ILLINOIS JOURNAL OF MATHEMATICS
Volume 33, Number 3, Fall 1989

RATIONAL PERIOD FUNCTIONS FOR THE MODULAR GROUP AND REAL QUADRATIC FIELDS

BY

YOUNGJU CHOIE

0. Introduction

Automorphic integrals with rational period functions, being generalization of automorphic forms on the discrete subgroup of $SL(2, R)$, share properties similar to those of forms. Examples are furnished by the Eichler integrals—automorphic integrals of negative integer weight with polynomial period functions—which have been the object of much attention in recent years [4], [5], [6], [9], [12], [15], [17], [19]. The questional naturally arises whether there exist automorphic integrals with rational period functions which are not polynomials. M. Knopp [13] has constructed modular integrals of weight $2k$ (k odd) which differ from Eichler integrals. In [14] it has been shown that the poles of any rational period function, $q_{T,2k}(z)$, for the modular group $\Gamma(1)$ must lie in $Q(\sqrt{N})$, $N \in Z^+$. However, the only previously known quadratic fields containing poles of $q_{T,2k}(z)$ for $\Gamma(1)$ were $Q(\sqrt{5})$, $Q(\sqrt{3})$, and $Q(\sqrt{21})$, and these examples were known only for odd k.

The main object of this paper is the construction of $q_{T,2k}(z)$ for $\Gamma(1)$ with k any integer (*even* or *odd*), having poles in an *arbitrary* real quadratic field, $Q(\sqrt{N})$. We have developed three distinct new methods to achieve this goal. First, we have constructed $q_{T,2k}(z)$ for $\Gamma(1)$ by using the coset decomposition of $\Gamma'(1)$, the commutator subgroup of $\Gamma(1)$. Since $\Gamma'(1)$ is a free group, the necessary and sufficient conditions for the existence of a rational period function $q_{T,2k}(z)$ of a modular integral on $\Gamma'(1)$ reduce to a single condition on rational period functions for $\Gamma(1)$. Then rational period functions of a modular integral on $\Gamma(1)$ can be constructed by showing how to satisfy the above condition. This construction can be generalized to incorporate the class of Hecke group. By use of an operator of Bogo-Kuyk [1], $q_{T,2k}(z)$ for $\Gamma(1)$ can be constructed from those on the Hecke groups for $\lambda = \sqrt{2}$ and $\sqrt{3}$.

The second method entails the use of Pell's equation to construct $q_{T,2k}(z)$. This construction gives $q_{T,2k}(z)$ for $\Gamma(1)$ and *any integer* k with poles in an

Received July 10, 1987.

arbitrary real quadratic fields. Also, we show that the collection $\{q_{T,2k}\}_N$ of rational period function with poles in the real quadratic field $Q(\sqrt{N})$ is infinite dimensional over C. Finally, we generalize the method by M. Knopp [13] to construct rational period functions. The appended tables list a number of specific examples of $q_{T,2k}(z)$.

1. Definitions

Let $\mathscr{H}$ be the complex upper half plane and let Γ be a Fuchsian group acting on $\mathscr{H}$.

Let $F(z)$ be a meromorphic function in $\mathscr{H}$ satisfying the transformation formula

$$(1.1) \qquad (cz+d)^{-2k}F(Mz) = F(z) + q_M(z),$$

where k is a rational integer and for each element $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$, $q_M(z)$ is a rational function of z. Assume also that F is meromorphic in the local uniformizing variable at each parabolic cusp of a fundamental region for Γ. Then F is called an *automorphic integral* of *weight* $2k$ *for* Γ, with *rational period functions* $q_M(z)$. In the case when $\Gamma = \Gamma(1)$, the modular group, we call F a *modular integral* of *weight* $2k$. (Note that if $q_M(z) = 0$ for each $M \in \Gamma$, then F is simply an automorphic form of weight $2k$ for Γ.)

2. Rational period functions

The Hecke group $G(\lambda_n)$, $\lambda_n = \cos(\pi/n)$, is the group of all linear fractional transformations generated by the two transformations

$$S_n = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which satisfy the relations

$$(1.2) \qquad T^2 = (S_nT)^n = (TS_n)^n = I.$$

(Note. We identify $I = -I$ as linear fractional transformations.)

As is well known [5], the Hecke group $G(\lambda_n)$ is the free product of $\langle T \rangle$ and $\langle TS_n \rangle$.

