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FIXED POINTS OF ISOMETRIES AT INFINITY IN HOMOGENEOUS SPACES

BY

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Introduction

Let M be a simply connected homogeneous riemannian manifold of nonpositive curvature. Since M admits a simply transitive and solvable Lie group G of isometries, it can be represented as the Lie group G endowed with a left invariant metric of nonpositive curvature. If $\mathfrak{g}$ is the Lie algebra of G then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{a}$ where $\mathfrak{a}$, the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$ with respect to the metric, is an abelian subalgebra of $\mathfrak{g}$.

In this paper, we describe the set of fixed points of G at infinity and we classify all isometries defined by elements of G when M has no de Rham flat factor; more precisely, we show that the elements of $[G, G]$ are parabolic and the hyperbolic elements of G are those conjugate to $\exp(\mathfrak{a})$.

In Section 1, we study the action of right invariant vector fields on the geodesics $\gamma_Z(t) = \exp tZ$ with $Z \in \mathfrak{a}$. All stable Jacobi fields on γ_Z are determined on certain regular elements Z of $\mathfrak{a}$ (Corollary 1.3). Section 2 is devoted to describe, for each Z in $\mathfrak{a}$, the subgroups of G that fix $\gamma_Z(\infty)$ (Corollary 2.6). In the third section, the set of fixed points of G at infinity is described (Theorem 3.4) and all isometries coming from left translations by elements of G are classified (Corollaries 3.7 and 3.9). In particular, if M is not a visibility manifold and $I(M)$ (or $I_0(M)$) has a fixed point at infinity (for instance if M is not symmetric) this point is necessarily a flat point at infinity (Corollary 3.5).

Finally, in Section 4 we summarize some results about the points at infinity that can be joined by a geodesic to a fixed point of G.

Preliminaries

Let M denote a complete and simply connected riemannian manifold of nonpositive curvature ($K \leq 0$). All geodesics in M are assumed to have unit

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speed and the distance induced by the riemannian metric is denoted by $d(\ ,\ )$. $M(\infty)$ will denote the set of points at infinity, that is the set of asymptote classes of geodesics. The space $\overline{M} = M \cup M(\infty)$ together with the cone topology is a compactification of M that is homeomorphic to the closed ball of dimension n = dimension M. The angle at $p \in M$ subtended by q and r in $\overline{M}$ is defined by $\sphericalangle_p(q, r) = \sphericalangle_p(\gamma'_{pq}(0), \gamma'_{pr}(0))$ where γ_{pq} denotes the unique geodesic in M joining p with q. $I(M)$ and $I_0(M)$ will denote the isometry group of M and the connected component of the identity respectively. Both are Lie groups with the compact open topology. Isometries of M extend to homeomorphisms of $\overline{M}$ by defining $\varphi(\gamma(\infty)) = (\varphi \circ \gamma)(\infty)$ for any isometry φ and any point $\gamma(\infty)$ in $M(\infty)$. Any subgroup D of $I(M)$ determines a limit set $L(D)$, which is the set of points in $M(\infty)$ that are cluster points of an orbit $D(p)$. The definition of $L(D)$ does not depend on the choice of the point p in M and $L(D)$ is nonempty if and only if $\overline{D}$ is noncompact in $I(M)$.

A Jacobi vector field J on a geodesic γ of M is said to be *stable* (*unstable*) if there exists a constant $c > 0$ such that $|J(t)| \leq c$ for all $t \geq 0$ (all $t \leq 0$).

We recall the following known fact under the hypothesis $K \leq 0$. If γ is a geodesic in M then for every $v \in T_{\gamma(0)}M$ there is a unique stable Jacobi vector field J on γ such that $J(0) = v$ (See [9, Lemma 2.2]). Moreover, if J is an unstable vector field on γ, since $|J(t)|$ is a convex function of $t \in \mathbf{R}$ it then follows that $\lim_{t \to +\infty} |J(t)| = \infty$, if $|J(t)|$ is nonconstant.

Assume that $M = G$ is a solvable Lie group with a left invariant metric $\langle\ ,\ \rangle$ of nonpositive curvature and let $\mathfrak{g}$ be the Lie algebra of G. We recall that if $X, Y, Z \in \mathfrak{g}$ then the riemannian connection ∇ is given by

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle$$

and the sectional curvature K reduces to

$$\begin{aligned} |X \wedge Y|^2 K(X, Y) &= \langle R(X, Y)Y, X\rangle \\ &= \tfrac{1}{4}|(\mathrm{ad}_X)^*Y + (\mathrm{ad}_Y)^*X|^2 - \langle (\mathrm{ad}_X)^*X, (\mathrm{ad}_Y)^*Y\rangle \\ &\quad - \tfrac{3}{4}|[X, Y]|^2 - \tfrac{1}{2}\langle [[X, Y], Y], X\rangle \\ &\quad - \tfrac{1}{2}\langle [[Y, X], X], Y\rangle \end{aligned}$$

where $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ and * denotes the adjoint with respect to the metric.

Since $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$, $\mathfrak{a}$ is a totally geodesic subalgebra of $\mathfrak{g}$ (that is $\nabla_X Y \in \mathfrak{a}$ whenever $X, Y \in \mathfrak{a}$) and the connected Lie subgroup A with Lie algebra $\mathfrak{a}$ is a flat totally geodesic submanifold of G. Observe that $A = \exp(\mathfrak{a})$ where $\exp\colon \mathfrak{g} \to G$ is the exponential map of G.

We recall that since G is simply connected, G can be represented by $G = [G, G]\exp(\mathfrak{a})$ with $[G, G] \cap \exp(\mathfrak{a}) = \{e\}$ (e is the identity of G) where

$[G, G]$ is the connected (and closed) Lie subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Moreover, since $[G, G]$ is nilpotent and simply connected, the map exp: $[\mathfrak{g}, \mathfrak{g}] \to [G, G]$ is a diffeomorphism (See [11, Chap. 3]).

We note that in general, in a Lie group G if for any $g \in G$, L_g and R_g denote the left and right translations respectively and $I_g = L_g \circ R_{g^{-1}}$, then the adjoint representation of G defined by $\mathrm{Ad}(g) = (dI_g)_e$ satisfies $I_g(\exp X) = \exp(\mathrm{Ad}(g)X)$ and $\mathrm{Ad}(\exp X) = \mathrm{Exp}(\mathrm{ad}_X)$ for every X in $\mathfrak{g}$, where Exp denote the exponential map in $\mathrm{Gl}(\mathfrak{g})$ and ad_X the adjoint map of $\mathfrak{g}$.

For details and references in the subject the reader is referred to [1], [4], [6], [8].

