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PARTIALLY ISOMETRIC APPROXIMATION OF POSITIVE OPERATORS

BY

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1. Introduction

Consider the problems of minimizing the quantity

$$\|A - U\|_p$$

where A is a fixed positive operator and where U varies over the set of (i) all unitaries, (ii) all isometries, and (iii) all partial isometries (subject to the condition that $A - U \in \mathscr{C}_p$ where $\mathscr{C}_p$ denotes the von Neumann-Schatten p class). In the language introduced by Halmos [6], problems (i), (ii) and (iii) concern, respectively, unitary, isometric and partially isometric approximants in $\mathscr{C}_p$ of a positive operator. Problem (i) has been solved by Aiken, Erdos and Goldstein [1]. This paper tackles problems (ii) and (iii).

Aiken, Erdos and Goldstein proved that if the operator A is positive and U varies over all those unitaries such that $A - U \in \mathscr{C}_p$, where $1 \le p < \infty$, then $\|A - U\|_p$ is minimized when $U = I$ and, providing the underlying Hilbert space is finite-dimensional, maximized when $U = -I$ [1, corollary 3.6]. Further, if A is strictly positive and $1 < p < \infty$ these minimum and maximum points are unique [1, Theorem 3.5]. They also obtained the corresponding inequality for the operator norm [1, Theorem 3.1]: if A is positive then for all unitaries U in $\mathscr{L}(H)$

$$\|A - I\| \le \|A - U\| \le \|A + I\|. \tag{1.1}$$

A feature of their work is the use of noncommutative differential calculus. They found an explicit formula [1, Theorem 2.1] for the derivative of the map $X \mapsto \|X\|_p^p$, where $X \in \mathscr{C}_p$ with $1 < p < \infty$ (see Theorem 2.3 below). In searching for a global minimizer of $\|A - U\|_p$ one can thus restrict attention

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to those operators that are local extrema of the map $U \mapsto \|A - U\|_p^p$; cf. [1, Theorem 3.5].

Interestingly, the problem of minimizing $\|A - U\|_p$ arises from quantum chemistry: see [1], [2] and compare with [5].

In §2 of this paper we recall various preliminaries about partial isometries and the von Neumann-Schatten p-classes. In §3 we turn to isometric approximation of positive operators: for $\mathscr{C}_p$, where $0 < p \leq \infty$, this problem turns out to be exactly that of unitary approximation; whilst (1.1) also holds for all isometries in $\mathscr{L}(H)$.

Partially isometric approximation, dealt with in §4 and §5, is harder (perhaps because the initial and final spaces of a non-normal partial isometry do not coincide). §4 deals with the local theory pertaining to the map

$$F_p\colon U \mapsto \|A - U\|_p^p \tag{1.2}$$

where U varies over those partial isometries such that $A - U \in \mathscr{C}_p$, where $1 < p < \infty$, and A is positive. This local theory is utilized in §5, in particular in Lemma 5.4 and Theorem 5.6. Theorem 5.6 says that if F_p attains a global minimum then

$$\|A - E_{1/2}\|_p \leq \|A - U\|_p \tag{1.3}$$

(where $E_{1/2}$ is a certain projection introduced in Definition 5.5); and, for strictly positive A such that $\frac{1}{2} \notin \sigma_p(A)$, equality occurs in (1.3) if and only if $U = E_{1/2}$.

The problem of partially isometric approximation thus becomes an existence problem. If the underlying space is finite-dimensional then F_p attains a global minimum (and, at $U = -I$, a global maximum) so that (1.3) holds, with a similar result in the operator norm (see Theorem 5.7). In infinite dimensions, the Halmos/Bouldin theory of normal spectral approximants [7], [3] shows that (1.3) holds provided $p \geq 2$ and U is a *normal* partial isometry (see Theorem 5.11).

The positivity condition on A can be weakened. There is an infinite-dimensional result (Theorem 5.10) about approximating a normal operator A by normal partial isometries; and a finite-dimensional result (5.8) about approximating an arbitrary operator A by partial isometries.

After writing this paper I learnt that Wu [11] had obtained formulas for the operator norm distance, $\inf\|A - U\|$, where A is arbitrary and where U varies over (i) the isometries, (ii) the isometries and the co-isometries, and (iii) the partial isometries. For the cases considered in this paper, it can be checked that the relevant distance is attained when $U = I$ or when $U = E_{1/2}$.

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2. Preliminaries

Throughout this paper the term *Hilbert space* means complex Hilbert space with inner product denoted by $\langle \ , \ \rangle$, *basis* means orthonormal basis, *operator* means bounded linear operator and $\mathscr{L}(H)$ denotes the set of all operators on the Hilbert space H. *Projection* means orthogonal projection. The *spectrum* of an operator A is denoted by $\sigma(A)$ and its *point spectrum* by $\sigma_p(A)$. A self-adjoint operator A in $\mathscr{L}(H)$ is *positive* if $\langle Af, f\rangle \geq 0$ for all f in H and *strictly positive* if, further, $\operatorname{Ker} A = \{0\}$.

An operator is a *partial isometry* if it is isometric on the orthogonal complement of its kernel: thus, U is a partial isometry if $\|Uf\| = \|f\|$ for all f in $(\operatorname{Ker} U)^{\perp}$. If U is a partial isometry then U^*U and UU^* are, respectively, the projections onto $(\operatorname{Ker} U)^{\perp}$ (called the *initial space* of U and onto $\operatorname{Ran} U$ (called the *final space* of U). For a partial isometry U we shall write $E_U = U^*U$ and $F_U = UU^*$ (so that $E_{U^*} = F_U$). Thus, a partial isometry U is normal if and only if $E_U = F_U$, that is, if and only if its initial and final spaces coincide. Note also that an operator U is a partial isometry if and only if $U = UU^*U$.