Since the Hecke group $G(\lambda_n)$ is generated by S_n and T, the condition (1.1) is equivalent to

$$(1.3) \qquad \begin{aligned} F(z+\lambda_n) &= F(z) + q_{S_n}(z), \\ z^{-2k}F\left(-\frac{1}{z}\right) &= F(z) + q_{\lambda_n, T}(z), \end{aligned}$$

with q_{S_n}, $q_{\lambda_n, T}$ rational function in z. Since a rational function F satisfies (1.3) trivially, we impose the further restriction that F be periodic with period λ_n, that is, that $q_{S_n} = 0$. Then (1.3) can be written as

$$(1.4) \qquad F(z+\lambda_n) = F(z),\ z^{-2k}F\left(-\frac{1}{z}\right) = F(z) + q_{\lambda_{n,T}}(z).$$

If we introduce the customary notation for the stroke operator,

$$F|_{-2k}M = (cz+d)^{-2k}F(Mz), \quad M = \begin{pmatrix} * & * \\ c & d \end{pmatrix},$$

the condition (1.1) becomes $F|_{-2k}M = F + q_M$, $M \in G(\lambda_n)$, and from this follows the (cocycle) condition

$$q_{M_1M_2} = q_{M_1}|_{-2k}M_2 + q_{M_2} \quad \text{for } M_1, M_2 \in G(\lambda_n), \quad k \in Z.$$

Also (1.4) becomes

$$(1.5) \qquad F|_{-2k}S_n = F,\ F|_{-2k}T = F + q_{\lambda_{n,T}}.$$

For convenience, I shall write $F|M$ instead of $F|_{-2k}M$ throughout.

Suppose now that $q_{\lambda_{n,T}}(z)$ is a rational function as in (1.5), for some F meromorphic in $\mathscr{H}$. Then the defining relations (1.2) in $G(\lambda_n)$ imply that

$$(1.6) \qquad \begin{gathered} q_{\lambda_{n,T}}|T + q_{\lambda_{n,T}} = 0 \\ q_{\lambda_{n,T}}|(S_nT)^{n-1} + q_{\lambda_{n,T}}|(S_nT)^{n-2} + \cdots + q_{\lambda_{n,T}}|(S_nT) + q_{\lambda_{n,T}} = 0, \end{gathered}$$

are necessary conditions upon $q_{\lambda_{n,T}}$ for the existence of F meromorphic in $\mathscr{H}$ such that (1.5) holds. On the other hand, Knopp has shown, through the use of Eichler's "generalized Poincaré series" (see [12]), that (1.6) is in fact *sufficient* for the existence of such a function F, and that moreover F can be taken holomorphic in $\mathscr{H}$. Since any two integrals with the same period function differ by a form we conclude that the collection of "distinct" automorphic integrals with rational period functions is in 1-1 correspondence with the collection of rational period functions $q_{\lambda_{n,T}}$.

Now, we consider a rational period function $q_{\lambda_{n,T}}$ satisfying the two relations in (1.6) for $q_{\lambda_{n,T}} = q_{\lambda,T}$:

$$\begin{gathered} q_{\lambda,T} + q_{\lambda,T}|T = 0, \\ q_{\lambda,T}|(S_nT)^{n-1} + \cdots + q_{\lambda,T} = 0. \end{gathered}$$

These two identities yield the further one

$$(1.7) \qquad q_{\lambda,T} = q_{\lambda,T}|S_n + q_{\lambda,T}|(S_nT)S_n + \cdots + q_{\lambda,T}|(S_nT)^{n-2}S_n.$$

The following theorem is given by M. Knopp [14].

THEOREM 1. (a) *If Z_0 is a finite pole of any rational function satisfying* (1.6) *for $\lambda = 1$, then there is a squarefree positive integer N such that $Z_0 \in Q(\sqrt{N})$.*

(b) *If the finite pole Z_0 is in Q, then $Z_0 = 0$.*

Proof. See [14].

The following is a straightforward generalization of Theorem 1 given by H. Meier and G. Rosenberger [7].

COROLLARY 2. *If $Z_0 \in C$ is a pole of any rational function satisfying* (1.6) *then there is a positive number $N \in Z[\lambda^2]$ such that $Z_0 \in Q(\sqrt{N}, \lambda)$ or $Z_0 \in \lambda Q(\lambda^2)$ for $\lambda_n = \lambda$.*

3. Construction of an automorphic integral for the Hecke group from an automorphic integral for the commutator subgroup of the Hecke group

We state the following result without proof.

