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THE MAXIMAL OPERATORS RELATED TO THE CALDERÓN-ZYGMUND METHOD OF ROTATIONS

BY

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1. Introduction and result

Let a_i, $i = 1, \dots, n$, be positive numbers, $0 < a_1 < a_2 < \cdots < a_n$. Define

$$\delta_t x = \left(t^{a_1}x_1, \dots, t^{a_n}x_n\right), \quad t > 0,$$

where $x = (x_1, \dots, x_n) \in R^n$. Let $\tau = a_1 + \cdots + a_n$. and v always denote a unit vector,

$$v = (v_1, \dots, v_n) \in S^{n-1},$$

and $d\sigma(v)$ denote the Lebesgue measure on S^{n-1}. Let $L^p(L^q(S^{n-1})R^n)$ denote mixed norm Lebesgue spaces. More precisely, if

$$\|g\|_{L^p(L^q)} = \left\| \|g\|_{L^q(S^{n-1})} \right\|_{L^p(R^n)} = \left[\int_{R^n}\left(\int_{S^{n-1}} |g(v, x)|^q \, d\sigma(v)\right)^{p/q} dx\right]^{1/p} < \infty,$$

then we say $g(v, x) \in L^p(L^q)$. Define

$$M_v f(x) = \sup_{r>0} \frac{1}{r}\int_0^r |f(x - \delta_t v)| \, dt.$$

R. Fefferman [2] proved that if $a = \cdots = a_n = 1$ then $M_v f$ is bounded on $L^p(L^2)$, for $p > 2n/n + 1$. Further developments are found in [1] and [3].

In this paper, we prove the following theorem.

THEOREM. *If $f \in L^p(R^n)$, then*

$$\|M_v f\|_{L^p(L^q)} \le C\|f\|_p,$$

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provided

$$(1)\qquad 1 < q \le \frac{2\left(n-1+\frac{1}{n}\right)}{n-1} \quad \textit{and} \quad \frac{q\left(n-1+\frac{2}{n}\right)}{n-1+\frac{2}{n}q} < p \le \infty,$$

$$(2)\qquad \frac{2\left(n-1+\frac{1}{n}\right)}{n-1} < q \quad \textit{and} \quad \frac{q(n-1)}{n-1+\frac{2}{n}} < p \le \infty.$$

Let $\hat{f}$ denote the Fourier transform of f, and $\check{f}$ denote the corresponding inverse Fourier transform. C will denote some constants which may depend on n, p, $a_1, \ldots, a_n$ and may change at different occurrences.

2. Proof of the theorem

Let $f \ge 0$. It is clear that

$$M_v f(x) = \sup_{r>0} \frac{1}{r} \int_0^r f(x - \delta_t v)\, dt \le 2 \sup_k \Delta_k * f(x),$$

where

$$\int_{R^n} \Delta_k(x) g(x)\, dx = \int_1^2 g(\delta_{2^k t} v)\, dt,$$

and k is integer. Let us define a family of operators $\{T^\alpha_{k,v} f\}_\alpha$, where α is a complex number. Let

$$\begin{aligned}(T^\alpha_{k,v} f) \wedge (x) &= \int_1^2 \exp(i(\delta_{2^k t} v)\cdot x)\, dt \left(1 + |\delta_{2^k} x|^2\right)^{-\alpha/2} \hat{f}(x) \\ &= m_k(v, x) \hat{f}(x).\end{aligned}$$

Clearly $T^0_{k,v} f(x) = \Delta_k * f(x)$. In order to prove the theorem, we need the following three lemmas.

LEMMA 1. *If* $n > \operatorname{Re}\alpha > -1/n$ *then* $\|\sup_k |T^\alpha_{k,v} f|\,\|_{L^2(L^2)} \le C\|f\|_2$.

LEMMA 2. *Let* $1 < p \le \infty$. *Then* $\|M_v f\|_{L^p(L^1)} \le C\|f\|_p$.

LEMMA 3. *If* $n > \operatorname{Re}\alpha > n-1$ *then* $\|\sup_k |T^\alpha_{k,v} f|\,\|_{L^p(L^q)} \le C\|f\|_p$, *for* $1 \le q \le \infty$, $1 < p \le \infty$.