The *polar decomposition* says that every operator A in $\mathscr{L}(H)$ can be expressed uniquely as $A = U_0|A|$ where $|A| = (A^*A)^{1/2}$ and where U_0 is the partial isometry such that $\operatorname{Ker} U_0 = \operatorname{Ker}|A|$ (and where $\operatorname{Ran} U_0 = \operatorname{Ran} A$) [8, Chapter 16]. Note: the partial isometry U_0 can be extended to a unitary, say $\hat{U}_0$, which agrees with U_0 on $(\operatorname{Ker} U_0)^{\perp}$ (and which can be any isometry mapping $\operatorname{Ker} U_0$ onto $(\operatorname{Ran} A)^{\perp}$) if and only if $\dim \operatorname{Ker} U_0 = \dim(\operatorname{Ran} U_0)^{\perp}$, that is, if and only if, $\dim \operatorname{Ker} A = \dim \operatorname{Ker} A^*$ [9, p. 586]. (In finite dimensions the condition $\dim \operatorname{Ker} U_0 = \dim(\operatorname{Ran} U_0)^{\perp}$ is automatically met.)

We now give a brief resumé of the properties we require of the von Neumann–Schatten p-classes [4, Chapter XI]. For a compact operator A, let $s_1(A), s_2(A), \ldots$ denote the (positive) eigenvalues of $|A|$ arranged in decreasing order and repeated according to multiplicity. If, for some $p > 0$,

$$\sum_{i=1}^{\infty} s_i(A)^p < \infty$$

we say that A is in the von Neumann-Schatten p class $\mathscr{C}_p$ and write

$$\|A\|_p = \left[\sum_{i=1}^{\infty} s_i(A)^p\right]^{1/p}.$$

If $1 \leq p < \infty$, it can be shown that $\| \ . \ \|_p$ is a norm and under this norm $\mathscr{C}_p$ is a Banach space; if $0 < p < 1$, $\mathscr{C}_p$ is a metric space with metric given by

$$d(A, B) = \sum_{i=1}^{\infty} s_i(A - B)^p.$$

For all p, where $0 < p < \infty$, $\mathscr{C}_p$ is a two-sided ideal of $\mathscr{L}(H)$ and $\|A\|_p = \|A^*\|_p$ if $A \in \mathscr{C}_p$. If $0 < p_1 \leq p_2 < \infty$ then $\mathscr{C}_{p_1} \subseteq \mathscr{C}_{p_2}$. We identify $\mathscr{C}_\infty$ with the two-sided ideal of compact operators in $\mathscr{L}(H)$. The algebra $\mathscr{L}(H)/\mathscr{C}_\infty$ is the *Calkin algebra*.

The class $\mathscr{C}_1$ is called the *trace class*. If $A \in \mathscr{C}_1$ and if $\{\phi_n\}$ is a basis of the Hilbert space H then the quantity $\tau(A)$, called the *trace* of A and defined by

$$\tau(A) = \sum_n \langle A\phi_n, \phi_n \rangle,$$

can be shown to be finite and independent of the particular basis chosen. If $A \in \mathscr{C}_1$ and $S \in \mathscr{L}(H)$ then $\tau(SA) = \tau(AS)$. The rank 1 operator $x \mapsto \langle x, e\rangle f$, where e and f are fixed vectors in H, will be denoted by $e \otimes f$. Note that

$$A(e \otimes f)B = (B^*e) \otimes (Af) \quad \text{and} \quad \tau(e \otimes f) = \langle f, e \rangle.$$

If A is a compact normal operator and (λ_n) is the sequence of non-zero eigenvalues of A arranged in decreasing order of magnitude and repeated according to multiplicity then, for $0 < p < \infty$, $A \in \mathscr{C}_p$ if and only if $\sum_n |\lambda_n|^p < \infty$ and when $A \in \mathscr{C}_p$,

$$\|A\|_p^p = \sum_{n=1}^{\infty} |\lambda_n|^p. \tag{2.1}$$

From [10, Theorem (1.9)] we shall require the following result: if $A + B \in \mathscr{C}_p$, where $0 < p < \infty$, and if $\operatorname{Ran} A \perp \operatorname{Ran} B$ and $\operatorname{Ran} A^* \perp \operatorname{Ran} B^*$ then $A \in \mathscr{C}_p$, $B \in \mathscr{C}_p$ and

$$\|A + B\|_p^p = \|A\|_p^p + \|B\|_p^p. \tag{2.2}$$

Next we state the Aiken, Erdos and Goldstein differentiation result. The real part of a complex number z will be denoted by $\operatorname{Re} z$.

THEOREM 2.3 [1, Theorem 2.1]. *If $1 < p < \infty$ then the map $\mathscr{C}_p \to \mathbf{R}^+$ given by $X \mapsto \|X\|_p^p$ is Fréchet differentiable with derivative D_X at X given by*

$$D_X(T) = p \operatorname{Re} \tau\big(|X|^{p-1}U^*T\big)$$

where $X = U|X|$ is the polar decomposition of X. If the underlying Hilbert space is finite-dimensional, the same result holds for $0 < p \leq 1$ at every invertible element X.

We shall require the notion of retraction. If Λ is a non-empty closed subset of the complex plane $\mathbf{C}$ then a retraction for Λ is a function F mapping $\mathbf{C}$

onto Λ such that

$$|z - F(z)| \leq |z - \lambda|$$

for each λ in Λ, where z is an arbitrary complex number. It can be shown that every non-empty closed subset Λ of $\mathbf{C}$ has a Borel measurable retraction [7, p. 55, Lemma]. Also Λ has a unique retraction F if and only if Λ is convex if and only if F is continuous [7, p. 54]. We next state Halmos' main result on retractions and Bouldin's $\mathscr{C}_p$ variant of it.

THEOREM 2.4. *Let F be a Borel measurable retraction for a non-empty closed subset Λ of $\mathbf{C}$, let A be a normal operator and let $\mathscr{N}(\Lambda)$ denote the set of all those normal operators each of whose spectrum is in Λ. Then:*

(a) [7, p. 55, Theorem] *$F(A) \in \mathscr{N}(\Lambda)$ and for all X in $\mathscr{N}(\Lambda)$,*

$$\|A - F(A)\| \leq \|A - X\|.$$

(b) [3, Theorem 2] *In addition, if X varies such that $A - X \in \mathscr{C}_p$, where $2 \leq p < \infty$, then $F(A)$ is also such that $A - F(A) \in \mathscr{C}_p$ and*

$$\|A - F(A)\|_p \leq \|A - X\|_p; \tag{1}$$

further, $F(A)$ is the unique choice of X producing equality in (1) *if and only if every point of $\sigma(A)$ has a unique closest point in Λ.*

The operator $F(A)$ occurring in Theorem 2.4 (a)/(b) is called a *normal spectral approximant* of A (in norm/in $\mathscr{C}_p$)).

