ILLINOIS JOURNAL OF MATHEMATICS
Volume 33, Number 2, Summer 1989

SYSTEMS OF EQUATIONS OVER FINITELY GENERATED SOLVABLE GROUPS

BY

WOJTEK JASTRZEBOWSKI[1] AND JOHN LAWRENCE[2]

Let G be a group. An equation in n variables over G is an expression of the form $p(x_1, \dots, x_n) = 1$, where $p(x_1, \dots, x_n)$ is an element of the coproduct $G * \langle x_1, \dots, x_n \rangle$. Consider the set $L^*(G, n)$ of solution sets of systems of such equations. $L^*(G, n)$ is a complete lattice with the greatest lower bound of a set of elements defined in the obvious way and the least upper bound defined as the greatest lower bound of the upper bounds.

Now consider equations without coefficients, that is, where $p(x_1, \dots, x_n)$ is an element of the free group $\langle x_1, \dots, x_n \rangle$. Denote by $L(G, n)$ the (complete) lattice of solution sets of systems of equations over G. If G is finitely generated by k elements, there is a natural map from $L(G, n + k)$ to $L^*(G, n)$ which takes k variables to generators of G. Thus if $L(G, n + k)$ is of finite length, so is $L^*(G, n)$. For this reason, when we are dealing with finitely generated groups, we may assume that our equations are without coefficients.

Our study of $L(G, n)$ was motivated by the study of equations over a free monoid and this in turn was motivated by Ehrenfeucht's Conjecture (see [1]). J. Lewin and T. Lewin have proved that a noncommutative free monoid can be embedded into a Cartesian power of a finitely generated solvable group if and only if the group is not nilpotent-by-finite. As the lattice $L(\prod G, n)$ (where $\prod G$ is a Cartesian power of G) is isomorphic to $L(G, n)$, a careful study of $L(G, n)$ might yield some information about the lattice of solution sets of systems of equations over a free monoid.

In [1], a characterization of the Abelian groups G for which $L(G, n)$ is of finite length for all n was given. In particular $L(G, n)$ is of finite length if G is torsion-free Abelian or finitely generated Abelian. Moreover, for Abelian groups, if $L(G, 1)$ is of finite length, $L(G, n)$ is of finite length for all n.

In this paper G will denote a group, G^n will denote the direct product of n copies of G, G' will denote the commutator subgroup of G and $Z(G)$ will

Received April 3, 1987.
[1]Supported by a NSERC Undergraduate Research Award.
[2]Supported by a NSERC grant.

denote the centre of G. The generalized commutator $[x_1, x_2, \dots, x_n]$ is defined inductively by

$$[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$$

and

$$[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n].$$

We start off by considering subgroups of finite index.

THEOREM 1. *Let G be a group with a normal subgroup H of finite index. If $L(H, n)$ is of finite length for all n, then $L(G, n)$ is of finite length for all n.*

Proof. Let l be the index of H in G and let

$$\{c_1 = 1, c_2, \dots, c_l\}$$

be a fixed set of coset representatives.

Suppose that $p_i(x_1, \dots, x_n) = 1$, $i = 1, 2, \dots, r$, is a sequence of equations with $\alpha_j = (a_{j1}, \dots, a_{jn}) \in G^n$ a solution to $p_i = 1$ if $i < j$ and α_j not a solution to $p_j = 1$, $j = 1, 2, \dots, r$.

Under the natural map $\bar{}\ : G \to G/H$ consider

$$\bar{\alpha}_j = (\bar{a}_{j1}, \dots, \bar{a}_{jn}) \in (G/H)^n.$$

For some subset $\mathscr{S} \subset \{1, 2, \dots, r\}$ of cardinality $s \geq rl^{-n}$ we will have $\bar{\alpha}_j$ constant for all $j \in \mathscr{S}$. Note that if $\mathscr{S} = \{i_1, i_2, \dots, i_s\}$, then α_{i_m} is a solution to $p_{i_\alpha} = 1$ if $j < m$ but α_{i_m} is not a solution to $p_{i_m} = 1$. For this reason, we may assume, without loss of generality, that $\mathscr{S} = \{1, 2, \dots, s\}$.