Using the analytic interpolation theorem with Lemmas 1 and 3, we have

$$\|M_v f\|_{L^p(L^q)} \le C\|f\|_p,$$

for

$$\frac{2\left(n-1+\frac{1}{n}\right)}{n-1+\frac{2}{n}} < q, p < \frac{2\left(n-1+\frac{1}{n}\right)}{n-1}.$$

Next, by interpolation between Lemma 2 and the trivial case, $\|M_v f\|_{L^\infty(L^q)} \leq C\|f\|_{L^\infty}$, for $1 \leq q \leq \infty$, we have $\|M_v f\|_{L^p(L^q)} \leq C\|f\|_p$, if $1 < q \leq p < \infty$. Therefore the theorem follows by applying the real interpolation theorem to the above results.

Proof of Lemma 1. One takes a smooth function, $p \in C_0^\infty(R^n)$, with compact support and $\int p(x)\,dx = 1$. Let

$$p_k(x) = \frac{1}{(2^k)^r} p(\delta_{2^{-k}} x).$$

Then

$$\sup_k |T^\alpha_{k,v} f| \leq \left(\sum_k |T^\alpha_{k,v} f - p_k * f|^2\right)^{1/2} + M_1 M_2 \cdot M_n f,$$

where $M_i f$ is the Hardy-Littlewood maximal operators acting on the x_i variable. It is well-known that $M_i f$ is bounded on $L^p(R^n)$. To prove that $\sup_k |T^\alpha_{k,v} f|$ is bounded on $L^2(L^2)$, it is sufficient to show that

$$(1.1) \qquad \sum_k \int_{S^{n-1}} |m_k(v,x) - \hat{p}_k(x)|^2 \, d\sigma(v)$$

is bounded for every $x \in R^n$. We claim that

$$\int_{S^{n-1}} |m_k(v,x) - \hat{p}_k(x)|^2 \, d\sigma(v) \leq C \min\left\{|\delta_{2^k} x|^2, |\delta_{2^k} x|^{-(2/n + 2\operatorname{Re}\alpha)}\right\}.$$

By dilation invariance, we can assume $k = 0$. If $|x|$ is near zero, $m_0(v, 0) = \hat{p}(0) = 1$, $m_0(v, x)$ and $\hat{p}(x)$ are smooth functions. Therefore,

$$|m_0(v,x) - \hat{p}(x)| \leq C|x|.$$

It is clear that $\hat{p}(x) \leq 1/|x|^{1/n}$. On the other hand, if $|x|$ is large, then

$$\begin{aligned} m_0(v,x) &= \int_1^2 e^{i(\delta_t v)\cdot x}\,dt\left(1 + |x|^2\right)^{-\alpha/2} \\ &= \int_1^2 e^{ir(t)\cdot\xi}\,dt\left(1 + |x|^2\right)^{-\alpha/2} \end{aligned}$$

where $r(t) = (t^{a_1}, \ldots, t^{a_n})\xi = (v_1 x_1, \ldots, v_n x_)$. The last equality is bounded by

$$|\xi|^{-1/n}|x|^{-\operatorname{Re}\alpha}.$$

This critical estimate for $m_0(v, x)$ is obtained by the Van der Corput's lemma (see [4], age 1257). Without loss of generality, we can assume $|x_1| = \max\{|x_1|, \ldots, |x_n|\}$. It is clear that $|x| \sim |x_1|$. So,

$$|m_0(v, x)| \leq C|v_1|x||^{-1/n}|x|^{-\operatorname{Re}\alpha}.$$

Hence,

$$\begin{aligned}\int_{S^{n-1}}|m_0(v, x)|^2\, d\sigma(v) &\leq |x|^{-2\operatorname{Re}\alpha}\int_{|v_1|x||<1} 1\, d\sigma(v) \\ &\quad + \int_{|v_1|x||>1}|m_0(v, x)|^2\, d\sigma(v) \\ &\leq C|x|^{-1-2\operatorname{Re}\alpha} + C'|x|^{-2\operatorname{Re}\alpha}\int_{1/|x|}^{1}(|x|\,|v_1|)^{-2/n}\, dv_1 \\ &\leq C|x|^{-(2/n+2\operatorname{Re}\alpha)}\end{aligned}$$