In the sequel G will be a solvable and simply connected Lie group with a left invariant metric of nonpositive curvature. The complexification of $\mathfrak{g}'$, $\mathfrak{g}'^{\mathbf{C}}$, decomposes as $\mathfrak{g}'^{\mathbf{C}} = \sum_\lambda \mathfrak{g}'^{\mathbf{C}}_\lambda$, where

$$\mathfrak{g}'^{\mathbf{C}}_\lambda = \left\{U \in \mathfrak{g}'^{\mathbf{C}} \colon \left(\mathrm{ad}_H - \lambda(H)I\right)^k U = 0 \text{ for some } k \geq 1 \text{ and for all } H \in \mathfrak{a}\right\}$$

is the associated root space for the root $\lambda \in (\mathfrak{a}^*)^{\mathbf{C}}$ under the abelian action of $\mathfrak{a}$ on $\mathfrak{g}'$. If $\lambda = \alpha \pm i\beta$ is a root of $\mathfrak{a}$ in $\mathfrak{g}'$,

$$\mathfrak{g}'_{\alpha\beta} = \mathfrak{g}'_{\alpha-\beta} = \mathfrak{g}' \cap \left(\mathfrak{g}'^{\mathbf{C}}_\lambda \oplus \mathfrak{g}'^{\mathbf{C}}_{\bar\lambda}\right)$$

is the generalized root space of $\mathfrak{a}$ in $\mathfrak{g}'$ and $\mathfrak{g}' = \sum_{\lambda=\alpha+i\beta} \mathfrak{g}'_{\alpha\beta}$.

We assume that G has no de Rham flat factor; hence the factors

$$\mathfrak{g}'_0 = \sum_{i\beta} \mathfrak{g}'_{0\beta}, \quad \mathfrak{a}_0 = \{H \in \mathfrak{a} \colon \alpha(H) = 0 \text{ for all roots } \alpha + i\beta\}$$

are zero and $\mathfrak{g}' = \sum_{\alpha \neq 0} \mathfrak{g}'_{\alpha\beta}$. (For a proof see [2, Theorem 4.6]).

1. Right invariant vector fields on G

If X is in $\mathfrak{g}$ then $\tilde{X}$ will denote the right invariant vector field on G such that $\tilde{X}_e = X$, that is $\tilde{X}_g = (dR_g)_e X$ for all $g \in G$. Since the one-parameter flow associated to $\tilde{X}$ is given by $\varphi_s(g) = \exp sXg$ ($s \in \mathbf{R}$) it follows that $\tilde{X}$ is a Killing vector field on G and consequently $\tilde{X}$ is a Jacobi vector field on any geodesic of G. We note the following.

(i) $|\tilde{X}_g| = |(dL_g)_e(\mathrm{Ad}(g^{-1}))X| = |\mathrm{Ad}(g^{-1})X|$ for all $g \in G$.

(ii) $|\tilde{X} \exp sXg| = |\tilde{X}g|$ for all $s \in \mathbf{R}$, $g \in G$.

(iii) If γ is a geodesic in G with $\gamma(0) = e$, $\gamma'(0) = Z$ then $\tilde{X}_{\gamma(t)}$ is a Jacobi field on γ such that

$$\tilde{X}_{\gamma(0)} = X, \quad \left(\nabla_{\gamma'}\tilde{X}\right)(0) = \nabla_X Z.$$

In fact, if Z denotes the left invariant vector field associated to Z, $(\nabla_{\gamma'}\tilde{X})(0) = (\nabla_Z \tilde{X})_e = \nabla_X Z$ since $[Z, \tilde{X}]_e = 0$.

(iv) $\left\langle \tilde{X}_{\gamma(t)}, \gamma'(t) \right\rangle = \langle X, Z \rangle$ for all $t \in \mathbf{R}$ $\left(\langle \nabla_{\gamma'}\tilde{X}, \gamma' \rangle = 0 \right)$.

Now we fix H in $\mathfrak{a}$, $H \neq 0$. For each $X \in \mathfrak{g}$, X not a multiple of H, J^X will denote the Jacobi vector field on $\gamma_H(t) = \exp tH$ defined by $J^X(t) = \tilde{X}_{\gamma_H(t)}$. Then, from the remark above, J^X satisfies:

(i) $J^X(0) = X$.

(ii) $(\nabla_{\gamma_H'} J^X)(0) = -D_H X$ where D_H is the symmetric part of ad_H.
Note that $\nabla_H = S_H$, the skew symmetric part of ad_H.

(iii) $|J^X(t)| = |\mathrm{Ad}(\exp - tH)X| = |\mathrm{Exp}(-t\,\mathrm{ad}_H)X|$.

(iv) $\langle J^X(t), \gamma_H'(t) \rangle = \langle X, H \rangle$ for all $t \in \mathbf{R}$.

We have the decomposition

$$\mathfrak{g}' = \mathfrak{g}_0^H \oplus \mathfrak{g}_+^H \oplus \mathfrak{g}_-^H$$

where

$$\mathfrak{g}_0^H = \sum_{\alpha(H)=0} \mathfrak{g}'_{\alpha\beta}, \quad \mathfrak{g}_+^H = \sum_{\alpha(H)>0} \mathfrak{g}'_{\alpha\beta}, \quad \mathfrak{g}_-^H = \sum_{\alpha(H)<0} \mathfrak{g}'_{\alpha\beta}.$$

We next study the behavior of J^X along γ_H. For that, the following lemma is very useful.

LEMMA 1.1. (i) *If $\alpha(H) = 0$ and $X \in \mathfrak{g}'_{\alpha\beta}$ then* $\lim_{t\to+\infty} \mathrm{Exp}(-t\,\mathrm{ad}_H)X$ *exists if and only if $\beta(H) = 0$ (in which case* $\mathrm{ad}_H|_{\mathfrak{g}'_{\alpha\beta}} = 0$).

(ii) *If $\alpha(H) > 0$ then*

$$\lim_{t\to+\infty} \mathrm{Exp}(-t\,\mathrm{ad}_H)X = 0 \quad \textit{for all } X \in \mathfrak{g}'_{\alpha\beta}.$$

(iii) *If $\alpha(H) < 0$ then*

$$\lim_{t\to+\infty} |\mathrm{Exp}(-t\,\mathrm{ad}_H)X| = \infty \left(\lim_{t\to+\infty} \mathrm{Exp}(t\,\mathrm{ad}_H)X = 0 \right) \quad \textit{for all } X \in \mathfrak{g}'_{\alpha\beta}.$$

Proof. (i) If $X \in \mathfrak{g}'_{\alpha\beta}$ then $U = X + iY \in \mathfrak{g}_\lambda'^{\mathbf{C}}$ $(\lambda = \alpha + i\beta)$ for some $Y \in \mathfrak{g}'$. Since $\alpha(H) = 0$, $\mathrm{ad}_H|_{\mathfrak{g}_\lambda'^{\mathbf{C}}} = i\beta(H)I$ and ad_H must be skew symmetric on $\mathfrak{g}'_{\alpha\beta}$ by Lemma 4.4 and Theorem 5.2 of [1]. Then,

$$\mathrm{Exp}(-t\,\mathrm{ad}_H)|_{\mathfrak{g}_\lambda'^{\mathbf{C}}} = e^{-i\beta(H)t}I$$

and

$$\operatorname{Exp}(-t\,\mathrm{ad}_H)X = (\cos t\beta(H))X + (\sin t\beta(H))Y.$$

Therefore, $\lim_{t\to+\infty}\operatorname{Exp}(-t\,\mathrm{ad}_H)X$ exists in G if and only if $\beta(H) = 0$, in which case $\mathrm{ad}_H|_{\mathscr{g}'_{\alpha\beta}} = 0$ and $\operatorname{Exp}(-t\,\mathrm{ad}_H)X = X$ for all $t \in \mathbf{R}$.