3. Isometric approximation of positive operators

We now extend the results of Aiken, Erdos and Goldstein to isometric approximation of positive operators. First, their preliminary result, which says that if $A - U \in \mathscr{C}_p$ for some unitary U and positive A then $A - I \in \mathscr{C}_p$ [1, Theorem 3.2], holds equally well for isometric U (all that is required is that U satisfies $U^*U = I$). Thus:

LEMMA 3.1. *If $\mathscr{J}$ is a two-sided ideal of $\mathscr{L}(H)$ and if $A - U \in \mathscr{J}$ for some positive operator A and some isometry U then $A - I \in \mathscr{J}$; in particular, if $A - U \in \mathscr{C}_p$, where $0 < p \leq \infty$, then $A - I \in \mathscr{C}_p$.*

Theorem 3.2 gives the extension to isometries. Theorem 3.2 depends on the Fredholm alternative [8, Problem 179] which says that if K is compact and if $0 \neq \lambda \in \mathbf{C}$ then either λ is an eigenvalue of K or $K - \lambda I$ is invertible.

THEOREM 3.2. *Let A be a positive operator and U be an isometry such that $A - U \in \mathscr{C}_p$, where $0 < p \leq \infty$. Then U is unitary.*

Proof. From Lemma 3.1 it follows that $A - I \in \mathscr{C}_p$. So

$$U - I = (A - I) - (A - U) \in \mathscr{C}_p.$$

Hence $U - I = K$, say, is compact. Now, $U = I + K$ is isometric and hence 1-1. So $\text{Ker}(I + K) = \{0\}$ and hence -1 is not an eigenvalue of the compact operator K (for otherwise, $\text{Ker}(I + K)$ would contain a non-zero eigenvector of K with corresponding eigenvalue -1). Therefore by the Fredholm alternative, $K - (-1)I$ $(= U)$ is invertible and hence unitary. ■

Conclusion so far: all of Aiken, Erdos and Goldstein's results about unitary approximation in $\mathscr{C}_p$, where $0 < p \leq \infty$, hold for isometric approximation.

The same is true of their operator norm result (1.1): for the proof of (1.1) depends on the equality

$$(\|Af\| - 1)^2 \leq \|(A - U)f\|^2 \leq (\|Af\| + 1)^2$$

(where $\|f\| = 1$), an inequality which holds if U is isometric.

The positivity condition on A can be weakened; cf. [1, p. 63]. Let A be any operator such that $\dim \text{Ker} A = \dim \text{Ker} A^*$; then $A = \hat{U}_0|A|$ for some unitary $\hat{U}_0$. Suppose U is unitary and such that $A - U \in \mathscr{C}_p$. Then the equality

$$\|A - U\|_p = \big\| |A| - \hat{U}_0{}^*U \big\|_p \tag{3.3}$$

shows that, for $1 \leq p < \infty$, $\hat{U}_0$ is a unitary approximant in $\| \;.\; \|_p$ to A. The same result holds if U is assumed isometric: for then $\hat{U}_0{}^*U$ is isometric and, as $|A|$ is positive, Theorem 3.2 applies: $\hat{U}_0{}^*U$, and hence U, is unitary. Obviously, there is the corresponding result in the operator norm: if A satisfies $\dim \text{Ker} A = \dim \text{Ker} A^*$ and if $\hat{U}_0$ is as above then

$$\|A - \hat{U}_0\| \leq \|A - U\| \leq \|A + \hat{U}_0\|$$

for all isometries U in $\mathscr{L}(H)$.

Finally, since the norms $\| \;.\; \|_p$ and $\| \;.\; \|$ are self-adjoint all the results mentioned so far about isometric approximation hold for *co*-isometric approximation (U is a co-isometry if U^* is an isometry).

4. Partially isometric approximation: Local theory

THEOREM 4.1. *Let A be a positive operator and let the map F_p be defined by*

$$F_p\colon U \mapsto \|A - U\|_p^p$$

where U varies over those partial isometries such that $A - U \in \mathscr{C}_p$, where $1 < p < \infty$. If V is a critical point of F_p then:

(a) $AV = V^*A$ *and* $AV^* = VA$;

(b) $E_V A = AE_V$;

(c) Ker V *resduces* A *and* Ker A *reduces* E_V;

(d) Ker A *reduces* V;

(e) $AV = VA$;

(f) V *is self-adjoint if* A *is strictly positive*.

Proof. (a) The proof of (a) is the longest (results (b) to (f) are simple deductions from it). The proof is analogous to that in [1, Lemma 3.3] for unitary operators. Thus, for an arbitrary unit vector z and an arbitrary real θ let the unitary operator $W_z(\theta)$ be defined by

$$W_z(\theta) = e^{i\theta}(z \otimes z) + I - (z \otimes z).$$

If V is a critical point of F_p then, as $W_z(\theta)$ is unitary, $W_z(\theta)V$ and $VW_z(\theta)$ are both partial isometries and, for each z,

$$\frac{dF_p}{d\theta}(W_z(\theta)V) \quad \text{and} \quad \frac{dF_p}{d\theta}(VW_z(\theta))$$

both vanish at $\theta = 0$. Applying the chain rule to the map

$$\theta \mapsto W_z(\theta)V \mapsto F_p(W_z(\theta)V)$$

and using Theorem 2.3 we get

$$0 = \frac{dF_p}{d\theta}(W_z(\theta)V)\bigg|_{\theta=0} = p\,\mathrm{Re}\,\tau\big[|A - V|^{p-1}U_1{}^*i(z \otimes z)V\big] \tag{1}$$

where $A - V = U_1|A - V|$ is the polar decomposition of $A - V$. From (1) (since $\tau[S(z \otimes z)V] = \langle VSz, z\rangle$ where $S \in \mathscr{L}(H)$), it follows that the operator $V|A - V|^{p-1}U_1{}^*$ is self-adjoint:

$$V|A - V|^{p-1}U_1{}^* = U_1|A - V|^{p-1}V^*. \tag{2}$$

Similarly, since

$$\frac{dF_p}{d\theta}(VW_z(\theta))\bigg|_{\theta=0} = 0,$$

it follows that

$$|A - V|^{p-1}U_1{}^*V = V^*U_1|A - V|^{p-1}. \tag{3}$$

Now, we will have $AV = V^*A$ if and only if

$$|A - V|U_1^*V = V^*U_1|A - V| \tag{4}$$

because $V^*V = (A - |A - V|U_1^*)V = V^*(A - U_1|A - V|)$ since $A = A^*$ and $V = A - U_1|A - V|$ (the polar decomposition of $A - V$).