Hence, we proved the claim. Note that $|\delta_t x| = |\delta_t \tilde{x}|$ where $\tilde{x} = (|x_1|, \ldots, |x_n|)$. It is clear that $|\delta_0 \tilde{x}| = 0$, $|\delta_t \tilde{x}| \to \infty$ if $t \to \infty$, $|\tilde{x}| \neq 0$ and $|\delta_t \tilde{x}|$ is an increasing function of t. Therefore for every $x \in R^n - \{0\}$ there exist k_0, j_0, $1 \leq j_0 \leq n$ such that

$$|x_i|/2^{k_0 a_i} < 2^{a_i}$$

for every $i = 1, \ldots, n$ and $1 < |x_{i_0}|/2^{k_0 a_{j_0}} < 2^{a_{j_0}}$. Changing index in the sum in (1.1) from k to $k + k_0$ it is sufficient to assume that $|x_i| < 2^{a_i}$ and $1 < |x_j| < 2^{a_j}$ for some j. Therefore,

$$\int_{S^{n-1}}|m_k(v, x) - \hat{p}_k(x)|^2\, d\sigma(v) \leq C\min\{2^{k\beta}, 2^{-k\gamma(2/n+2\operatorname{Re}\alpha)}\},$$

where β, γ are positive. Hence,

$$\sum_k \int_{S^{n-1}}|m_k(v, x) - \hat{p}_k(x)|^2\, d\sigma(v) < \infty$$

for every $x \in R^n$, if $\operatorname{Re}\alpha > -1/n$. Lemma 1 is proved.

Before proving Lemma 2, we need the following lemma.

LEMMA 4. *Suppose* $\| \sup_k |\Delta_k * f| \|_{L^q(L^1)} \leq C\|f\|_{L^q}$.

(I) *Suppose* $1/2 - 1/p_0 = 1/2q$. *For arbitrary functions* $u_k(v, x)$ *on* $S^{n-1} \times R^n$, *define the operators* L_k *by*

$$L_k u_k = \int_1^2 u_k(v, x + \delta_{2^k t} v)\, dt.$$

Then the following vector-valued inequality holds:

$$\left\| \left(\sum_k |L_k u_k|^2 \right)^{1/2} \right\|_{L^{p_0}(L^1)} \leq C \left\| \left\| \left(\sum_k |u_k|^2 \right)^{1/2} \right\|_{L^\infty(S^{n-1})} \right\|_{L^{p_0}(R^n)}.$$

(II) *Suppose* $|1/2 - 1/p_0| = 1/2q$ *and* $\{g_k\}_{k=1}^\infty$ *are arbitrary functions defined only on* R^n. *Then we have the same inequality as in* (*I*), *namely*

$$\left\| \left(\sum_k |\Delta_k * g_k|^2 \right)^{1/2} \right\|_{L^{p_0}(L^1)} \leq C \left\| \left(\sum_k |g_k|^2 \right)^{1/2} \right\|_{p_0}.$$

Proof of Lemma 4. (I) Since $p_0 > 2$, there exists a positive function, $h \in L^q(R^n)$, with unit norm, such that

$$\left\| \left(\sum_k |L_k u_k|^2 \right)^{1/2} \right\|^2_{L^{p_0}(L^1)} = \int_{R^n} \left\| \left(\sum_k |L_k u_k|^2 \right)^{1/2} \right\|^2_{L^1(S^{n-1})} h(x)\, dx$$

$$\leq C \int \int \sum_k L_k (u_k)^2\, d\sigma(v) h(x)\, dx \tag{1}$$

$$= C \int \int \sum_k u_k^2(v, y) \Delta_k * h(y)\, d\sigma(v)\, dy$$

$$\leq C \left\| \left\| \left(\sum_k u_k^2 \right) \right\|^{1/2}_{L^\infty(v)} \right\|^2_{p_0} \left\| \sup_k |\Delta_k * h| \right\|_{L^q(L^1)} \tag{2}$$

$$\leq C \left\| \left\| \left(\sum_k |u_k|^2 \right)^{1/2} \right\|_{L^\infty(v)} \right\|^2_{p_0},$$

where the inequalities (1) and (2) are established by Hölder's inequality.