(ii) From the definition of $\mathscr{g}'^{\mathbf{C}}_\lambda$, $N = (\mathrm{ad}_H - \lambda(H)I)|_{\mathscr{g}'^{\mathbf{C}}_\lambda}$ is a nilpotent operator on $\mathscr{g}'^{\mathbf{C}}_\lambda$. Then

$$\mathrm{ad}_H|_{\mathscr{g}'^{\mathbf{C}}_\lambda} = \lambda(H)I + N \quad\text{and}\quad \operatorname{Exp}(-t\,\mathrm{ad}_N)|_{\mathscr{g}'^{\mathbf{C}}_\lambda} = e^{-t\lambda(H)}\operatorname{Exp}(-tN).$$

Since $|e^{-it\beta(H)}| = 1$ it follows that

$$\lim_{t\to+\infty}\operatorname{Exp}(-t\,\mathrm{ad}_H)\Big|_{\mathscr{g}'^{\mathbf{C}}_\lambda} = 0$$

if and only if

$$\lim_{t\to+\infty} e^{-t\alpha(H)}\operatorname{Exp}(-tN) = 0 \quad \text{in } \mathscr{gl}\left(\mathscr{g}'^{\mathbf{C}}_\lambda\right).$$

We compute this limit in each coordinate (ij). Since N is nilpotent,

$$\operatorname{Exp}(-tN) = \sum_{k=0}^{s}(-1)^k\frac{t^k}{k!}N^k \quad (N^{s+1} = 0)$$

and

$$\operatorname{Exp}(-tN)_{ij} = \sum_{k=0}^{s}(-1)^k\frac{t^k}{k!}(N^k)_{ij} = p^s_{ij}(t)$$

is a polynomial in t of degree $s \geq 0$. Then

$$\lim_{t\to+\infty} e^{-t\alpha(H)}(\operatorname{Exp}(-tN))_{ij} = \lim_{t\to+\infty} e^{-t\alpha(H)}p^s_{ij}(t) = 0$$

since $\alpha(H) > 0$.

Hence, $\lim_{t\to+\infty}\operatorname{Exp}(-t\,\mathrm{ad}_H)U = 0$ for all $U \in \mathscr{g}'^{\mathbf{C}}_\lambda$ such that $\lambda = \alpha + i\beta$ is a root and consequently,

$$\lim_{t\to+\infty}\operatorname{Exp}(-t\,\mathrm{ad}_H)X = 0 \quad \text{for all } X \in \mathscr{g}'_{\alpha\beta}.$$

(iii) We will show that $\lim_{t\to+\infty}|\operatorname{Exp}(-t\,\mathrm{ad}_H)U| = \infty$ for any $U \in \mathscr{g}'^{\mathbf{C}}_\lambda$ such that $\lambda = \alpha + i\beta$ and $\alpha(H) < 0$; here we consider in $\mathscr{g}'^{\mathbf{C}}_\lambda$ the complex inner product inherited from $\langle\ ,\ \rangle$ in $\mathscr{g}$. If $\{e_1, \ldots, e_m\}$ is an orthonormal basis of $\mathscr{g}_\lambda'^{\mathbf{C}}$ and $U = \sum_{i=1}^m u_i e_i$ then

$$|\operatorname{Exp}(-t\,\mathrm{ad}_H)U|^2 = \sum_i \left|\sum_j \operatorname{Exp}(-t\,\mathrm{ad}_H)_{ij} u_j\right|^2.$$

Since

$$\operatorname{Exp}(-t\,\mathrm{ad}_H) = e^{-t\alpha(H)} e^{-it\beta(H)} \operatorname{Exp}(-tN),$$

as in (ii), we have

$$\begin{aligned}\left|\sum_j \operatorname{Exp}(-t\,\mathrm{ad}_H)_{ij} u_j\right| &= \left|e^{-t\alpha(H)} \sum_j p_{ij}^s(t) u_j\right| \\ &= e^{-t\alpha(H)} |q_{ij}^s(t)|\end{aligned}$$

where $q_{ij}^s(t) = \sum_j p_{ij}^s(t) u_j$ is a polynomial in t of degree $s \geq 0$. Since

$$\lim_{t \to +\infty} e^{-t\alpha(H)} |q_{ij}^s(t)| = \infty$$

it follows that

$$\lim_{t \to +\infty} |\operatorname{Exp}(-t\,\mathrm{ad}_H) X| = \infty \quad \text{for all } X \text{ in } \mathscr{g}'_{\alpha\beta}.$$

The fact that $\lim_{t \to +\infty} \operatorname{Exp}(t\,\mathrm{ad}_H)X = 0$ follows from (ii) since $\alpha(-H) > 0$.

We observe that the proof of (ii) and (iii) does not depend on the fact that $K \leq 0$.

Let $J(\gamma_H)$ denote the $2n$-dimensional space of Jacobi vector fields on γ_H. The subspaces of $J(\gamma_H)$ of *parallel*, *stable* and *unstable* vector fields on γ_H will be denoted by $J^P(\gamma_H)$, $J^S(\gamma_H)$ and $J^U(\gamma_H)$ respectively.

PROPOSITION 1.2. *The map J: $\mathscr{g} \to J(\gamma_H)$, $X \to J^X$ is an injective linear map satisfying*:

(i) $J(\mathscr{g}_0^H \oplus \mathscr{a}) \subset J_{\gamma_H}^P$.

(ii) $J(\mathscr{g}_+^H) \subset J_{\gamma_H}^S$ *and* $\lim_{t \to +\infty} |J^X(t)| = 0$ *for* $X \in \mathscr{g}_+^H$

(iii) $J(\mathscr{g}_-^H) \subset J_{\gamma_H}^U$ *and* $\lim_{t \to +\infty} |J^X(t)| = \infty$ *for* $X \in \mathscr{g}_-^H$

As a consequence, J^X is stable on γ_H if and only if $X \in \mathscr{g}_0^H \oplus \mathscr{g}_+^H \oplus \mathscr{a}$.

