isometrically in $\mathscr{L}(X_{p'}, B)$ we shall use the Banach lattice structure of $X_{p'}$.

DEFINITION 1. [7] Let X be a Banach lattice. An operator T in $\mathscr{L}(X, B)$ is called a cone absolutely operator (c.a.s.) if there exists a constant $C \ge 0$ such that for every positive finite family $x_1, x_2, \dots, x_k \ge 0$ in X we have

$$\sum_{i=1}^{k} \|Tx_i\|_B \le C \sup_{\|\xi\|_{X^*} \le 1} \sum_{i=1}^{k} |\langle \xi, x_i\rangle|. \tag{2.3}$$

We denote by $\Lambda^1(X, B)$ the space of c.a.s. operators and the norm in it is given by the infimum of the constants verifying (2.3).

Let us recall that the space of absolutely summing operators, $\pi^1(X, B)$ has the same definition without requiring that the family $x_1, \dots, x_k$ be positive (see [6]).

We shall use an equivalent norm in $\Lambda^1(X, B)$ (see [7]):

$$|||T||| = \sup\left\{ \sum_{i=1}^{k} \|T(x_i)\|_B : x_i \ge 0 \left\| \sum_{i=1}^{k} x_i \right\|_X \le 1 \right\}. \tag{2.4}$$

The following useful characterization may be found in [7] (or in [1] in the particular case of $X = L^p$).

PROPOSITION 2. *Let $T \in \mathscr{L}(X, B)$. T is c.a.s. if and only if there exists a positive functional ξ in X^* such that*

$$\|Tx\|_B \leq \langle \xi, |x| \rangle \quad \text{for all } x \in X. \tag{2.5}$$

Moreover ξ can be chosen with $\|\xi\|_{X^*} = |||T|||$.

The only cases we are interested in are $X = L^p(\mathbf{R}^n)$ or $X = C_0(\mathbf{R}^n)$ and for these we have the following result.

PROPOSITION 3.

$$\Lambda^1(L^1(\mathbf{R}^n), B) = \mathscr{L}(L^1(\mathbf{R}^n), B), \tag{2.6}$$

$$\Lambda^1(C_0(\mathbf{R}^n), B) = \pi^1(C_0(\mathbf{R}^n), B). \tag{2.7}$$

Proof. (a) Take T in $\mathscr{L}(L^1(\mathbf{R}^n), B)$ and $\varphi_1, \dots, \varphi_k \geq 0$ in $L^1(\mathbf{R}^n)$. Then

$$\sum_{i=1}^{k} \|T(\varphi_i)\|_B \leq \|T\| \sum_{i=1}^{k} \|\varphi_i\|_1 = \|T\| \left\| \sum_{i=1}^{k} \varphi_i \right\|_1 .$$

Therefore $|||T||| \leq \|T\|$ which proves (2.6) since the reverse inequality is true in general.

To show (2.7), since $\pi^1(X, B) \subseteq \Lambda^1(X, B)$, we take T in $\Lambda^1(C_0(\mathbf{R}^n), B)$ and $\varphi_n, \varphi_2, \dots, \varphi_k$ in $C_0(\mathbf{R}^n)$ not necessarily positive. Then

$$\begin{aligned}\sum_{i=1}^{k} \|T(\varphi_i)\|_B &\leq \sum_{i=1}^{k} \|T(\varphi_i^+)\|_B + \sum_{i=1}^{k} \|T(\varphi_i^-)\|_B \\ &\leq |||T||| \left(\left\| \sum_{i=1}^{k} \varphi_i^+ \right\|_\infty + \left\| \sum_{i=1}^{k} \varphi_i^- \right\|_\infty \right) \\ &\leq 2|||T||| \left\| \sum_{i=1}^{k} |\varphi_i| \right\|_\infty .\end{aligned}$$

On the other hand (see [4], page 162)

$$\left\| \sum_{i=1}^{k} \cdot |\varphi_i| \right\|_\infty = \sup_{\|\mu\|_{M(\mathbf{R}^n)} \leq 1} \sum_{i=1}^{k} |\langle \varphi_i, \mu \rangle|.$$

Therefore T is absolutely summing and $\|T\|_{\pi^1} \leq 2|||T|||$. ∎

Remark 2. $\Lambda^1(C_0(\mathbf{R}^n), B) = M_B(\mathbf{R}^n)$. (This is essentially proved in [4], page 163.)