(II) By (I), the vector valued inequality holds if $p_0 > 2$. On the other hand, if $p_0 < 2$, $1/p_0 + 1/p_0' = 1$ then

$$\left\| \left(\sum_k |\Delta_k * g_k|^2 \right)^{1/2} \right\|_{L^{p_0}(L^1)} = \sup \int_{R^n} \int_{S^{n-1}} \sum_k \Delta_k * g_k(x) u_k(v, x)\, d\sigma(v)\, dx$$

$$= \sup \int \int \sum_k g_k(x) L_k u_k(v, x)\, d\sigma(v)\, dx$$

$$\leq \sup \int \left(\sum |g_k|^2 \right)^{1/2} \left\| \left(\sum_k |L_k u_k(\cdot, x)|^2 \right)^{1/2} \right\|_{L^1(v)} dx \tag{3}$$

$$\leq \sup \left\| \left(\sum |g_k|^2 \right)^{1/2} \right\|_{p_0} \left\| \left(\sum_k |L_k u_k|^2 \right)^{1/2} \right\|_{L^{p_0'}(L^1)} \tag{4}$$

$$\leq \left\| \left(\sum |g_k|^2 \right)^{1/2} \right\|_{p_0}, \tag{5}$$

where the supremum is taken over all indicated $\{u_k\}_k$ with the unit norm in $L^{p'}(L^\infty(l^2))$. The inequalities (3) and (4) follow by Hölder's inequality, and the inequality (5) is supported by (I). Lemma 4 is proved.

The proof of Lemma 2 follows the ideas of J. Duoandikoetexa and J. L. Rubio de Francia [5], but in this lemma, we need to consider the extra $L^1(S^{n-1})$-norm. We will use the same notation as in the proof of Lemma 1. The proof of Lemma 2 will be obtained by induction.

Proof of Lemma 2. We separate the proof into two parts: the boundedness on $L^2(L^1)$ in the first part, and the boundness on $L^p(L^1)$ in the second part.

Part 1. It is enough to show

$$\left\| \left\| \sup_k |\Delta_k * f| \right\|_{L^1(v)} \right\|_{L^2(x)} \leq C \|f\|_2.$$

If $n = 1$, it is clear that

$$\Delta_k * f \leq CMf.$$

Suppose that for $n - 1$ dimensions, $\sup_k |\Delta_k * f|$ is bounded on $L^2(L^1(S^{n-2}), R^{n-1})$, $n \geq 2$. In n dimensions, let $v = (v', v_n) \in S^{n-1}$ where $v' = (v_1, \ldots, v_{n-1})$, and $x = (x', x_n) \in R^n$, where $x' \in R^{n-1}$, and $\delta_t v' =$

$(t^{a_1}v_1, \ldots, t^{a_{n-1}}v_{n-1})$. Let

$$\Delta_k^{n-1} * f(x) = \int_1^2 f(x' - \delta_{2^k t}v', x_n)\, dt.$$

We claim that $\sup_k |\Delta_k^{n-1} * f(x)|$ is bounded on $L^2(L^1(S^{n-1}), R^n)$. Let us make a change of variables for v, $v \in S^{n-1}$. Let $d\sigma_{n-2}$ denote the Lebesgue measure on S^{n-2}. Since

$$\left\| \sup_k |\Delta_k^{n-1} * f| \right\|_{L} L^2(L^1)$$

$$= \left[\int_{R^n} \left| \int_{-1}^1 \int_{\xi \in S^{n-2}} \sup_k \int_1^2 f\left(x' - (1 - S^2)^{1/2} \delta_{2^k t}\xi, x_n\right) dt \right.\right.$$
$$\left.\left. \times d\sigma_{n-2}(\xi)(1 - S^2)^{(n-3)/2}\, dS \right|^2 dx \right]^{1/2}$$

$$\leq \int_{-1}^1 \left[\int_R \int_{R^{n-1}} \left| \int_{\xi \in S^{n-2}} \left\{ \sup_k \int_1^2 f\left(x' - (1 - S^2)^{1/2} \delta_{2^k t}\xi, x_n\right) dt \right\} \right.\right.$$
$$\left.\left. \times d\sigma_{n-2}(\xi) \right|^2 dx'\, dx_n \right]^{1/2} (1 - S^2)^{n-3/2}\, dS.$$