Proof. Since any Jacobi field on γ_H is completely determined by $J(0)$ and $\nabla_{\gamma_H'} J(0) = -D_H X$, it follows that J is a linear map.

(i) We observe first that since $\mathscr{a}$ is abelian, if $X \in \mathscr{a}$, $J^X(t) = X_{\exp tH}$ where X, the left invariant field associated to X, is a parallel field on

$\gamma_H(\nabla_H X = 0)$.

If $\alpha(H) = 0$ then $\mathrm{ad}_H|_{\mathfrak{g}'_{\alpha\beta}}$ is skew-symmetric ($\mathrm{ad}_H|_{\mathfrak{g}'^{C}_{\lambda}} = i\beta(H)I$ if $\lambda = \alpha + i\beta$); hence $\mathrm{Exp}(-t\,\mathrm{ad}_H)|_{\mathfrak{g}'_{\alpha\beta}}$ is orthogonal, and for any $X \in \mathfrak{g}'_{\alpha\beta}$ we have

$$|J^X(t)| = |\mathrm{Exp}(-t\,\mathrm{ad}_H)X| = |X| \quad \text{for all } t \in \mathbf{R}.$$

Then

$$0 = \frac{d^2}{dt^2}|J^X(t)|^2 = 2|\nabla_{\gamma'_H} J^X(t)|^2 + 2\langle R(\gamma'_H, J^X)\gamma'_H, J^X\rangle(t)$$

and therefore J^X is parallel on γ_H. If $X \in \mathfrak{g}_0^H$ then $X = \Sigma_{\alpha(H)=0} X_{\alpha\beta}$ and

$$J^X = \sum_{\alpha(H)=0} J^{X_{\alpha\beta}}$$

is parallel on γ_H.

Now, (ii) and (iii) are immediate from Lemma 1.1, (ii) and (iii) respectively. The last assertion is clear since $J(\mathfrak{g}_0^H \oplus \mathfrak{g}_+^H \oplus \mathfrak{a}) \cap J(\mathfrak{g}_-^H) = 0$.

COROLLARY 1.3. *If $H \in \mathfrak{a}$ satisfies $\alpha(H) > 0$ whenever $\alpha + i\beta$ is a root and $\alpha \neq 0$, then $J(\mathfrak{g}) = J^S(\gamma_H)$.*

(*Such an H exists by* [1, Proposition 5.6].)

Proof. Since $\mathfrak{g}' = \mathfrak{g}_+^H$ it follows from Proposition 1.2 that $J(\mathfrak{g}) \subset J^S(\gamma_H)$. Conversely, the fact that there exists a unique stable Jacobi field J on γ_H with the initial condition $J(0)$ implies that $J = J^{J(0)}$ and the assertion follows.

2. Subgroups of G fixing $\gamma_H(\infty)$

Let x be a point at infinity and $G_x = \{g \in G\colon g(x) = x\}$. Since the action of G at infinity is continuous, G_x is a closed Lie subgroup of G with Lie algebra $\mathfrak{g}_x = \{X \in \mathfrak{g}\colon \exp sX(x) = x \text{ for all } s \in \mathbf{R}\}$.

We want to determine $\mathfrak{g}_x$ and G_x when $x = \gamma_H(\infty)$ for all $H \in \mathfrak{a}$. First we give some preliminaries.

LEMMA 2.1. *If $\lim_{t\to+\infty}\exp(-tH)g\exp tH$ exists in G, then g fixes $\gamma_H(\infty)$.* (*See* [8, *Proposition* 2.17.3]).

Proof. Suppose that $g_0 = \lim_{t\to+\infty}\exp(-tH)g\exp tH$; then

$$d(g\exp tH, \exp tH) = d(\exp(-tH)g\exp tH, e) \to d(g_0, e) \quad \text{as } t \to +\infty.$$

Hence the geodesics γ_H and $g \circ \gamma_H$ are asymptotic, which means that g fixes $\gamma_H(\infty)$.

The converse is valid in some cases as we will see later (Proposition 2.4).

LEMMA 2.2. *Let $g \in G$ be such that $g = \exp X \exp H_0$ where $X \in \mathfrak{g}'$, $H_0 \in \mathfrak{a}$. Then*

$$\lim_{t \to +\infty} \exp(-tH) g \exp tH$$

exists in G if and only if

$$\lim_{t \to +\infty} \mathrm{Ad}(\exp(-tH)) X = \lim_{t \to +\infty} \mathrm{Exp}(-t \,\mathrm{ad}_H) X$$

exists in $\mathfrak{g}'$.

Proof. We recall that $\mathrm{Ad}(\exp(-tH))X = \mathrm{Exp}(-t\,\mathrm{ad}_H)X$. Since

$$\exp(-tH) g \exp tH = (\exp(-tH) \exp X \exp tH) \exp H_0,$$

$\lim_{t \to +\infty} \exp(-tH) g \exp tH$ exists in G if and only if

$$\lim_{t \to +\infty} \exp(-tH) \exp X \exp tH$$

exists in G. We know that

$$\exp(-tH) \exp X \exp tH = I_{\exp(-tH)}(\exp X) = \exp(\mathrm{Ad}(\exp(-tH)) X)$$

and $\exp\colon \mathfrak{g}' \to [G, G]$ is a diffeomorphism, so the lemma follows since

$$\mathrm{Ad}(\exp(-tH)) X = \mathrm{Exp}(-t\,\mathrm{ad}_H) X \quad \text{and} \quad X \in \mathfrak{g}'.$$

PROPOSITION 2.3. *Let $X \in \mathfrak{g}$. If J^X is a stable field on γ_H then*

$$\exp sX(\gamma_H(\infty)) = \gamma_H(\infty) \quad \textit{for all } s \in \mathbf{R}.$$

Conversely, if $X \in \mathfrak{g}'$ satisfies $\exp X(\gamma_H(\infty)) = \gamma_H(\infty)$ then J^X is stable on γ_H. Hence $\exp sX(\gamma_H(\infty)) = \gamma_H(\infty)$ for all $s \in \mathbf{R}$.