Proof of (4). If $p = 2$ it is obvious that (4) holds; for then (3) is the same as (4).

For arbitrary p, where $1 < p < \infty$, the proof uses the functional calculus for self-adjoint operators. Write $X = |A - V|^{p-1}$ and $Y = U_1^*V$. Then (3) says that

$$XY = Y^*X \tag{5}$$

and (4) is the same as $X^{1/(p-1)}Y = Y^*X^{1/(p-1)}$. This will follow, by the functional calculus, from

$$X^nY = Y^*X^n, \quad n \in \mathbf{N}, \tag{6}$$

for the function $f\colon t \mapsto t^{1/(p-1)}$, $1 < p < \infty$, where $t \in \mathbf{R}^+ \supseteq \sigma(X)$, satisfies $f(0) = 0$ and so can be approximated uniformly by a sequence $\{p_i\}$ of polynomials without constant term. Thus, (6) will imply that $p_i(X)Y = Y^*p_i(X)$ and hence that $X^{1/(p-1)}Y = Y^*X^{1/(p-1)}$.

The proof of (6) is by induction, first for odd, and then for even, n. We need the following assertion: $YX = XY^*$. To prove this assertion, observe that since

$$\operatorname{Ker} U_1 = \operatorname{Ker}|A - V| = \operatorname{Ker} X \quad \left(\text{where } X = |A - V|^{p-1}\right)$$

then $(\operatorname{Ker} U_1)^\perp = \operatorname{Ran} X$ and hence that $U_1^*U_1$, the projection onto $(\operatorname{Ker} U_1)^\perp$, satisfies

$$U_1^*U_1X = X = XU_1^*U_1.$$

Then multiplying (2) on the left by U_1^* and on the right by U_1 we get

$$U_1^*VXU_1^*U_1 = U_1^*U_1XV^*U_1,$$

that is, $YX = XY^*$ (where $Y = U_1^*V$). Returning now to (6) in the case of n odd: for $n = 1$, (6) is just (5); whilst the inductive step follows from the assertion (in the form $XY^* = YX$) and from (5).

The final step—that (6) holds for even n, too, —follows by another application of the functional calculus. Since (6) holds for odd n then

$$(X^{2l})(X^{2k-1}Y) = (Y^*X^{2k-1})(X^{2l}), \quad \text{where } l \geq 0 \text{ and } k \geq 1.$$

Hence, for every polynomial q, $q(X^2)(X^{2k-1}Y) = (Y^*X^{2k-1})q(X^2)$. Take a sequence $\{q_j\}$ of polynomials converging uniformly to the positive square root function $t \mapsto t^{1/2}$ where $t \in \mathbf{R}^+$. Then, as $q_j(X^2)(X^{2k-1}Y) = (Y^*X^{2k-1})q_j(X^2)$ for every q_j, it follows, on taking limits, that

$$X(X^{2k-1}Y) = (Y^*X^{2k-1})X,$$

that is, $X^{2k}Y = Y^*X^{2k}$—which is (6) for even n. This proves that $AV = V^*A$.

Finally, since A is self-adjoint, V is a local extremum of

$$F_p\colon U \mapsto \|A - U\|_p^p$$

if and only if V^* is a local extremum of F_p. Hence $AV^* = VA$.

(b) From (a), $E_VA^2 = A^2E_V$ (for, by (a), $V^*(VA)A = V^*(AV^*)A = AVAV = A(AV^*)V$). Hence, E_V commutes with A (the positive square root of A^2).

(c) $E_VA = AE_V$ means that $\operatorname{Ker} V$ $(= (\operatorname{Ran} E_V)^\perp)$ reduces A. $\operatorname{Ker} A$ reduces E_V since if $f \in \operatorname{Ker} A$ (so that $Af = 0$) then $E_Vf \in \operatorname{Ker} A$ since $E_VA = AE_V$.

(d) $\operatorname{Ker} A$ is invariant under V; for if $f \in \operatorname{Ker} A$ then $Vf \in \operatorname{Ker} A$ because, by (a), $AVf = V^*Af = 0$. Similarly, $\operatorname{Ker} A$ is invariant under V^* (because $AV^* = VA$).

(e) From (a), $A^2V = AV^*A = VA^2$. Hence, as A is positive, $AV = VA$.

(f) From (a) and (e), $V^*A = AV = VA$. So, $(V^* - V)A = 0$, that is, V is self-adjoint on $\operatorname{Ran} A = (\operatorname{Ker} A)^\perp$. Hence, $V = V^*$ if A is strictly positive. ∎

Notice that the positivity of A, though required in parts (b), (c), (e) and (f) above is not required in part (a) which holds if A is self-adjoint. In fact, the differentiation argument of Theorem (4.1) yields the following result: let A be in $\mathscr{L}(H)$ and let U vary over those partial isometries such that $A - U \in \mathscr{C}_p$, where $1 < p < \infty$; then if V is a local extremum of the map $U \mapsto \|A - U\|_p^p$ it follows that $A^*V = V^*A$.

Observe that the proof of Theorem 4.1 (in particular, the argument involving approximation to the function $t \mapsto t^{1/(p-1)}$ where $t \geq 0$) does not work in the $0 < p \leq 1$ case. Of course, this does not preclude Theorem 4.1 from holding when $0 < p \leq 1$ provided F_p is differentiable at V.