The following step is to verify that the spaces $L^p_B(\mathbf{R}^n)$ are isometrically embedded into $\Lambda^1(X_{p'}, B)$.

From (2.2) we have $\|T_f(\varphi)\|_B \le \langle \|f\|, |\varphi|\rangle$ for all φ in $X_{p'}$ which implies, by (2.5), that $L^p_B(\mathbf{R}^n) \subseteq \Lambda^1(X_{p'}, B)$ and $|||T||| \le \|f\|_p$. But we also have the following:

Proposition 4. *For $1 \le p \le \infty$, if $f \in L^p_B(\mathbf{R}^n)$ then $|||T_f||| = \|f\|_p$.*

Proof. The cases $p = 1$ and $p = \infty$ are clear from Remark 2 and (2.6) respectively.

Take a simple function $s = \sum_{i=1}^k a_i \chi_{E_i}$, $a_i \in B$. Then

$$\begin{aligned}
\|s\|_p &= \left(\sum_{i=1}^k \left\|a_i m(E_i)^{1/p}\right\|_B^p\right)^{1/p} \\
&= \sup\left\{\sum_{i=1}^k \|a_i\|_B m(E_i)^{1/p}\alpha_i : \sum_{i=1}^k \alpha_i^{p'} = 1\right\} \\
&= \sup\left\{\sum_{i=1}^k \left\|T_s\left(\alpha_i m(E_i)^{-1/p'}\chi_{E_i}\right)\right\|_B : \sum_{i=1}^k \alpha_i^{p'} = 1\right\} \\
&\le |||T_s|||\sup\left\{\left\|\sum_{i=1}^k \alpha_i m(E_i)^{-1/p'}\chi_{E_i}\right\|_B : \sum_{i=1}^k \alpha_i^{p'} = 1\right\} \\
&= |||T_s|||.
\end{aligned}$$

Therefore $|||T_s||| = \|s\|_p$ and now the density of simple functions completes the proof. ■

The following property says that when the Banach space is a dual space then the space of c.a.s. operators is also a dual. The proof can be found in [7], page 277, for $1 \le p < \infty$ and the other part is a reformulation of Singer's theorem.

Proposition 5.

$$(2.8) \qquad (L^p_B(\mathbf{R}^n))^* = \Lambda^1(L^{p'}(\mathbf{R}^n), B^*), \quad 1 \le p < \infty,$$

$$(2.9) \qquad (C_{0,B}(\mathbf{R}^n))^* = \Lambda^1(C_0(\mathbf{R}^n), B^*).$$

3. Proof of the main theorem

Let us recall the notation $X_p = L^p(\mathbf{R}^n)$ for $1 \le p < \infty$ and $X_\infty = C_0(\mathbf{R}^n)$ and p' such that $1/p + 1/p' = 1$.

THEOREM 1. *For* $1 \leq p \leq \infty$, $h_B^p(\mathbf{R}_+^{n+1})$ *is isometric to* $\Lambda^1(X_{p'}, B)$.

Proof. The isometry will be achieved by consideration of the following extension of the Poisson integral: Given an operator T in $\mathscr{L}(X_{p'}, B)$ we can define the harmonic function $u = \mathscr{P}(T)$ by the formula

$$u(x, y) = T(P(x - \cdot, y)) \tag{3.1}$$

where $P(x, y)$ is the Poisson kernel on the half-space.

We shall prove that $\mathscr{P}$ maps $\Lambda^1(X_{p'}, B)$ onto $h_B^p(\mathbf{R}_+^{n+1})$. To do that let us take T in $\Lambda^1(X_p, B)$. Then according to (2.5) there exists a positive function g in $(X_{p'})^* = L^p(\mathbf{R}^n)$ for $1 \leq p < \infty$ or a positive measure μ in $(X_\infty)^* = M(\mathbf{R}^n)$ such that

$$\|u(x, y)\|_B \leq P(\cdot, y) * g(x) \quad \text{for } 1 < p \leq \infty, \tag{3.2}$$

or

$$\|u(x, y)\|_B \leq \int P(x - t)\, d\mu(t) \quad \text{for } p = 1. \tag{3.2'}$$

From (3.2), Minkowski's inequality implies $|u|_p \leq \|g\|_p = |||T|||$ and by Fubini's theorem and (3.2′) we get $|u|_1 \leq \|\mu\|_1 = |||T|||$.