We now want to transform our equations over G in variables X_i to equations over H in a new (finite) set of variables. In order to do this make the following replacements. Replace X_j by c_jY_j, where $\bar{c}_j = \bar{a}_{1j}$ and Y_j is a new variable, replace Y_ic_j by c_jZ_{ij}, $(i = 1, 2, \dots, n;\ j = 1, \dots, l)$, where Z_{ij} is a new variable and replace c_ic_j by $c_{\phi(i,j)}W_{ij}$ $(i, j = 1, \dots, l)$, where $\bar{c}_i\bar{c}_j = \bar{c}_{\phi(i,j)}$ and W_{ij} is a new variable. Note that for fixed n, the number of new variables introduced is a function of r and l only. Using these formal substitutions we transform a word in the X_i's to a word in c_jY_j's and then move the coset representatives to the left side by starting at the right and moving left. For example the word $X_1X_2X_3$ becomes $c_1Y_1c_2Y_2c_3Y_3$ which then becomes an expression of the form $c_*W_*Z_*W_*Z_*Y_3$. As $X_1X_2X_3 = 1$ has a solution, $c = 1$.

We now have a system of equations $p_t^*(Y_i, Z_{ij}, W_{ij}) = 1$, $t = 1, 2, \dots, s$ in $n + nl + l^2$ variables with a strictly descending chain of solution sets over the group H. As $L(H, n + nl + l^2)$ is of finite length, s is uniformly bounded. But $r \leq sl^n$; hence, r is uniformly bounded.

This completes the proof of the theorem.

If A is a torsion-free Abelian group, then $L(A, n)$ is of finite length for all n. We now generalize this.

THEOREM 2. *Suppose that G is a torsion-free nilpotent group. Then $L(G, n)$ is of finite length for all n.*

Proof. We use induction on the nilpotent length l of G. The induction has been started at $l = 1$. A polynomial in n variables, X_i, over a nilpotent group of nilpotent class l is equivalent to one of the form

$$X_1^{m_1} X_2^{m_2} \dots X_n^{m_n} *$$

where $*$ is a product of generalized commutators with entries X_i and X_i^{-1} and of length at most $l + 1$. Replace each of these generalized commutators by a variable Y_j. Thus $*$ is a word in the variables Y_j. Note that the number of new variables introduced is a function of n and l only.

Suppose that $p_i(X_1, \dots, X_n) = 1$, $i = 1, 2, \dots, s$ is a sequence of equations with a strictly descending chain of solution sets. If

$$p_i(X_1, \dots, X_n) = X_1^{m_{1i}} X_2^{m_{2i}} \dots X_n^{m_{ni}} *_i$$

where $*_i$ is a product of generalized commutators, we let

$$\alpha(p_i) = (m_{1i}, \dots, m_{ni}).$$

Suppose that for some integer l the n-tuple $(m_{1l}, \dots, m_{nl})$ is in the span of $\{(m_{1i}, \dots, m_{ni}) \colon 1 \leq i < l\}$. Then there is a positive integer r_l and integers r_i, $1 \leq i < l$, such that

$$r_l(m_1, \dots, m_{nl}) = \sum_{i=1}^{l-i} r_i(m_{1i}, \dots, m_{ni}).$$

Now suppose that

$$\beta = (a_1, \dots, a_n) \in G^n$$

is a solution to $p_i(X_1, \dots, X_n) = 1$ for $i = 1, 2, \dots, l - 1$. Then β is a solution to $p_l(X_1, \dots, X_n) = 1$ if and only if it is a solution to $[p_l(X_1, \dots, X_n)]^{r_l} = 1$ (it is here that we use the fact that the group G is torsion-free) and this occurs if and only if β is a solution to

$$[p_l(X_1, \dots, X_n)]^{r_l} \prod_{i=1}^{l-1} [p_i(X_1, \dots, X_n)]^{-r_i} = 1.$$

Note that the polynomial on the left is equivalent to one involving only the variables Y_j. We conclude that if $\alpha(p_l)$ is in the span of $\{\alpha(p_i)\colon 1 \leq i < l\}$, then the polynomial $p_l(X_1, \dots, X_n)$ may be replaced by one involving only the variables Y_j and we will still get a strictly descending chain.