THEOREM 3 (Nielsen). *Let G be a free product of n cyclic group c_i of order m_i generated by elements a_i $(1 \leq i \leq l)$. Then the commutator group G' is a free group of index $m = m_1 m_2 \dots m_l$ in G and the rank of G' is*

$$1 + m\left\{-1 + \sum_{i=1}^{l}\left(1 - \frac{1}{m_i}\right)\right\}.$$

G' is generated by the commutators $[a_i^{\mu}, a_j^{\phi}]$, where $1 \leq i \leq j \leq l$ and $0 < \mu < m_i$, $0 < \phi < m_j$. The factor group G/G' is isomorphic to the direct product of the cyclic groups $c_1, c_2, \dots, c_l$.

Let us introduce some notation.

Notation. $G'(\lambda_n)$ is the commutator subgroup of $G(\lambda_n)$. In particular, $G'(\lambda_3) = \Gamma'(1)$ is the commutator subgroup of $G(\lambda_3) = \Gamma(1)$. Let q_{λ_n} denote the rational period function of an automorphic integral f_{λ_n} of weight $2k$ (k is an integer) on $G(\lambda_n)$; i.e., $f_{\lambda_n}|T = f_{\lambda_n} + q_{\lambda_{n,T}}$. In the case $n = 3$, we shall write $q_{\lambda_{3,T}} = q_T$. If we do not specify n, we shall write $\lambda_n = \lambda$, $q_{\lambda_{n,T}} = q_{\lambda,T}$, f_{λ_n} and $G(\lambda_n) = G(\lambda)$.

We construct automorphic integrals of weight $2k$ with rational period functions on the commensurable Hecke groups $G(\lambda)$ $(\lambda = 1, \sqrt{2}, \sqrt{3})$ from those on $G'(\lambda)$. Since $G'(\lambda)$ is a free group by Theorem 3, the two conditions in (1.6) for the rational period function reduce to only one condition and consequently we can obtain automorphic integrals with rational period functions on $G'(\lambda)$. From these we construct automorphic integrals on $G(\lambda)$ itself by summing over cosets of $G(\lambda)/G'(\lambda)$. Applying an appropriate operator (the Bogo-Kuyk operator [1]) from $G(\lambda)$ ($\lambda = \sqrt{2}, \sqrt{3}$) to $\Gamma(1)$, we then obtain further rational period functions for $\Gamma(1)$.

(a) The case $\lambda = 1$. Construction of modular integrals with rational period functions for $\Gamma(1)$ from modular integrals with rational period functions for $\Gamma'(1)$.

Note. By Theorem 3, $\Gamma'(1)$ is generated by $\langle a_1, b_1 \rangle$ where

$$a_1 = S^2TS, \quad b_1 = STS^2. \tag{1.8}$$

Construction. Let f be an automorphic integral of weight $2k$, $k \in Z$, with rational period functions of $\Gamma'(1)$: f is a meromorphic function on $\mathscr{H}$ satisfying the condition (1.1), that is, $f|M = f + q_M$, where $M \in \Gamma'(1)$, and q_M is a rational function. Further, f is meromorphic in the local uniformizing parameter at each cusp of a fundamental region for $\Gamma'(1)$. Since $\Gamma'(1)$ is generated by a_1, b_1 in (1.8), the condition (1.1) is equivalent to

$$f|a_1 = f + q_{a_1}, \quad f|b_1 = f + q_{b_1}. \tag{1.9}$$

Since $\Gamma'(1)$ is a free group, there is no element of finite order.

Now, consider the full group $\Gamma(1)$. We know that $\Gamma(1) = \Sigma_{j=0}^{5} \Gamma'(1)S^j$. Let us define the function

$$\tilde{f} = \sum_{j=0}^{5} f|S^j,$$

where f is the given automorphic integral on $\Gamma'(1)$. First note that the function $\tilde{f}$ is meromorphic in $\mathscr{H}$. At the cusps of a fundamental region for the modular group the behavior of $\tilde{f}$ is determined by the behavior of f at the cusps of a fundamental region for the commutator subgroup of the modular group. Also, we have

(i)

$$\begin{aligned} \tilde{f}|S &= \sum_{j=1}^{5} f|S^j + f|S^6 \quad \left(\text{since } S^6 \in \Gamma'(1)\right) \\ &= \sum_{j=1}^{5} f|S^j + (f + q_{S^6}) = \sum_{j=1}^{5} f|S^j + q_{S^6} \end{aligned} \tag{1.10}$$

where q_{S^6} is a rational period function of f.