On the other hand, assume $\varphi_1, \varphi_2, \ldots, \varphi_k \geq 0$ in $X_{p'}$. Since

$$P(\cdot, y) * \varphi_i \to \varphi_i$$

in $X_{p'}$ for all $1 \leq p' \leq \infty$ and T is continuous we have

$$\begin{aligned}\sum_{i=1}^{k} \|T(\varphi_i)\|_B &= \lim_{y \to 0} \sum_{i=1}^{k} \|T(P(\cdot, y) * \varphi_i)\|_B \\ &\leq \sup_{y>0} \sum_{i=1}^{k} \left\| T\left(\int P(\cdot - t, y) \varphi_i(t) \right) \right\|_B .\end{aligned}$$

Now use Hille's theorem [4, page 47] to put T inside the integral and use the fact that $\varphi_i \geq 0$ to get

$$\begin{aligned}\sum_{i=1}^{k} \|T(\varphi_i)\|_B &\leq \sup_{y>0} \sum_{i=1}^{k} \int \|T(P(\cdot - t, y)\|_B \varphi_i(t)\, dt \\ &= \sup_{y>0} \int \|u(t, y)\|_B \left(\sum_{i=1}^{k} \varphi_i(t) \right) dt \\ &\leq |u|_p \left\| \sum_{i=1}^{k} \varphi_i \right\|_{p'} .\end{aligned}$$

Therefore if $T \in \Lambda^1(X_{p'}, B)$ then $u = \mathscr{P}(T) \in h^p_B(\mathbf{R}^{n+1}_+)$ and $|u|_p = |||T|||$. To finish the proof we must show that $\mathscr{P}$ is surjective.

Let us take u in $h^p_B(\mathbf{R}^{n+1}_+)$. This means that $u(\cdot, y)$ is a family of B-valued functions uniformly bounded in $L^p_B(\mathbf{R}^n)$. Since $L^p_B(\mathbf{R}^n) \subseteq L^p_{B^{**}}(\mathbf{R}^n)$, then Propositions 4 and 5 allow us to look at $u(\cdot, y)$ as a family contained in a ball of the dual space $\Lambda^1(X_{p'}, B^{**})$. Therefore there exists an operator T and a sequence y_n such that $u(\cdot, y_n)$ coverges to T in w^*-topology.

Due to the identification of these duals, we may write that for every φ in $X_{p'}$, and ξ in B^*,

$$\left\langle \xi, \int u(t, y_n)\varphi(t)\, dt \right\rangle \underset{n\to\infty}{\longrightarrow} \langle T(\varphi), \xi\rangle. \tag{3.3}$$

The harmonicity of u implies that

$$\left\langle \xi, \int u(t, y')P(x - t, y)\, dt \right\rangle = \langle \xi, u(x, y + y')\rangle. \tag{3.4}$$

By taking $\varphi(t) = P(x - t, y)$ for fixed $x \in \mathbf{R}$ and $y > 0$ then (3.3) and (3.4) imply $\mathscr{P}(T) = u$.

Finally, if we show that the range of T is actually in B the proof will be complete. To see that, it suffices to use the continuity of T and the fact that

$$P(\cdot, y) * \varphi \to \varphi$$

in $X_{p'}$ as y goes to zero. Then

$$\begin{aligned} T(\varphi) &= \lim_{y\to 0} T(P(\cdot, y) * \varphi) = \lim_{y\to 0} T\left(\int P(\cdot - t, y)\varphi(t)\, dt\right) \\ &= \lim_{y\to 0} \int T\left(P(\cdot - t, y)\varphi(t)\, dt = \lim_{y\to 0} \int u(t, y)\varphi(t)\, dt, \end{aligned}$$

and since $u(t, y)$ belongs to B there, so does $T(\varphi)$. ■

Let me finish by showing that the Radon-Nikodym property of B is the necessary and sufficient condition for the Poisson integral to be an isometry between $h^p_B(\mathbf{R}^{n+1}_+)$ and $L^p_B(\mathbf{R}^n)$. For the disc D this was proved by Bukhvalov and Danilevich [3] and the author gave a different approach in [1]. This result will follow from the next theorem which was proved in [2] for a finite measure space.