5. Partially isometric approximation; Global theory

First, we have the following partially isometric analogue of Lemma 3.1. Lemma 5.1 is stated for self-adjoint, rather than positive, operators A: the positivity of A is only used in the proof of Lemma 3.1 to ensure that $A + I$ is invertible [1, Theorem 3.2].

LEMMA 5.1. *If $\mathscr{J}$ is a two-sided ideal of $\mathscr{L}(H)$ and if $A - U \in \mathscr{J}$ for some self-adjoint operator A and some partial isometry U then $A^2 - E_U \in \mathscr{J}$ and $A^2 - F_U \in \mathscr{J}$ (where $E_U = U^*U$ and $F_U = UU^*$); in particular, if $A - U \in \mathscr{C}_p$, where $0 < p \leq \infty$, then $A^2 - E_U \in \mathscr{C}_p$ and $A^2 - F_U \in \mathscr{C}_p$.*

Proof. It follows (cf. [1, Theorem 3.2]) from the self-adjointness of A and the ideal property of $\mathscr{J}$ that $A^2 - E_U = (A + U^*)(A - U) + (AU - U^*A) \in \mathscr{J}$. Similarly, $A^2 - F_u \in \mathscr{J}$.

LEMMA 5.2. *If $A - U \in \mathscr{C}_p$, where $0 < p \leq \infty$, for some positive operator A and some partial isometry U then*

$$A^2 - A \in \mathscr{C}_\infty, \quad A - E_U \in \mathscr{C}_\infty \quad \textit{and} \quad A - F_U \in \mathscr{C}_\infty.$$

Proof. If $A - U \in \mathscr{C}_p$, where $0 < p \leq \infty$, then, by Lemma 5.1,

$$A^2 - E_U \in \mathscr{C}_p \quad \left(\text{and } A^2 - F_U \in \mathscr{C}_p\right)$$

and hence $A^2 - E_U \in \mathscr{C}_\infty$. Therefore, if π denotes the canonical homomorphism on $\mathscr{L}(H)$ onto to the Calkin algebra $\mathscr{L}(H)/\mathscr{C}_\infty$ then $\pi(A^2) = \pi(E_U)$. Using the homomorphism property of π (in particular that $(\pi(X))^2 = \pi(X^2)$) we have

$$\left(\pi(A)\right)^4 = \left(\pi(E_U)\right)^2 = \pi(E_U) = \left(\pi(A)\right)^2.$$

Write $a = \pi(A)$. Then $a^4 = a^2$. Taking positive square roots of this (here we use the positivity of A), we get $a^2 = a$, that is, $\pi(A^2) = \pi(A)$. So, $A^2 - A \in \mathscr{C}_\infty$. Hence,

$$A - E_U = (A - A^2) + \left(A^2 - E_U\right) \in \mathscr{C}_\infty$$

and, similarly, $A - F_U \in \mathscr{C}_\infty$. ■

PROPOSITION 5.3. *Let K be a compact normal operator in $\mathscr{L}(H)$ and let H be the direct sum, $H = \oplus H_i$, of a (possibly countably infinite) number of subspaces H_i each of which reduces K. Then there exists a basis $\{\phi_n\}$ of H consisting of eigenvectors of K and such that each ϕ_n is in only one H_i.*

Proof. Fix i. Since H_i reduces K, the restriction of K to H_i, $K|_{H_i}$, is a compact normal operator in $\mathscr{L}(H_i)$. Hence there exists a basis of H_i consisting of eigenvectors of $K|_{H_i}$. Now let i vary and take the union of all such bases. This union, $\{\phi_n\}$ say, is a basis of H consisting of eigenvectors of K and such that each ϕ_n is in one H_i. ■

The next preliminary result, Lemma 5.4, deals with minimizing $\|A - V\|_p$ in the special case when the map $U \mapsto \|A - U\|_p^p$ has a critical point at $U = V$.

LEMMA 5.4. *Let A be a positive operator. Let V be a critical point of*

$$F_p\colon U \mapsto \|A - U\|_p^p,$$

where U varies over those partial isometries such that $A - U \in \mathscr{C}_p$, where $1 < p < \infty$. Then:

(a) *$E_V A = A E_V$, $A - E_V \in \mathscr{C}_p$, and*

$$\|A - E_V\|_p \le \|A - V\|_p; \tag{1}$$

further, for strictly positive A there is equality in (1) *if and only if $V = E_V$.*

(b) *If the underlying space H is finite-dimensional then*

$$\|A - E_V\|_p \le \|A - V\|_p \le \|A + E_V\|_p \tag{2}$$

and, further, for strictly positive A the left hand (right hand) inequality in (2) *is an equality if and only if $V = E_V$ ($V = -E_V$).*

Proof. (a) The proof is suggested by the proof of [1, Theorem 3.5]. Let V be a critical point of F_p. Then by Theorem 4.1(b), (c) and (e), $E_V A = A E_V$, Ker V reduces A (and hence $A - E_V$) and $AV = VA$. Also, by Lemma 5.2, $A - E_V$ is compact since A is positive.

Suppose now A is strictly positive. Then by Theorem 4.1(f), $V = V^*$ so that $A - V$ is reduced by Ker V and $A - V$ commutes with $A - E_V$. Hence, since the compact normal operators $(A - V)|_{\operatorname{Ker} V}$ and $(A - E_V)|_{\operatorname{Ker} V}$, in $\mathscr{L}(\operatorname{Ker} V)$, commute then there exists a basis of Ker V consisting of common eigenvectors of

$$(A - V)|_{\operatorname{Ker} V} \quad \text{and} \quad (A - E_V)|_{\operatorname{Ker} V}.$$

There is a similar result about common eigenvectors of

$$(A - V)|_{(\operatorname{Ker} V)^\perp} \quad \text{and} \quad (A - E_V)|_{(\operatorname{Ker} V)^\perp}.$$