After a change of variables for x, (i.e., $x' \to (1 - S^2)^{1/2}x'$, $x_n \to x_n$), we let

$$f_S(x', x_n) = f\left((1 - S^2)^{1/2}x', x_n\right).$$

By the induction hypothesis, the term in parentheses is bounded on

$$L^2\left(L^1(S^{n-2}), R^{n-1}\right)$$

for almost every $x_n \in R$. Therefore, the last inequality is not bigger then

$$\int_{-1}^1 \|f_S\|_{L^2(R^n)} (1 - S^2)^{(n-1)/4} (1 - S^2)^{(n-3)/2}\, dS.$$

Then we change variables again. The inequality above becomes

$$\int_{-1}^1 \|f\|_2 (1 - S^2)^{(n-3)/2}\, dS \leq C\|f\|_2,$$

if $n \geq 2$. Therefore, we have

$$\left\| \left\| \sup_k |\Delta_k^{n-1} * f| \right\|_{L^1(S^{n-1})} \right\|_{L^2(R^n)} \leq C\|f\|_2, \tag{2.1}$$

Let p be a smooth function in R, (this p is different from the p in Lemma 1), $p \in C_0^\infty(R)$, and $\int p(x)\,dx = 1$. Let

$$p_k(x) = \frac{1}{2^{ka_n}} p\left(\frac{x}{2^{ka_n}}\right).$$

Let

$$\left(\Delta_k^{n-1} \otimes p_k\right) * f = \int_R \int_1^2 f(x' - \delta_{2^k t} v', x_n - y_n) p_k(y_n)\, dt\, dy_n.$$

Let us write

$$\Delta_k * f \leq \left|\Delta_k * f - \left(\Delta_k^{n-1} \otimes p_k\right) * f\right| + \sup_k |\Delta_k^{n-1} * M_n f|,$$

where M_n is the Hardy-Littlewood maximal operator acting on x_n variable. From (2.1) the second term of the right hand side of the above inequality is bounded on $L^2(L^1)$. On the other hand, the first term is dominated by the square function,

$$G(f) = \left(\sum_k \left|\Delta_k * f - \left(\Delta_k^{n-1} \otimes p_k\right) * f\right|^2\right)^{1/2}.$$

To show $G(f)$ is bounded on $L^2(L^1)$, by the Minkowski's inequality and the Plancherel's theorem, it is sufficient to show that

$$\int \Bigg(\int \int \sum_k |\hat{\Delta}_k(v - x) - \hat{\Delta}_k^{n-1}(v', x') \hat{p}(2^{ka_n} x_n)|^2 |\hat{f}(x', x_n)|^2\, dx'\, dx_n \Bigg)^{1/2} d\sigma(v)$$

is bounded by $C\|f\|_2$. Since

$$\begin{aligned} &|\hat{\Delta}_k(v, x) - \hat{\Delta}_k^{n-1}(v', x') \hat{p}(2^{ka_n} x_n)| \\ &= \left| \int_1^2 \exp(ix\delta_{2^k t} v) - \exp(ix'\delta_{2^k t} v') \hat{p}(2^{ka_n} x_n)\, dt \right| \\ &\leq \int_1^2 |\exp(ix_n v_n 2^{ka_n} t^{a_n}) - \hat{p}(2^{ka_n} x_n)|\, dt \\ &\leq C|2^{ka_n} x_n v_n| + C'|2^{ka_n} x_n| \leq C|2^{ka_n} x_n|. \end{aligned}$$