(ii)

$$\begin{aligned}\tilde{f}|T &= f|(TS^{-3}) \cdot S^3 + f|(STS^{-4})S^4 + f|(S^2TS^{-5})S^5 \\ &\quad + f|(S^3T) + f|(S^4TS^{-1})S + f|(S^5TS^{-2})S^2 \\ &\qquad \left(\text{since } TS^{-3}, STS^{-4}, S^2TS^{-5}, S^3T, S^4TS^{-1}, S^5TS^{-2} \in \Gamma'(1)\right) \\ &= [f + q_{TS^{-3}}]|S^3 + [f + q_{STS^{-4}}]|S^4 + [f + q_{S^2TS^{-5}}]|S^5 \\ &\quad + [f + q_{S^3T}] + [f + q_{S^4TS^{-1}}]|S + [f + q_{S^5TS^{-2}}]|S^2.\end{aligned}$$

But $TS^{-3} = b_1a_1^{-1}$, $STS^{-4} = b_1S^{-6}$, $S^2TS^{-5} = a_1S^{-6}$ by (1.8), which implies that

(1.11)

$$\begin{aligned}\tilde{f}|T &= \left[f + q_{b_1a_1^{-1}}\right]|S^3 + \left[f + q_{b_1S^{-6}}\right]|S^4 + \left[f + q_{a_1S^{-6}}\right]|S^5 \\ &\quad + \left[f + q_{a_1b_1^{-1}}\right] + \left[f + q_{S^6b_1^{-1}}\right]|S + \left[f + q_{S^6a_1^{-1}}\right]|S^2 \\ &= \sum_{j=0}^{5} f|S^j + q_{b_1a_1^{-1}}|S^3 + q_{a_1b_1^{-1}} + q_{b_1S^{-6}}|S^4 + q_{S^6b_1^{-1}}|S + q_{a_1S^{-6}}|S^5 \\ &\quad + q_{S^6a_1^{-1}}|S^2.\end{aligned}$$

If we put $q_{S^6} \equiv \tilde{q}_S$ in (1.10), (i) becomes

$$\tilde{f}|S = \tilde{f} + q_{S^6} = \tilde{f} + \tilde{q}_S. \tag{1.12}$$

If we put

$$q_{b_1a_1^{-1}}|S^3 + q_{a_1b_1^{-1}} + q_{b_1S^{-6}}|S^4 + q_{S^6b_1^{-1}}|S + q_{a_1S^{-6}}|S^5 + q_{S^6a_1^{-1}}|S^2 \equiv \tilde{q}_T$$

in (1.11), (ii) becomes

$$\begin{aligned}\tilde{f}|T &= \tilde{f} + q_{b_1a_1^{-1}}|S^3 + q_{a_1b_1^{-1}} + q_{b_1S^{-6}}|S^4 + q_{S^6b_1^{-1}}|S + q_{a_1S^{-6}}|S^5 \\ &\quad + q_{S^6a_1^{-1}}|S^2 = \tilde{f} + \tilde{q}_T,\end{aligned} \tag{1.13}$$

where $\tilde{q}_S$, $\tilde{q}_T$ are rational functions. Since any rational $\tilde{f}$ trivially satisfies (1.12) we impose the further restriction that $\tilde{f}$ be a periodic function, that is, $\tilde{f}(z+1) = \tilde{f}(z)$. This implies $\tilde{q}_S = q_{S^6} = 0$ in (1.12). Then $\tilde{f}$ is a modular integral with a rational period function on $\Gamma(1)$. For $\tilde{q}_T$ satisfies the two relations in (1.6) if $\tilde{q}_S = q_{S^6} \equiv 0$.