Therefore, as in the proof of Proposition 5.3, there exists a basis $\{\phi_n\}$ of H consisting of common eigenvectors of $A - V$ and $A - E_V$ such that $\phi_n \in$ Ker V or $\phi_n \in (\operatorname{Ker} V)^\perp$ for each n. Thus each ϕ_n is an eigenvector of E_V, A and V. Let λ_n, ξ_n and ν_n be the corresponding eigenvalues of A, E_V and V respectively. Then, for each n, $|\lambda_n - \nu_n| \ge |\lambda_n - \xi_n|$ (for if $\phi_n \in$ Ker V then $\xi_n = \nu_n = 0$; whilst if $\phi_n \in (\operatorname{Ker} V)^\perp$ then $\xi_n = 1 = |\nu_n|$ which, since $\lambda_n \ge 0$, gives the desired inequality). Therefore, as the normal operator $A - V \in \mathscr{C}_p$ then by (2.1),

$$\|A - V\|_p^p = \sum_n |\lambda_n - \nu_n|^p \geq \sum_n |\lambda_n - \xi_n|^p. \tag{3}$$

Hence, by (2.1) again, the normal operator $A - E_V \in \mathscr{C}_p$ and

$$\|A - E_V\|_p^p = \sum_n |\lambda_n - \xi_n|^p$$

which gives the inequality (1).

Suppose next there is equality in (1). Then there is equality throughout (3) and hence, for each n, $|\lambda_n - \nu_n| = |\lambda_n - \xi_n|$. If $\nu_n = \xi_n = 0$ this equality automatically holds; whilst if $|\nu_n| = 1 = \xi_n$ then $|\lambda_n - \nu_n| = |\lambda_n - 1|$ which forces $\operatorname{Re}\nu_n = 1$ (because $\lambda_n \neq 0$ since A is *strictly* positive) and hence $\nu_n = 1$. So, $\nu_n = \xi_n$ for every n, that is, $V = E_V$.

Next we extend the inequality (1) to positive (as distinct from strictly positive) A. Since by Theorem 4.1(c) and (d), $\operatorname{Ker} A$ reduces E_V and V, therefore $\operatorname{Ker} A$ reduces $A - E_V$ and $A - V$. Decompose $A - V$ into its restrictions to $\operatorname{Ker} A$ and $(\operatorname{Ker} A)^\perp$, viz.

$$(A - V)|_{KerA} \,(= S) \quad \text{and} \quad (A - V)|_{(KerA)^\perp} \,(= T, \ \text{say}).$$

Since $S + T \in \mathscr{C}_p$ and since $\operatorname{Ran} S \perp \operatorname{Ran} T$ and $\operatorname{Ran} S^* \perp \operatorname{Ran} T^*$ it follows that (2.2) applies: $S \in \mathscr{C}_p$, $T \in \mathscr{C}_p$ and

$$\|A - V\|_p^p = \|S\|_p^p + \|T\|_p^p. \tag{4}$$

Now, $S = (A - V)|_{\operatorname{Ker} A} = -V|_{\operatorname{Ker} A}$ and $|V|^p = |E_V|^p$ and so, since $\|X\|_p^p = \tau(|X|^p)$ if $X \in \mathscr{C}_p$,

$$\|S\|_p^p = \left\|(A - E_V)|_{\operatorname{Ker} A}\right\|_p^p.$$

As for T, since A is strictly positive on $(\operatorname{Ker} A)^\perp$ the first part of the proof shows that $(A - E_V)|_{(\operatorname{Ker} A)^\perp} \in \mathscr{C}_p$ and that

$$\|T\|_p^p = \left\|(A - V)|_{(\operatorname{Ker} A)^\perp}\right\|_p^p \geq \left\|(A - E_V)|_{(\operatorname{Ker} A)^\perp}\right\|_p^p.$$

Substituting back into (4) and again using the equality (2.2) we obtain, as desired the inequality (1).

(b) The proof is similar to that of (a) and so is omitted. ■

The next example shows that the inequality $\|A - E_V\|_p^p \leq \|A - V\|_p^p$ does not hold for all partial isometries V such that $A - V \in \mathscr{C}_p$.

Take

$$V = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \end{bmatrix}, \quad E_V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$

so that V is a partial isometry (with initial space the x-axis) and A is positive. It is easily checked that V, E_V and A do not satisfy the inequality

$$\|A - E_V\|_2^2 \leq \|A - V\|_2^2.$$

We next define the projection $E_{1/2}$. The notation $\bar{S}\{\phi_n\}$ refers to the closure of the span of the vectors ϕ_n.

DEFINITION 5.5. Let A in $\mathscr{L}(H)$ be positive and such that there exists a basis $\{\phi_n\}$ of H consisting of eigenvectors of A. The operator $E_{1/2}$ is defined as the projection onto the subspace $M_{1/2}$ given by

$$M_{1/2} = \bar{S}\{\phi_n \colon \lambda_n \geq \tfrac{1}{2} \text{ where } A\phi_n = \lambda_n\phi_n\}.$$

Not surprisingly, the same results hold in the rest of this paper if $E_{1/2}$ is replaced by $E'_{1/2}$, where $E'_{1/2}$ is defined in the same way as $E_{1/2}$ except that the condition $\lambda_n \geq \frac{1}{2}$ is replaced by $\lambda_n > \frac{1}{2}$.

We now come to the first main result on global minimization.

THEOREM 5.6. *Let A be a positive operator. Let U vary over those partial isometries such that $A - U \in \mathscr{C}_p$, where $1 < p < \infty$. If the map*

$$F_p \colon U \mapsto \|A - U\|_p^p$$

attains a global minimum then there exists a basis of the underlying space consisting of eigenvectors of A and

$$\|A - E_{1/2}\|_p \leq \|A - U\|_p \tag{1}$$

where $E_{1/2}$ is as in Definition 5.5; and, further, for strictly positive A such that $\frac{1}{2} \notin \sigma_p(A)$, equality occurs in (1) if and only if $U = E_{1/2}$.

Proof. Let F_p attain a global minimum at V, say, so that

$$\|A - V\|_p \leq \|A - U\|_p.$$

Since, for $1 < p < \infty$, a global minimum is a critical point it follows from

Lemma 5.4 (a) that $E_V A = AE_V$, $A - E_V \in \mathscr{C}_p$ and

$$\|A - E_V\|_p = \|A - V\|_p \leq \|A - U\|_p.$$

(The equality is because F_p attains a global minimum at V.) The inequality (1) will now follow (on taking $E = E_V$) from the assertion below.