Proof. Assume that J^X is stable. If $\alpha(s, t) = \exp sX \exp tH$ then the first assertion follows since,

$$\begin{aligned} d(\exp sX \exp tH, \exp tH) &\le \int_0^{|s|} \left| \frac{\partial}{\partial u} \alpha(u, t) \right| du = \int_0^{|s|} |\tilde{X}_{\alpha(s,u)}|\, du \\ &= \int_0^{|s|} |\tilde{X}_{\exp tH}|\, du = |s| |J^X(t)| \quad \text{for all } s \in \mathbf{R}. \end{aligned}$$

Conversely, if $\exp X(\gamma_H(\infty)) = \gamma_H(\infty)$ then $d(\exp(-tH)\exp X \exp tH, e) \le c$ for some $c > 0$ and any $t \ge 0$. By [10, Theorem 2.2, Ch. II], there exists a subsequence $\{n_k\} \subset \mathbf{N}$ and an element $g_0 \in [G, G]$ so that

$$g_{n_k} = \exp(-n_k H)\exp X \exp n_k H \to g_0 \quad \text{as } k \to \infty .$$

Since $g_{n_k} = \exp(\mathrm{Ad}(\exp(-n_k H))X)$ and $\exp\colon \mathscr{g}' \to [G, G]$ is a diffeomorphism it follows that

$$|J^X(n_k)| = |\mathrm{Ad}(\exp(-n_k H))X| \to |Z| \text{ as } k \to \infty \quad \text{if } g_0 = \exp Z,\ Z \in \mathscr{g}'.$$

Then, the convex function $|J^X(t)|$ is bounded above for $t \ge 0$ and J^X is stable on γ_H.

PROPOSITION 2.4. *Suppose $H \in \mathscr{a}$ satisfies one of the following conditions*:
(1) $\alpha(H) \ne 0$ *whenever $\alpha + i\beta$ is a root.*
(2) $\alpha(H) = 0$ *implies* $\beta(H) = 0$ *whenever $\alpha + i\beta$ is a root.*
Then, $\lim_{t \to +\infty} \exp(-tH) g \exp tH$ *exists in G if and only if* $g(\gamma_H(\infty)) = \gamma_H(\infty)$.
In the particular case when all roots are real the last assertion holds.

Proof. Assume that $g(\gamma_H(\infty)) = \gamma_H(\infty)$ where $g = \exp X \exp H_0$ with $X \in \mathscr{g}'$, $H_0 \in \mathscr{a}$. Since $\exp H_0(\gamma_H(\infty)) = \gamma_H(\infty)$ then $\exp X(\gamma_H(\infty)) = \gamma_H(\infty)$ and consequently J^X is stable on γ_H. Hence $X = X_0 + X_1$ where $X_0 \in \mathscr{g}_0^H$, $X_1 \in \mathscr{g}_+^H$ and $\mathrm{Exp}(-t\, \mathrm{ad}_H)X = \mathrm{Exp}(-t\, \mathrm{ad}_H)X_0 + \mathrm{Exp}(-t\, \mathrm{ad}_H)X_1$. By Lemma 1.1,

$$\lim_{t \to +\infty} \mathrm{Exp}(-t\, \mathrm{ad}_H)X_1 = 0$$

and therefore

$$\lim_{t \to +\infty} \mathrm{Exp}(-t\, \mathrm{ad}_H)X$$

exists in $\mathscr{g}'$ if and only if

$$\lim_{t \to +\infty} \mathrm{Exp}(-t\, \mathrm{ad}_H)X_0$$

exists in $\mathscr{g}'$.

If $\alpha(H) \ne 0$ for all roots $\alpha + i\beta$ then $\mathscr{g}_0^H = 0$ and consequently $X_0 = 0$.

If $\alpha(H) = 0$ for some root $\alpha + i\beta$, then $\beta(H) = 0$ and by Lemma 1.1,

$$\mathrm{Exp}(-t\, \mathrm{ad}_H)X_0 = X_0 \quad \text{for all } t \in \mathbf{R}.$$

In both cases $\lim_{t \to +\infty} \operatorname{Exp}(-t \operatorname{ad}_H) X$ exists and from Lemma 2.2,

$$\lim_{t \to +\infty} \exp(-tH) \exp X \exp tH$$

exists in G; since $\exp H_0 \exp tH = \exp tH \exp H_0$ ($\mathfrak{a}$ is abelian) the assertion follows. Lemma 2.1 completes the proof.

Next, we describe $\mathfrak{g}_x$ and G_x when $x = \gamma_H(\infty)$.

PROPOSITION 2.5. *If $\gamma_H(t) = \exp tH$ ($H \in \mathfrak{a}$) and $x = \gamma_H(\infty)$ then*

$$\mathfrak{g}_x = \mathfrak{g}_0^H \oplus \mathfrak{g}_+^H \oplus \mathfrak{a}.$$

Proof. If $H_0 \in \mathfrak{a}$, since $\mathfrak{a}$ is abelian,

$$\begin{aligned} d(\exp sH_0 \exp tH, \exp tH) &= d(\exp(-tH) \exp sH_0 \exp tH, e) \\ &= d(\exp sH_0, e). \end{aligned}$$

Hence $\exp sH_0(x) = x$ for all $s \in \mathbf{R}$.

If $X \in \mathfrak{g}_0^H \oplus \mathfrak{g}_+^H$ then J^X is stable on γ_H (Proposition 1.2) and by Proposition 2.3, $\exp sX(x) = x$ for all $s \in \mathbf{R}$.

For the converse, let $X \in \mathfrak{g}_x$ and write $X = X_1 + X_2 + H_0$ with $X_1 \in \mathfrak{g}_0^H \oplus \mathfrak{g}_+^H$, $X_2 \in \mathfrak{g}_-^H$ and $H_0 \in \mathfrak{a}$. By the work above, $X_1 + H_0 \in \mathfrak{g}_x$ and hence $X_2 = X - (X_1 + H_0) \in \mathfrak{g}_x$. By Proposition 2.3, J^X is stable on γ_H and so $X_2 = 0$. Hence $X \in \mathfrak{g}_0^H \oplus \mathfrak{g}_+^H \oplus \mathfrak{a}$.

COROLLARY 2.6. *For each $H \in \mathfrak{a}$, if $x = \gamma_H(\infty)$ then*

(i) $[G, G]_x = \exp(\mathfrak{g}_0^H \oplus \mathfrak{g}_+^H)$

and

(ii) $G_x = [G, G]_x \cdot \exp(\mathfrak{a})$.

Proof. (i) It is obvious from Proposition 2.5 that $\exp(\mathfrak{g}_0^H \oplus \mathfrak{g}_+^H) \subset [G, G]_x$. For the converse, from Proposition 2.3, if $\exp X$ fixes x ($X \in \mathfrak{g}'$) then J^X is stable on γ_H and so $X \in \mathfrak{g}_0^H \oplus \mathfrak{g}_+^H$ (Proposition 1.2). Then (i) follows since $[G, G] = \exp(\mathfrak{g}')$.

(ii) This is immediate from (i), since $G = [G, G]\exp(\mathfrak{a})$ and $\exp(\mathfrak{a})$ fixes x.

The following results, which is a consequence of Proposition 2.5, allows us to describe the set of fixed points of G at infinity; this description will be completed in section 3 (Theorem 3.4).