Now applying the consistency condition

$$q_{M_1M_2} = q_{M_1}|M_2 + q_{M_2} \quad \text{for } M_1, M_2 \in \Gamma'(1),$$

especially $q_M|M^{-1} = -q_{M^{-1}}$, we see that (for $S^6 = a_1 b_1^{-1} a_1^{-1} b_1$) $\tilde{q}_S = q_{S^6} \equiv 0$ is equivalent to

$$q_{a_1}|b_1^{-1}a_1^{-1}b_1 - q_{b_1}|b_1^{-1}a_1^{-1}b_1 - q_{a_1}|a_1^{-1}b_1 + q_{b_1} = 0.$$

Thus,

$$q_{a_1} - q_{a_1}|b_1 = q_{b_1} - q_{b_1}|b_1^{-1}a_1 b_1, \tag{1.14}$$

or

$$q_{a_1} - q_{a_1}|TS^{-1}TS = q_{b_1} - q_{b_1}|S^{-3}TSTS^2.$$

Furthermore,

$$\begin{aligned} \tilde{q}_T = {} & q_{a_1} + q_{a_1}|S^{-1} - q_{a_1}|T - q_{a_1}|S^{-1}T \\ & + q_{b_1}|S^{-2} + q_{b_1}|S^{-3} - q_{b_1}|S^{-2}T - q_{b_1}|S^{-3}T. \end{aligned} \tag{1.15}$$

For, from (1.13),

$$\begin{aligned} \tilde{q}_T &= q_{b_1 a_1^{-1}}|S^3 + q_{a_1 b_1^{-1}} + q_{b_1 S^{-6}}|S^4 + q_{S^6 b_1^{-1}}|S + q_{a_1 S^{-6}}|S^5 + q_{S^6 a_1^{-1}}|S^2 \\ &= -q_{a_1}|a_1^{-1}S^3 + q_{b_1}|a_1^{-1}S^3 + q_{a_1}|b_1^{-1} - q_{b_1}|b_1^{-1} + q_{a_1}|S^{-1} \\ &\quad - q_{a_1}|a_1^{-1}S^2 + q_{b_1}|S^{-2} - q_{b_1}|b_1^{-1}S \quad (\text{since } q_{S^6} = 0) \\ &= -q_{a_1}|T + q_{b_1}|S^{-3} + q_{a_1} - q_{b_1}|S^{-3}T + q_{a_1}|S^{-1} \\ &\quad - q_{a_1}|a_1^{-1}S^2 + q_{b_1}|S^{-2} - q_{b_1}|b_1^{-1}S, \end{aligned}$$

because (1.14) implies that

$$\begin{aligned} & q_{a_1}|a_1^{-1}S^3 + q_{b_1}|a_1^{-1}S^3 + q_{a_1}|b_1^{-1} - q_{b_1}|b_1^{-1} \\ & \quad = -q_{a_1}|T + q_{b_1}|S^{-3} + q_{a_1} - q_{b_1}|S^{-3}T. \end{aligned}$$

On the basis of the above construction, we state the following theorem.

THEOREM 4. *If we have any two rational functions q_{a_1}, q_{b_1} that satisfy the single relation*

$$q_{a_1} - q_{a_1}|TS^{-1}TS = q_{b_1} - q_{b_1}|S^{-3}TSTS^2$$

in (1.14), *then*

$$\begin{aligned} \tilde{q}_T = {} & q_{a_1} + q_{a_1}|S^{-1} - q_{a_1}|T - q_{a_1}|S^{-1}T + q_{b_1}|S^{-3} \\ & - q_{b_1}|S^{-2}T - q_{b_1}|S^{-3}T + q_{b_1}|S^{-2} \end{aligned}$$

in (1.15) *is a rational period function.*

Proof. From the above construction, $\tilde{q}_T$ in (1.15) satisfies the two relations in (1.6).

Later, we shall see the direct application of Theorem 4 (Theorem 8).

This method can be generalized to construct automorphic integrals with rational period functions for $G(\lambda)$ from automorphic integrals with rational period functions for $G'(\lambda)$.

(b) The case $\lambda = \sqrt{2}$. Construction of automorphic integrals with rational period functions for $G(\sqrt{2})$ from automorphic integrals with rational period functions for $G'(\sqrt{2})$.