Now by induction on the nilpotent length of G, there is a uniform bound on the length of a strictly descending chain of solution sets for a sequence of equations involving only the Y_i's, for in this case we are essentially solving in the commutator subgroup of G. However $\alpha(p_l)$ is in the span of $\{\alpha(p_i)\colon 1 \leq i < l\}$ except in at most $n + 1$ cases, so there is a uniform bound on the length of our chain $s - 1$.

This completes the proof of the theorem.

THEOREM 3. *If G is a finitely generated nilpotent-by-finite group, then $L(G, n)$ is of finite length for all n.*

Proof. Suppose that G is a finite extension of the nilpotent group H. As G is finitely generated, so is H. The group H embeds into a direct product $A \times B$, where A is torsion-free nilpotent and B is finite [2, Theorem 2.1]. By Theorem 2, $L(A, n)$ has finite length and by two applications of Theorem 1, we see that $L(G, n)$ has finite length.

If $L(G, n)$ is of finite length, what can be said about G? If G is torsion-free, then $L(G, 1)$ has finite length. If G is a non-Abelian free group, then $L(G, 2)$ has finite length, while $L(G, 3)$ does not [1].

Example. Let $G = \mathbf{Z} \wr \mathbf{Z}$ denote the restricted wreath product of the infinite cyclic groups. This group is finitely generated metabelian. Let $e_i(X, Y) = X^i Y^{-1}$ and let

$$E_i(X, Y) = \left[X, Y, e_1(X, Y), \dots, \widehat{e_i}(X, Y), \dots, e_n(X, Y)\right], \quad i = 1, \dots, n,$$

where $\widehat{e_i}(X, Y)$ means that $e_i(X, Y)$ is omitted in the above equation. Then the system of equations

$$E_1(X, Y) = 1,\ E_2(X, Y) = 1, \dots,\ E_n(X, Y) = 1$$

has the property that for each k, there exists $(a_k, b_k) \in G^2$ which is a solution to $E_i(X, Y) \cdot = 1$, $i \neq k$ but is not a solution to $E_k(X, Y) = 1$.

THEOREM 4. *Suppose that G is a group for which $L(G, 3)$ has finite length k. Let $n_1\ _k$ be a sequence of distinct integers. Then for some l, $1 < l \leq k$, G satisfies the identity*

$$[X, Y^{n_1 - n_l}, Y^{n_2 - n_l}, \dots, Y^{n_{l-1} - n_l}] = 1.$$

Proof. Consider the system of equations

$$E_i\colon [X, Y^{n_1}Z^{-1}, \dots, Y^{n_i}Z^{-1}] = 1, \quad i = 1, 2, \dots, k.$$

If $i < j$, then the solution set of E_i is contained in the solution set of E_j. Thus, taking i from k to 1 we have a descending chain of solution sets. If we add the equations $X = 1$ and $Y = 1$ at the end, then we are looking at $k + 2$ solution sets. As $L(G, 3)$ has finite length k, for some l the solution set of E_l equals the solution set of E_{l-1} $(1 < l \le k)$. If a and b are elements of G, then $X = a$, $Y = b$, $Z = b^{n_l}$ is a solution to E_l; hence, it is a solution to E_{l-1}. It follows that for all a, b in G, we have

$$[a, b^{n_1 - n_l}, \dots, b^{n_{l-1} - n_l}] = 1.$$

This completes the proof of the theorem.