THEOREM 2. *Let $1 < p \leq \infty$. The following statements are equivalent.*

(a) *Every operator T in $\Lambda^1(L^{p'}(\mathbf{R}^n), B)$ is representable by a function f in $L^p_B(\mathbf{R}^n)$*

(b) *B has the Radon-Nikodym property.*

Proof. Let us assume (a). By Theorem 5 on page 63 of [4] we have to show that any operator

$$T: L^1(\Omega_1) \to B$$

can be represented by a function f in $L^1_B(\Omega_1)$, where $\Omega_1 = \{x \in \mathbf{R}^n : |x| \le 1\}$. Let us consider the projection $\pi : L^{p'}(\mathbf{R}^n) \to L^p(\Omega_1)$ given by $\pi(\varphi) = \varphi \cdot \chi_{\Omega_1}$. It is simple to verify that $T\pi \in \Lambda^1(L^{p'}(\mathbf{R}^n), B)$ if $T \in \mathscr{L}(L^1(\Omega_1), B)$. By assumption we get f in $L^p_B(\mathbf{R}^n)$ and now take $f\chi_{\Omega_1}$ to represent T.

Conversely, suppose B has the RNP and take T in $\Lambda^1(L^{p'}(\mathbf{R}^n), B)$. Let

$$\Omega_k = \{x \in \mathbf{R}^n : |x| \le k\}$$

and define $i_k : L^{p'}(\Omega_k) \to L^p(\mathbf{R}^n)$ by $L_k(\varphi) = \varphi_k$ where $\varphi_k = \varphi$ on Ω_k and $\varphi_k = 0$ on Ω_k^c. Then we can easily show that $T_k = i_k \cdot T$ belongs to $\Lambda^1(L^{p'}(\Omega_k), B)$. Now by using Theorem 2 in [2] we get f_k in $L^p_B(\Omega_k)$ such that

$$T_k(\varphi) = \int_{\Omega_k} \varphi(t) f_k(t)\, dt \quad \text{for all } \varphi \text{ in } L^p(\Omega_k).$$

Consider $f = f_k$ on Ω_k. This function is measurable and it is well defined since $f_{k+1} = f_k$ on Ω_k. Now Fatou's lemma implies that

$$\int_{\mathbf{R}^n} \|f(t)\|_B^p\, dt \le \lim_{k \to \infty} \int_{\Omega_k} \|f(t)\|_B^p\, dt \le |||T_k||| \le |||T|||.$$

Therefore f belongs to $L^p_B(\mathbf{R}^n)$ and the proof is completed. ∎

Notice that due to the representation $u(x, y) = T(P(x - \cdot, y))$ we can say that T is representable if and only if $u(x, y) = P(\cdot, y) * f(x)$ for some f in $L^p_B(\mathbf{R}^n)$.

In other words, Theorems 1 and 2 imply the following result.

COROLLARY 1. *Let $1 < p \le \infty$. Then $h^p_B(\mathbf{R}^{n+1}_+) = L^p_B(\mathbf{R}^n)$ (by the Poisson integral) if and only if B has the RNP.*

Let us mention that from (2.8) and taking into account Theorem 2 we get the following result due to Lai [5].

COROLLARY 2. *Let $1 \le p < \infty$. Then $(L^p_B(\mathbf{R}^n))^* = L^{p'}_{B^*}(\mathbf{R}^n)$ if and only if B^* has the RNP.*

Acknowledgement. This work was done while the author was visiting the University of Illinois during the special year in Analysis, 1986.

REFERENCES

1. O. BLASCO, *Boundary values of vector valued harmonic functions considered as operators*, Studia Math., vol. 86 (1987), pp. 19–33.
2. ______, *Positive p-summing operators on L_p-spaces*, Proc. of Amer. Math. Soc., vol. 100 (1987), pp. 275–280.
3. A.V. BUKHVALOV and A.A. DANILEVICH, *Boundary properties of analytic and harmonic functions with values in Banach spaces*, Mat. Zametki, vol. 31 (1982), pp. 203–214; English. Transl., Math. Notes, vol. 31 (1982), 104–110.
4. J. DIESTEL and J.J. UHL, *Vector measures*, Math. Surveys, no. 15, Amer. Math. Soc., Providence, R.I., 1977.
5. H. C. LAI, *Duality of Banach function spaces and the Radon-Nikodyin property*, Acta Math. Hungar, vol. 47 (1986), pp. 45–52.
6. J. LINDENSTRAUSS and L. TZAFRIRI, *Classical Banach spaces*, vol. II, Springer-Verlag, Berlin, 1979.
7. H.H. SCHAEFER, *Banach lattices and positive operators*, Springer-Verlag, Berlin, 1974.
8. E.M. STEIN and G. WEISS, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, New Jersey, 1975.

UNIVERSIDAD DE ZARAGOZA
ZARAGOZA, SPAIN