PROPOSITION 2.7. *If $H \in \mathfrak{a}$ then $G(\gamma_H(\infty)) = \gamma_H(\infty)$ if and only if $\alpha(H) \geq 0$ whenever $\alpha + i\beta$ is a root of $\mathfrak{a}$.*

Proof. Let $x = \gamma_H(\infty)$. Note that $G(x) = x$ if and only if $G = G_x$. Since G_x is a closed Lie subgroup of G, the last assertion is equivalent to $\mathscr{g} = \mathscr{g}_x$ or $\mathscr{g}' = \mathscr{g}_0^H \oplus \mathscr{g}_+^H$. But this occurs if and only if $\mathscr{g}_-^H = 0$ which is equivalent to $\alpha(H) \geq 0$ for all roots $\alpha + i\beta$.

3. Fixed points of G at infinity

Let $\mathscr{a}' = \{H \in \mathscr{a}: \alpha(H) \geq 0$ whenever $\alpha + i\beta$ is a root$\}$. In this section we describe the fixed points of G at infinity as the set of $\gamma_H(\infty)$ with $H \in \mathscr{a}'$. Our starting point is the following lemma which is very useful for such a description; its proof is in [7, Lemma 5.4.a].

LEMMA 3.1. *Let γ be a geodesic in M and $z \neq \gamma\,(\pm\infty)$ in $M(\infty)$ such that*

$$\sphericalangle_{\gamma(t)}(\gamma(\infty), z) = \beta \quad \textit{for every } t \in \mathbf{R}.$$

Then γ is the boundary of a flat half plane that contains all rays joining $\gamma(t)$ with z.

That is, there exists an isometric totally geodesic imbedding

$$F: \mathbf{R} \times [0, \infty) \to M$$

with $F(t, 0) = \gamma(t)$ for all $t \in \mathbf{R}$. In our case, F is defined by

$$F(s, t) = \exp_{\gamma(t)}(sZ(t))$$

where $Z(t)$ is the initial speed of the ray joining $\gamma(t)$ with z.

PROPOSITION 3.2. *Let γ be a geodesic in G with $\gamma(0) = e$. If $H \in \mathscr{a}$ is such that* $\exp sH$ *fixes $x = \gamma(\infty)$ for all $s \in \mathbf{R}$ then $[H, \gamma'(0)] = 0$ and $\tilde{H}$, the right invariant field on G with $\tilde{H}_e = H$, is parallel on $\gamma|_{[0,\infty)}$.*

Proof. Since $\mathscr{a}$ is abelian we may assume $X = \gamma'(0) \notin \mathscr{a}$. We will show first that $\gamma_H(s) = \exp sH$ $(|H| = 1)$ is the boundary of a flat half plane containing the rays joining $\exp sH$ with x.

Let $\alpha(s, t) = \exp sH\gamma(t)$ the geodesic variation of $\gamma(t)$ whose variational field on γ is $\tilde{H}_{\gamma(t)}$. We observe that the geodesics

$$\alpha_s(t) = \exp sH\gamma(t)$$

satisfy

$$\alpha_s(0) = \exp sH, \quad \alpha_s(\infty) = \exp sH(x) = x$$

and

$$\alpha_s'(0) = \frac{d}{dt}\left(L_{\exp sH}\gamma(t)\right)\Big|_{t=0} = \left(dL_{\exp sH}\right)_e X.$$

Then

$$\sphericalangle_{\gamma_H(s)}\left(\gamma_H(\infty), x\right) = \sphericalangle_{\exp sH}\left(\left(dL_{\exp sH}\right)_e H, \left(dL_{\exp sH}\right)_e X\right) = \sphericalangle_e(H, X)$$

and from Lemma 3.1, γ_H is the boundary of the flat half plane $\mathbf{P}$ determined by $\alpha(s, t)$ for $s \in \mathbf{R}$, $t \geq 0$. Hence

$$X_{\exp sH} = \frac{\partial \alpha}{\partial s}(s, 0),$$

the left invariant field restricted to γ_H, is parallel on γ_H and then $S_H X = \nabla_H X = 0$. Now, since $K(H, X) = |S_H X|^2 - \|[H, X]\|^2 = 0$, it follows that $[H, X] = 0$.

Next, we will see that $\tilde{H}_{\gamma(t)}$ is parallel on γ for $t \geq 0$. In fact, since

$$R\left(\tilde{H}_{\gamma(t)}, \gamma'(t)\right)\gamma'(t) = 0 \quad \text{for all } t \geq 0$$

($\mathbf{P}$ is flat and $\tilde{H}_{\gamma(t)} \in T_{\gamma(t)}\mathbf{P}$) and $\tilde{H}_{\gamma(t)}$ is a Jacobi field on γ it follows that $\nabla_{\gamma'}\tilde{H}(t)$ is parallel on γ for all $t \geq 0$. Then,

$$\left|\nabla_{\gamma'}\tilde{H}(t)\right| = \left|\nabla_{\gamma'}\tilde{H}(0)\right| \quad \text{for all } t \geq 0$$

and from the fact that $\nabla_{\gamma'}\tilde{H}(0) = \nabla_H X = 0$ (see (iii) at the beginning of Section 1) we have $\nabla_{\gamma'}\tilde{H}(t) = 0$ for all $t \geq 0$ and the last assertion follows.

COROLLARY 3.3. *If H_0 is a regular element of $\mathfrak{a}$ (that is, $\operatorname{ad}_{H_0}|_{\mathfrak{g}'}$ is non-singular) then $\{\exp sH_0\}_{s \in \mathbf{R}}$ does not have fixed points at infinity distinct from $\gamma_H(\infty)$ with $H \in \mathfrak{a}$.*

Proof. Assume $\exp sH_0(x) = x$ for all $s \in \mathbf{R}$ and let γ be the geodesic in G with $\gamma(0) = e$, $\gamma(\infty) = x$. Let $\gamma'(0) = X = X_1 + H_1$, where $X_1 \in \mathfrak{g}'$ and $H_1 \in \mathfrak{a}$. By Proposition 3.2, $0 = \operatorname{ad}_{H_0}(X) = \operatorname{ad}_{H_0}(X_1)$, which implies that $X_1 = 0$ and $X = H_1 \in \mathfrak{a}$ by the regularity of H_0. Since $\exp(\mathfrak{a})$ is totally geodesic in G it follows that $\gamma(t) = \exp tH_1$ and $x = \gamma_{H_1}(\infty)$.

THEOREM 3.4. *Let G be a solvable and simply connected Lie group with a left invariant metric of nonpositive curvature without de Rham flat factor. Then the fixed points of G at infinity are $\gamma_H(\infty)$ with $H \in \mathfrak{a}'$.*

Proof. Assume that $G(x) = x$ and let H_0 be a regular element of $\mathfrak{a}$ (such an H_0 exists since G has no flat factor). Since $\exp sH_0(x) = x$ for all $s \in \mathbf{R}$, it follows from Corollary 3.3 and Proposition 2.7 that $x = \gamma_H(\infty)$ for some $H \in \mathfrak{a}$ satisfying $\alpha(H) \geq 0$ for all roots $\alpha + i\beta$. By Proposition 2.7, Theorem 3.4 holds.