ASSERTION. *Let E be a projection such that $EA = AE$ and $A - E \in \mathscr{C}_p$, where $1 < p < \infty$. Then:*

(a) *There exists a basis $\{\phi_n\}$ of the underlying space H consisting of eigenvectors of A and such that $\phi_n \in \operatorname{Ran} E$ or $\phi_n \in (\operatorname{Ran} E)^\perp$ for each n.*

(b) $A - E_{1/2} \in \mathscr{C}_p$ *and*

$$\|A - E_{1/2}\|_p \leq \|A - E\|_p; \tag{2}$$

and, provided $\frac{1}{2} \notin \sigma_p(A)$, equality holds in (2) if and only if $E = E_{1/2}$.

Proof of assertion. (a) Since $EA = AE$ the compact normal operator $A - E$ is reduced by $\operatorname{Ran} E$. Therefore, by Proposition 5.3, there exists a basis $\{\phi_n\}$ of H consisting of eigenvectors of $A - E$ and such that $\phi_n \in \operatorname{Ran} E$ or $\phi_n \in (\operatorname{Ran} E)^\perp$ for each n.

(b) Each ϕ_n is thus an eigenvector of E, A, $E_{1/2}$, $A - E$ and $A - E_{1/2}$. If, for each n, $A\phi_n = \lambda_n\phi_n$, $E\phi_n = \xi_n\phi_n$ and $E_{1/2}\phi_n = e_n\phi_n$ then $|\lambda_n - \xi_n| \geq |\lambda_n - e_n|$. (To prove this inequality consider the four cases: ϕ_n is/is not in $M_{1/2}/\operatorname{Ran} E$). Hence, using (2.1)

$$\|A - E\|_p^p = \sum|\lambda_n - \xi_n|^p \geq \sum|\lambda_n - e_n|^p.$$

This proves that $A - E_{1/2} \in \mathscr{C}_p$ and gives the inequality (2).

Next, if equality holds in (2) then $|\lambda_n - \xi_n| = |\lambda_n - e_n|$ for each n and since $\frac{1}{2} \notin \sigma_p(A)$, this forces $\operatorname{Ran} E = M_{1/2}$; for if either $\phi_n \in \operatorname{Ran} E$ and $\phi_n \notin M_{1/2}$, or, if $\phi_n \notin \operatorname{Ran} E$ and $\phi_n \in M_{1/2}$ (when $\lambda_n > \frac{1}{2}$) we would have $|\lambda_n - \xi_n| > |\lambda_n - e_n|$. This proves the assertion.

Finally, let A be strictly positive and such that $\frac{1}{2} \notin \sigma_p(A)$. If there is equality in (1) for some partial isometry U then $\|A - E_{1/2}\|_p = \|A - E_U\|_p = \|A - U\|_p$. The second equality implies, by Lemma 5.4 (a), that $U = E_U$; and the first equality implies, by the assertion, that $E_U = E_{1/2}$. So, $U = E_{1/2}$. ∎

The assumption of finite-dimensionality is critical in Theorem 5.7.

THEOREM 5.7. *Let the underlying space H be finite-dimensional. Let A be a positive operator and let $E_{1/2}$ be as in Definition 5.5. Then for all partial* *isometries U in $\mathscr{L}(H)$,*

$$\|A - E_{1/2}\|_p \leq \|A - U\|_p \leq \|A + I\|_p, \quad \text{where } 1 < p < \infty, \tag{1}$$

$$\|A - E_{1/2}\| \leq \|A - U\| \leq \|A + I\|. \tag{2}$$

Further, for strictly positive A the right hand inequality in (1) *is an equality if and only if $U = -I$ and, further, for strictly positive A such that $\frac{1}{2} \notin \sigma_p(A)$ the left hand inequality in* (1) *is an equality if and only if $U = E_{1/2}$.*

Proof. The set of all partial isometries is closed and bounded [8, Problem 129] and hence, since H is finite-dimensional, compact. It follows as in [1, Theorem 3.5] that the (continuous) map F_p: $U \mapsto \|A - U\|_p^p$ is bounded and attains its bounds. The left hand inequality in (1), and the corresponding uniqueness assertion, now follows from Theorem 5.6.

To prove the right hand inequality in (1) let W be a global maximum, and hence a critical point, of F_p. Then by Lemma 5.4, $E_W A = A E_W$ and we have the equality

$$\|A - W\|_p = \|A + E_W\|_p$$

which, for strictly positive A, forces $W = -E_W$. It can be shown (by considering the eigenvalues of A) that if E is a projection such that $EA = AE$ and if H is finite-dimensional then $\|A + E\|_p$ attains its maximum at $E = I$ and at no other point. This gives the right hand inequality in (1) and the corresponding uniqueness assertion.

As H is finite-dimensional, the operator norm inequality (2) follows from (1). ■

In finite dimensions the condition on A of positivity can be dropped: in that case $A = \hat{U}_0|A|$ where $\hat{U}_0$ is unitary. Let $\{\phi_n\}$ be a basis of H consisting of eigenvectors of $|A|$ and let $\hat{E}_{1/2}$ be the projection onto to the subspace

$$\bar{S}\{\phi_n\colon \lambda_n \geq \tfrac{1}{2}, |A|\phi_n = \lambda_n\phi_n\}.$$

Then if U(and hence $\hat{U}_0^*U$) is a partial isometry it follows, cf. (3.3), from Theorem 5.7 that $\|A - U\|_p$, where $1 < p \leq \infty$, is minimized when $U = \hat{U}_0\hat{E}_{1/2}$ and maximized when $U = -\hat{U}_0$ (here, $\|\ .\ \|_\infty$ denotes the operator norm $\|\ .\ \|$ on $\mathscr{L}(H)$). Thus,

$$\|A - \hat{U}_0\hat{E}_{1/2}\|_p \leq \|A - U\|_p \leq \|A + \hat{U}_0\|_p \quad \text{where } 1 < p \leq \infty \tag{5.8}$$

(with the now obvious necessary conditions for equality when $1 < p < \infty$).