On the other hand, as in the proof for Lemma 1, using the Van der Corput's lemma, and $\hat{p}(x) \leq 1/|x|$, we have, if $|2^{ka_n}x_n|$ is large,

$$\left|\hat{\Delta}_k(v,x) - \hat{\Delta}_k^{n-1}(v',x')\hat{p}(2^{ka_n}x_n)\right| \leq C|\xi|^{-1/n} + |2^{ka_n}x_n|^{-1} \leq C|2^{ka_n}x_n v_n|^{-1/n}$$

where $\xi = (2^{ka_1}x_1v_1, \ldots, 2^{ka_n}x_n v_n)$. Hence, we get

(2.2)
$$\left|\hat{\Delta}_k(v,x) - \hat{\Delta}_k^{n-1}(v',x')\hat{p}_k(x_n)\right| \leq C|v_n|^{-1/n}\min\left\{2^{ka_n}|x_n|, |2^{ka_n}x_n|^{-1/n}\right\}.$$

So

$$\sum_k \left|\hat{\Delta}_k(v,x) - \hat{\Delta}_k^{n-1}(v',x')\hat{p}_k(x_n)\right|^2 \leq C|v_n|^{-2/n}.$$

Hence $G(f)$ is bounded on $L^2(L^1)$, if $n \geq 2$. That is to say

$$\left\|\sup_k|\Delta_k * f|\right\|_{L^2(L^1)} \leq C\|f\|_2. \tag{2.3}$$

Part 2. Now, we start to prove that $\sup_k|\Delta_k * f|$ is bounded on $L^p(L^1)$, $1 < p \leq \infty$, again, via in duction argument at dimension n. As above, when $n = 1$, $\sup_k|\Delta_k * f| \leq CM(f)$. Suppose that for dimension $n-1$,

$$\left\|\left\|\sup_k|\Delta_k * f|\right\|_{L^1(S^{n-2})}\right\|_{L^p(R^{n-1})} \leq C\|f\|_{L^p(R^{n-1})}, \quad 1 < p \leq \infty.$$

(Note that

$$\left\|\left\|\sup_k|\Delta_k^{n-1} * f|\right\|_{L^1(S^{n-1})}\right\|_{L^p(R^n)} \leq C\|f\|_{L^p(R^n)}. \tag{2.4}$$

by the same proof as in (2.1) of part I.)

Let us consider a partition of unity on $(0,\infty)$. That is, there exists a function $h \in C_0^\infty(R)$ supported in $[2^{-a_n}, 2^{a_n}]$ and such that $\sum_j h(2^{a_n j}t) = 1$. Define

$$\widehat{S_j f}(x', x_n) = h(2^{a_n j}|x_n|)\hat{f}(x', x_n).$$

Then

$$\sup_k|\Delta_k * f| \leq \left(\sum_k \left|\left(\Delta_k - \Delta_k^{n-1} \otimes p_k\right) * f\right|^2\right)^{1/2} + \sup_k \left|\left(\Delta_k^{n-1} \otimes p_k\right) * f\right|$$

$$(2.5) \qquad \leq \sum_j \left(\sum_k \left|\left(\Delta_k - \Delta_k^{n-1} \otimes p_k\right) * S_{j+k} f\right|^2\right)^{1/2} + \sup_k \left|\left(\Delta_k^{n-1} \otimes p_k\right) * f\right|.$$

By (2.4) the second term of the last inequality is bounded on $L^p(L^1)$. Let

$$T_j f = \left(\sum_k \left|\left(\Delta_k - \Delta_k^{n-1} \otimes p_k\right) * S_{j+k} f\right|^2\right)^{1/2}.$$

First, let us compute the $L^2(L^1)$-norm of $T_j f$. We have

$$\|T_j f\|_{L^2(L^1)} \leq \left\| \|T_j f\|_{L^2(R^n)} \right\|_{L^1(S^{n-1})}$$
$$\leq \left\| \left(\sum_k \int_{R^{n-1}} \int_{1/2^{a_n} < |2^{a_n(j+k)} x_n| < 2^{a_n}} \left|\hat{\Delta}_k(v, x) - \hat{\Delta}_k^{n-1}(v', x') \hat{p}_k(x_n)\right|^2 \times \left|\hat{f}(x', x_n)\right|^2 dx_n\, dx' \right)^{1/2} \right\|_{L^1(S^{n-1})}.$$