COROLLARY 3.5. *Let M be a simply connected homogeneous space of non-positive curvature that has no de Rham flat factor. If $I(M)$ $(I_0(M))$ has a fixed point x in $M(\infty)$ then x is a flat point, provided that M is not a visibility manifold.*

We recall that x is a flat point (at infinity) if every geodesic γ belonging to x is the boundary of a flat half plane. (See [4, Section 3].)

Proof. We note that M is isometric to a solvable Lie group G with a left invariant metric, where G is a closed connected Lie subgroup of $I(M)$. Since M does not satisfy the visibility axiom then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{a}$ where $\dim \mathfrak{a} \geq 2$ (see [5, Theorems 2.4 and 3.1]) and therefore the fixed points of G at infinity given by Theorem 3.4 are flat points: if $x = \gamma_H(\infty)$, $H \in \mathfrak{a}'$ then $g \circ \gamma_H$ is asymptotic to γ_H through $g \in G$.

Since a fixed point of $I(M)$ $(I_0(M))$ corresponds to a fixed point of G at infinity, the corollary follows immediately.

We note that if M is not symmetric then $I(M)$ has a fixed point in $M(\infty)$ (see [8, Remark 1.9.18]).

THEOREM 3.6. *If G has no flat de Rham factor and $g \in G$ translates a geodesic γ with $\gamma(0) = e$, then $\gamma(t) = \exp tH$ and $g = \exp t_0H$ for some $H \in \mathfrak{a}$, $t_0 \in \mathbf{R}$.*

In particular, the only one-parameter subgroups of G which are geodesics are $\gamma_H(t) = \exp tH$ with $H \in \mathfrak{a}$.

Proof. Let $t_0 > 0$ be such that $g\gamma(t) = \gamma(t + t_0)$ for all $t \in \mathbf{R}$. Since G has no flat factor, we can choose $H \in \mathfrak{a}$ such that $\alpha(H) > 0$ for all roots $\lambda = \alpha + i\beta$ (such an H exists by Proposition 5.6 of [1]).

If $x = \gamma_H(\infty)$ we may assume $x \neq \gamma(\pm\infty)$ (otherwise there is nothing to prove); so $G(x) = x$ and from the fact, $g(\gamma(\infty)) = \gamma(\infty)$ we have

$$\sphericalangle_{\gamma(t)}(x, \gamma(\infty)) = \sphericalangle_{g^n\gamma(t)}(g^n(x), g^n(\gamma(\infty))) = \sphericalangle_{\gamma(t+nc)}(x, \gamma(\infty))$$

for any $t \in \mathbf{R}$ and $n \in \mathbf{N}$. Since the function $t \to \sphericalangle_{\gamma(t)}(x, \gamma(\infty))$ is nondecreasing, it converges to some number $\beta > 0$ as $n \to \infty$. Hence, for every

$t \in \mathbf{R}$,

$$\measuredangle_{\gamma(t)}(x, \gamma(\infty)) = \lim_{n \to \infty} \measuredangle_{\gamma(t+nc)}(x, \gamma(\infty)) = \beta.$$

Then, Lemma 3.1 implies that γ is the boundary of a flat half plane $\mathbf{P}$ which is determined by $\alpha(s, t) = \gamma(t)\exp sH$ for $s \geq 0$ and $t \in \mathbf{R}$ ($\alpha_t(0) = \gamma(t)$, $\alpha_t(\infty) = x$).

We compute

$$\frac{\partial}{\partial t}\alpha(s, t)\bigg|_{t=0} = \left(dR_{\exp sH}\right)_e \cdot \gamma'(0) = J^X(s)$$

if $X = \gamma'(0)$. Then $R(J^X(s), \gamma_H'(s))\gamma_H'(s) = 0$ for $s \geq 0$ and since J^X is a Jacobi field on γ_H, $\nabla_{\gamma_H'} J^X$ is parallel on $\gamma_H|_{[0,\infty)}$; therefore $J^X(s)$ is parallel for $s \geq 0$ (J^X is stable on γ_H by Corollary 1.3). By writing $X = X_1 + H_0$ with $X_1 \in \mathfrak{g}'$ and $H_0 \in \mathfrak{a}$, we have

$$J^{X_1}(s) = J^X(s) - J^{H_0}(s)$$

which is parallel for $s \geq 0$. Since $\lim_{s \to +\infty}|J^{X_1}(s)| = 0$ it follows that $X = H_0$; consequently, $\gamma(t) = \exp tH_0$ and $g = \exp t_0 H_0$ since $g = g\gamma(0) = \gamma(t_0)$.

The last assertion is immediate since if $\gamma(t) = \exp tX$ is a geodesic then $\exp X$ translates γ.

COROLLARY 3.7. *The axial (hyperbolic) elements of G are those which are conjugate to elements in* $\exp(\mathfrak{a})$.

Proof. If $g \in G$ translates a geodesic γ with $\gamma(0) = g_0$ then $g_0^{-1}gg_0$ translates the geodesic $\alpha(t) = g_0^{-1}\gamma(t)$ with $\alpha(0) = e$. By Theorem 3.6,

$$\alpha(t) = \exp tH \quad \text{and} \quad g_0^{-1}gg_0 = \exp t_0 H$$

for some $H \in \mathfrak{a}$, $t_0 > 0$. Then $g = g_0(\exp t_0 H)g_0^{-1}$ which is conjugate to $\exp t_0 H \in \exp(\mathfrak{a})$.

Clearly, if $g = g_0 \exp Hg_0^{-1}$ ($H \in \mathfrak{a}$) then g translates $\gamma(t) = g_0 \exp tH$.

Remark 3.8. We recall that g (or L_g) is hyperbolic (parabolic) if the function d_g ($h \to d(h, gh)$) assumes (does not assume) the infimum. It is clear that if d_g assumes the minimum at g_0 then g translates the geodesic joining g_0 with gg_0 in G. Conversely, if g translates a geodesic γ then d_g assumes the infimum on each point of γ (see [3, Proposition 4.2]).

COROLLARY 3.9. *The elements of* $[G, G]$ *different from the identity are all parabolic. Moreover, the parabolic elements of* G *are those which are not conjugate to an element of* $\exp(\mathfrak{a})$.

Proof. Let $g \neq e$ be in $[G, G]$. If g is hyperbolic then g translates a geodesic γ with $\gamma(0) = g_0$; hence $g = g_0 \exp H g_0^{-1}$ for some $H \in \mathfrak{a}$ and therefore $g_0^{-1} g g_0 = \exp H$. Since $[G, G]$ is normal in G and $[G, G] \cap \exp(\mathfrak{a}) = \{e\}$, we have a contradiction and consequently g is parabolic. From the fact that any $g \neq e$ in G is either hyperbolic or parabolic (Remark 3.8) the corollary follows.