We return to the infinite-dimensional case. As for maximizing $\|A - U\|$, as in [1, Theorem 3.1] if A is positive then for all partial isometries U in $\mathscr{L}(H)$,

$$\|A - U\| \leq \|A + I\|. \tag{5.9}$$

To get the infinite-dimensional approximation results we appeal to the Halmos/Bouldin theorem on normal spectral approximation (Theorem 2.4).

First, there is the following result about approximating a normal operator by normal partial isometries.

THEOREM 5.10. *Let A be a normal operator and define the function F: $\mathbf{C} \to \Lambda$, where $\Lambda = \{0\} \cup C$ with $C = \{z: |z| = 1\}$, by*

$$F(re^{i\theta}) = \begin{cases} e^{i\theta} & \text{if } r \geq \frac{1}{2} \\ 0 & \text{if } r < \frac{1}{2}. \end{cases}$$

Then:

(a) *$F(A)$ is a normal partial isometry and for all normal partial isometries U,*

$$\|A - F(A)\| \leq \|A - U\|. \tag{1}$$

(b) *Further, for all normal partial isometries U such that $A - U \in \mathscr{C}_p$, where $2 \leq p < \infty$, then $A - F(A) \in \mathscr{C}_p$ and*

$$\|A - F(A)\|_p \leq \|A - U\|_p. \tag{2}$$

Proof. First, the spectrum of a normal partial isometry is a non-empty closed subset of $\{0\} \cup C$. This is because (i) a normal partial isometry is the direct sum of the zero operator, and a unitary, and conversely [8, Problem 204]; and (ii) the spectrum of the direct sum of two operators is the union of their individual spectra.

Conversely, if the spectrum of some normal operator U is a non-empty closed subset of $\{0\} \cup C$ then the underlying space H can be decomposed so that U is the direct sum of a normal quasi-nilpotent operator, i.e. the zero operator, and a unitary. Hence, U is a normal partial isometry.

The mapping F: $\mathbf{C} \to \Lambda$ is clearly a retraction. Therefore, by Theorem 2.4 (a) it follows that $\sigma(F(A)) \subset \Lambda$ so that by the above argument $F(A)$ is a normal partial isometry, and $F(A)$ satisfies the inequality (1). By Theorem 2.4 (b) it follows that $A - F(A) \in \mathscr{C}_p$ and $F(A)$ satisfies the $\mathscr{C}_p$ inequality (2). ■

Of course, the same results hold in Theorem 5.10 if F is replaced by the function F': $\mathbf{C} \to \Lambda$ defined in the same way as F except that the condition $r \geq \frac{1}{2}$ ($r < \frac{1}{2}$) is replaced by $r > \frac{1}{2}$ ($r \leq \frac{1}{2}$).

Finally, we come to the following result about normal partially isometric approximation in $\mathscr{C}_p$ of positive operators.

THEOREM 5.11. *Let A in $\mathscr{L}(H)$ be positive. Then*:

(a) *If there exists a basis of H consisting of eigenvectors of A then for all normal partial isometries U,*

$$\|A - E_{1/2}\| \leq \|A - U\|$$

where $E_{1/2}$ is as in Definition 5.5.

(b) *For all normal partial isometries U such that $A - U \in \mathscr{C}_p$, where $2 \leq p < \infty$, there exists a basis of H consisting of eigenvectors of A and*

$$\|A - E_{1/2}\|_p \leq \|A - U\|_p; \tag{1}$$

further, for strictly positive A such that $\frac{1}{2} \notin \sigma_p(A)$, equality occurs in (1) *if and only if $U = E_{1/2}$.*

Proof. (a) If $\{\phi_n\}$ is a basis of H consisting of eigenvectors of A, with $A\phi_n = \lambda_n\phi_n$ where $\lambda_n \geq 0$, then, with F as in Theorem 5.10, $F(A)\phi_n = F(\lambda_n)\phi_n = E_{1/2}\phi_n$ and hence $F(A) = E_{1/2}$. The result now follows from Theorem 5.10 (a).

(b) By Theorem 5.10(b), the map F_p: $U \mapsto \|A - U\|_p^p$ attains a global minimum. The result now follows from Theorem 5.6. ■

Observe that we cannot deduce from Theorem 5.11 an inequality like (5.8) dealing with approximation to non-positive A (because, in the notation of (5.8), the partial isometries $\hat{U}_0^* U$ need not be normal).

In the light of Theorem 5.7, Theorem 5.11 raises the following questions: in the infinite-dimensional case, what happens if the partial isometries are not normal and/or if $p < 2$?

REFERENCES

1. J.G. AIKEN, J.A. ERDOS and J.A. GOLDSTEIN, *Unitary approximation of positive operators*, Illinois J. Math., vol. 24 (1980), pp. 61–72.
2. ______, *On Löwdin orthogonalization*, Internat. J. Quantum Chem., vol. 18 (1980), pp. 1101–1108.
3. R. BOULDIN, *Best approximation of a normal operator in the Schatten p-norm*, Proc. Amer. Math. Soc., vol. 80 (1980), pp. 277–282.
4. N. DUNFORD and J.T. SCHWARTZ, *Linear operators, Part II*, Interscience, New York, 1963.
5. J.A. GOLDSTEIN and M. LEVY, *Hilbert-Schmidt approximation problems arising in quantum chemistry*, Adv. in Appl. Math., vol. 5 (1984), pp. 216–225.
6. P.R. HALMOS, *Positive approximants of operators*, Indiana Univ. Math. J., vol. 21 (1972), pp. 951–960.
7. ______, *Spectral approximants of normal operators*, Proc. Edinburgh Math. Soc., vol. 19 (1974), pp. 51–58.
8. ______, *A Hilbert space problem book*, second edition, Springer-Verlag, New York, 1982.
9. P.R. HALMOS and J.E. MCLAUGHLIN, *Partial isometries*, Pacific J. Math., vol. 13 (1963), pp. 585–596.
10. P.J. MAHER, *Some operator inequalities concerning generalized inverses*, Illinois J. Math., to appear.
11. P.Y. WU, *Approximation by partial isometries*, Proc. Edinburgh Math. Soc., vol. 29 (1986), pp. 255–261.

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