THEOREM 5. *Let G be a finitely generated solvable group satisfying an identity $[x, y^{n_1}, \dots, y^{n_k}] = 1$ where $n_i > 0$. Then G is nilpotent-by-finite.*

Proof. Induct on solvability length of G. If $G' = 1$ then nothing to prove. Hence assume there is a normal Abelian subgroup A of G such that G/A is nilpotent-by-finite. Pass to a subgroup of finite index, if necessary, and assume G/A is torsion-free nilpotent. Now G satisfies the maximal condition for normal subgroups, so we may assume that every proper quotient of G is nilpotent-by-finite.

There is a central series

$$A \lhd X_1 = \langle A, g_1 \rangle \lhd X_2 = \langle X_1, g_2 \rangle \dots \lhd X_r = \langle X_{r-1}, g_r \rangle = G$$

from A to G with infinite cyclic factors. Let

$$d = \operatorname{lcm}\{n_1, \dots, n_k\}.$$

Then every element of $\langle A, G^{d^r} \rangle = J$ is the d-th power of an element of G modulo A. (See [5]; originally proved by Malcev in 50's.)

Let $Y_i = X_i \cap J$. Then for any $y_i \in Y_i$ there exists $x_i \in X_i$ such that

$$x_i^d \equiv y_i \pmod{A}.$$

Pick $a \ne 1$ from A. Then $[a, x_i^{n_1}, \dots, x_i^{n_k}] = 1$ implies $C_A(y_i) \ne 1$. In particular $C_1 = C_A(Y_1) \ne 1$. Since $C_1 \lhd G$, $1 \ne C_2 = C_{C_1}(Y_2) \lhd G$. Inductively we obtain

$$1 \ne C_r = C_{C_{r-1}}(Y_r) \lhd G.$$

But $C_r \leq Z(J)$. By our choice of G, J/C_r is nilpotent by finite. Hence J is nilpotent-by-finite.

COROLLARY. *Suppose that G is a finitely generated solvable group satisfying an identity of the form $[X, Y^n, Y^n, \dots, Y^n] = 1$. Then G is nilpotent-by-finite.*

The above corollary generalizes the special case where $n = 1$. In this case it is known that the group must be nilpotent (see [4] for example).

THEOREM 6. *Suppose that G is a finitely generated solvable group such that $L(G, 3)$ is of finite length. Then G is nilpotent-by-finite.*

Proof. This follows from Theorem 5.

Acknowledgement. The authors would like to thank the referee for simplifying the proof and generalizing the statement of Theorem 5.

REFERENCES

1. M. ALBERT and J. LAWRENCE, *The descending chain condition on solution sets for system of equations in groups*, Proc. Edinburgh Math. Soc., vol. 29 (1986), pp. 69–73.
2. G. BAUMSLAG, *Lecture notes on nilpotent groups*, Conference Board of Mathematical Sciences, Amer. Math. Soc., 1969.
3. G. BIRKHOFF, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, Vol. 25, Third Ed., 1973.
4. K.W. GRUENBERG, *Two theorems on Engel groups*, Proc. Cambridge Philos. Soc., vol. 49 (1953), pp. 377–380.
5. J.C. LENNOX, *A note on a centrality property of finitely generated soluble groups*, Proc. Cambridge Philos. Soc., vol. 75 (1974), pp. 23–24.
6. J. LEWIN and T. LEWIN, *Semigroup laws in varieties of groups*, Proc. Cambridge Philos. Soc., vol. 65 (1969), pp. 1–9.
7. R. LYNDON, *Equations in groups*, Bol. Soc. Brasil Mat., vol. 11 (1980), pp. 79–102.
8. I. MACDONALD, *The theory of groups*, Clarendon Press, Oxford, 1968.
9. J. MILNOR, *Growth of finitely generated solvable groups*, J. Differential Geometry, vol. 2 (1968), pp. 447–449.
10. D. ROBINSON, *Finiteness conditions and generalized solvable groups*, Springer-Verlag, New York, 1972.

UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO, CANADA