4. Joining points at infinity

In this section we summarize some results about the points at infinity which can be joined to a fixed point of G. We recall that two points $x \neq y$ in $M(\infty)$ can be joined (by a geodesic of M) if there exists a geodesic γ of M with $\gamma(\infty) = x$ and $\gamma(-\infty) = y$. If M is a visibility manifold, any two distinct points in $M(\infty)$ can be joined. Two points x, y in $M(\infty)$ are said to be *G-dual* if there exists a sequence $\{g_n\} \subset G$ such that $g_n(p) \to x$ and $g_n^{-1}(p) \to y$ as $n \to \infty$ for some p in M.

Let $M = G$ be a solvable Lie group with a left invariant metric of non-positive curvature.

LEMMA 4.1. *Let* x *be a fixed point of* G *at infinity. If* y *can be joined to* x *then* $g_n^{-1} \to x$ *as* $n \to \infty$ *for every sequence* $\{g_n\} \subset G$ *such that* $g_n \to y$ *as* $n \to \infty$. *In particular, if* y *can be joined to* x *then* y *is G-dual to* x.

Proof. Let γ be a geodesic joining y with x and set $g_0 = \gamma(0)$; then

$$\begin{aligned}
\sphericalangle_{g_0}\left(g_n^{-1}g_0, x\right) &= \sphericalangle_{g_0}\left(g_n^{-1}g_0, g_n^{-1}(x)\right) \\
&= \sphericalangle_{g_n g_0}\left(g_0, x\right) \\
&\leq \pi - \sphericalangle_{g_0}\left(g_n g_0, x\right) \\
&= \sphericalangle_{g_0}\left(g_n g_0, y\right) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence, $g_n^{-1} g_0 \to x$ as $n \to \infty$ and consequently $g_n^{-1} \to x$ as $n \to \infty$.

PROPOSITION 4.2. *Let* $x = \gamma_H(\infty)$ *where* $H \in \mathfrak{a}'$. *Then*:

(i) *The set of points* (*at infinity*) *to which* x *can be joined is the orbit* $G(\gamma_H(-\infty))$.

(ii) *If* y *can be joined to* x *then the closure of the orbit* $G(y)$ *is the set of points at infinity that are G-dual to* x. *In particular, this set coincides with the closure of the orbit* $G(\gamma_H(-\infty))$.

Proof. Let $y = \gamma_H(-\infty)$.

(i) It is clear that for any $g \in G$, $g \circ \gamma_H$ is a geodesic joining $g(y)$ and x. Conversely, if γ is a geodesic joining z to x and $\gamma(0) = g$, then $\gamma(t) = g \exp tH$ $(g(x) = x)$ which means $z = \gamma(-\infty) = g(\gamma_H(-\infty))$.

(ii) Assume that z is G-dual to x and we will show that $z \in G(y)^-$ (the closure of the orbit in $M(\infty)$). Let $\{g_n\} \subset G$ be such that $g_n \to z$ and $g_n^{-1} \to x$ as $n \to \infty$. Then,

$$\sphericalangle_e(g_n, g_n(y)) = \sphericalangle_{g_n^{-1}}(e, y) \leq \pi - \sphericalangle_e(g_n^{-1}, y)$$
$$= \sphericalangle_e(g_n^{-1}, x) \to 0 \quad \text{as } n \to \infty.$$

Hence, $z = \lim_{n \to \infty} g_n = \lim_{n \to \infty} g_n(y)$.

Now (ii) follows from Lemma 4.1 since the set of points at infinity which are G-dual to x is G-invariant and closed.

Now, we consider the limit sets $L([G,G])$ and $L(A)$ of the subgroups $[G,G]$ and $A = \exp(\mathfrak{a})$ respectively. We observe first that $L(A)$ coincides with the set $A(\infty) = \{\gamma_H(\infty)\colon\ H \in \mathfrak{a}\}$. It follows from Lemma 4.1 that if $x = \gamma_H(\infty)$ is a fixed point of G then the only point of $L(A)$ that can be joined to x by a geodesic is $\gamma_H(-\infty)$. In fact, if $z = \gamma_{H_0}(\infty)$ can be joined to x, since $\exp nH_0 \to z$ as $n \to \infty$, then $\exp(-nH_0) \to x$ as $n \to \infty$ and therefore $H_0 = -H$.

In the special case $x = \gamma_H(\infty)$ with $\alpha(H) > 0$ whenever $\alpha + i\beta$ is a root, no point in $L([G,G])$ can be joined to x.

PROPOSITION 4.3. *Let $x = \gamma_H(\infty)$ be such that $\alpha(H) > 0$ whenever $\alpha + i\beta$ is a root. Then*:

(i) $\gamma_H(-\infty) \notin L([G,G])$.

(ii) $L([G,G]) \cap G(\gamma_H(-\infty)) = \varnothing$. *In particular, there are no points of $L([G,G])$ which can be joined to x.*

Proof. (i) Let T be the center of $[G,G]$ ($T \neq \{e\}$ since $[G,G]$ is nilpotent); it is clear that any element of T fixes all points of $L([G,G])$. Assume that $\gamma_H(-\infty) \in L([G,G])$. If $g = \exp X$, $X \in \mathfrak{g}'$ is an element of T $(g \neq e)$, $g(\gamma_H(-\infty)) = \gamma_H(-\infty)$ and $g(\gamma_H(\infty)) = \gamma_H(\infty)$; it follows then from Proposition 1.2 that $X \in \mathfrak{g}_0^H$, contradicting the choice of $H(\mathfrak{g}_0^H = 0)$. Hence

$$\gamma_H(-\infty) \notin L([G,G]).$$

(ii) This is an immediate consequence of (i) since $L([G,G])$ is invariant under G.

We observe that as a consequence of (i), $L([G,G]) \cap L(A)$ is contained in the set $\{\gamma_H(\infty)\colon\ \alpha(H) \geq 0 \text{ for some root } \alpha + i\beta\}$.

Remark 4.4. If G satisfies the visibility axiom it follows from Proposition 4.3 (ii) that $L([G, G])$ contains a unique point x which is a fixed point of G. ($L([G, G])$ is G-invariant). Let $x = \gamma_H(\infty)$, $H \in a'$; then $L(A) = \gamma_H(\pm\infty)$ and by Proposition 4.2 (i), $G(\gamma_H(-\infty)) = M(\infty) - \{\gamma_H(\infty)\}$. Since all elements of $[G, G]$ are parabolic (Corollary 3.9), it follows from Theorem 6.5 of [6] that x is the unique fixed point of every $g \neq \text{id}$ in $[G, G]$. (Compare with [5, Theorem 1.5].)

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