From (2.2), we have

$$(2.6) \qquad \|T_j f\|_{L^2(L^1)} \leq C \min\left\{2^{-a_n j}, \left(2^{a_n j}\right)^{1/n}\right\} \|f\|_2.$$

Next, applying (2.3) and (2.4) (just with $p = 2$) to Lemma 4 (II) (let $g_k = S_{j+k} f$), and using the classical Littlewood-Paley theorem and the vector valued maximal operators, we have

$$\left\| \left(\sum_k |\Delta_k * S_{j+k} f|^2 \right)^{1/2} \right\|_{L^{p_0}(L^1)} \leq C\|f\|_{p_0},$$

and

$$\left\| \left(\sum_k \left|\left(\Delta_k^{n-1} \otimes p_k\right) * S_{j+k} f\right|^2 \right)^{1/2} \right\|_{L^{p_0}(L^1)} \leq C \left\| \left(\sum_k |M_n S_{j+k} f|^2 \right)^{1/2} \right\|_{p_0} \leq C\|f\|_{p_0},$$

where $|1/2 - 1/p_0| = 1/4$. Hence

$$\|T_j f\|_{L^{p_0}(L^1)} \leq C\|f\|_{p_0}.$$

Interpolation between above inequality and (2.6), yields

$$\|T_j f\|_{L^p(L^1)} \leq C\min\{2^{-\varepsilon a_n j}, 2^{\varepsilon a_n j/n}\}\|f\|_p, \tag{2.7}$$

where $|1/p - 1/2| < 1/4$ and $0 < \varepsilon < 1$, ε depending on p. Hence, from (2.5), we have

(2.8)
$$\left\|\sup_k |\Delta_k * f|\right\|_{L^p(L^1)} \leq \left\|\sup_k \left|(\Delta_k^{n-1} \otimes p_k) * f\right|\right\|_{L^p(L^1)} + \sum_j \|T_j f\|_{L^p(L^1)} \leq C\|f\|_p,$$

if $|1/p - 1/2| < 1/4$. Again, applying (2.8) to Lemma 4 (II), finding the new range of p of inequality (2.7) and repeating the procedure as above, we conclude that (2.8) holds if $1 < p < \infty$. Lemma 2 is proved.

Proof of Lemma 3. If $n > \operatorname{Re}\alpha > 0$. Let G^α denote the Bessel potentials, (See [6], page 132). Then

$$\widehat{G^\alpha}(y) = \left(1 + |y|^2\right)^{-\alpha/2}$$

and let

$$\widehat{G_k^\alpha}(y) = \left(1 + |\delta_{2^k} y|^2\right)^{-\alpha/2}.$$

Then

$$T_{k,v}^\alpha f(x) = \int_1^2 \int_{R^n} f(x - y - \delta_{2^k t} v) G_k^\alpha(y)\, dy\, dt$$
$$= \int_{R^n} f(x - \delta_{2^k} y) \int_1^2 G^\alpha(y - \delta_t v)\, dt\, dy.$$

It is well known that $G^\alpha(x)$ is controlled by

$$\frac{1}{|x|^{n - \operatorname{Re}\alpha}} \quad \text{as } |x| \to 0$$

and is rapidly decreasing as $|x| \to \infty$. Therefore, $\int_1^2 G^\alpha(y - \delta_t v)\,dt$ is dominated by

$$\int_1^2 \frac{\chi_{|y|<C}}{|y - \delta_t v|^{n - \operatorname{Re}\alpha}}\,dt + C' \frac{\chi_{|y|>1}}{|y|^{n+1}}.$$

Since $v = (v_1, \ldots, v_n) \in S^{n-1}$, there exists v_i, say v_1, such that $v_1^2 \geq 1/n$. So the integration term is not bigger then

$$\int_1^2 \frac{\chi_{|y|<C}}{|y_1 - t^{a_1} v_1|^{n - \operatorname{Re}\alpha}}\,dt.$$

It is easy to see that if $n - 1 < \operatorname{Re}\alpha < n$, the above integral is bounded by a constant which doesn't depend on $v \in S^{n-1}$. Hence $T^\alpha_{k,v} f(x) \leq CMf(x)$. Lemma 3 is proved.

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