Note. By Theorem 3, $G'(\sqrt{2})$ is generated by $\langle a_2, b_2, c_2 \rangle$ where

$$(1.16) \quad a_2 = S_4TS_4^{-1}T, \quad b_2 = TS_4^{-1}TS_4, \quad c_2 = S_4TS_4^2TS_4, \quad S_4 = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}.$$

Construction. Let f_2 be an automorphic integral of weight $2k$, $k \in Z$, with rational period functions on $G'(\sqrt{2})$; f_2 is a meromorphic function on $\mathscr{H}$ satisfying the condition (1.1), that is, $f_2|M = f_2 + q_{2,M}$, where $M \in G'(\sqrt{2})$, and $q_{2,M}$ is a rational function. Further, f_2 is meromorphic in the local uniformizing parameter at each cusp of a fundamental region for $G'(\sqrt{2})$. Since $G'(\sqrt{2})$ is generated by a_2, b_2, c_2 given in (1.16) the condition (1.1) is equivalent to

$$(1.17) \quad f_2|a_2 = f_2 + q_{2,a_2}, \quad f_2|b_2 = f_2 + q_{2,b_2}, \quad f_2|c_2 = f_2 + q_{2,c_2}.$$

Since $G'(\sqrt{2})$ is a free group, there is no element of finite order.

Now, consider the full group $G(\sqrt{2})$. We know that

$$G(\sqrt{2}) = \left(\bigcup_{j=0}^{3} G'(\sqrt{2})S_4^j \right) \cup \left(\bigcup_{j=0}^{3} G'(\sqrt{2})TS_4^j \right).$$

Let us define the function

$$\tilde{f}_2 = \sum_{j=0}^{3} \left(f_2|S_4^j + f_2|TS_4^j \right),$$

where f_2 is the above automorphic integral with rational period functions on $G'(\sqrt{2})$.

First note that the function $\tilde{f}_2$ is meromorphic on $\mathscr{H}$. At the cusps of a fundamental region for the Hecke group $G(\sqrt{2})$ the behavior of $\tilde{f}_2$ is determined by the behavior of f_2 at the cusps of a fundamental region for the

corresponding commutator subgroup $G'(\sqrt{2})$. And we have
(i)

$$\tilde{f}_2|S_4 = \sum_{j=0}^{3}\left(f_2|S_4^j + f_2|TS_4^j\right) + q_{2,S^4} + q_{2,TS_4^4T}|T \tag{1.18}$$

since S_4^4, $TS_4^4T \in G'(\sqrt{2})$.
(ii)

$$\begin{aligned}\tilde{f}_2|T = {} & f_2|T + f_2|S_4TS_4^{-1}T|TS_4 + f_2|S_4^2TS_4^{-2}T|TS_4^2 \\ & + f_2|S_4^3TS_4^{-3}T|TS_4^3 + f_2 + f_2|TS_4TS_4^{-1}|S_4 + f_2|TS_4^2TS_4^{-2}|S_4^2 \\ & + f_2|TS_4^3TS_4^{-3}|S_4^3.\end{aligned}$$

Since, by (1.16), we know that

$$S_4TS_4^{-1}T = a_2, \quad S_4^2TS_4^{-2}T = a_2c_2^{-1}a_2,$$
$$S_4^3TS_4^{-3}T = S_4^4c_2^{-1}a_2, \quad S_4^4 = a_2c_2^{-1}b_2,$$

the above implies that

$$\begin{aligned}\tilde{f}_2|T = {} & \tilde{f}_2 + q_{2,a_2}|TS_4 + q_{2,a_2c_2^{-1}a_2}|TS_4^2 + q_{2,S_4^4c_2^{-1}a_2}|TS_4^3 \\ & + q_{2,a_2^{-1}}|S_4 + q_{2,a_2^{-1}c_2a_2^{-1}}|S_4^2 + q_{2,a_2^{-1}c_2S_4^{-4}}|S_4^3.\end{aligned} \tag{1.19}$$

If we put $q_{2,a_2c_2^{-1}b_2} + q_{2,a_2^{-1}c_2b_2^{-1}}|T = \tilde{q}_{2,S_4}$ in (1.18) and

$$\begin{aligned} & q_{2,a_2}|TS_4 + q_{2,a_2c_2^{-1}a_2}|TS_4^2 + q_{2,S_4^4c_2^{-1}a_2}|TS_4^3 + q_{2,a_2^{-1}}|S_4 \\ & + q_{2,a_2^{-1}c_2a_2^{-1}}|S_4^2 + q_{2,a_2^{-1}c_2S_4^{-4}}|S_4^3 = \tilde{q}_{2,T},\end{aligned}$$

then (1.18) and (1.19) become

$$\tilde{f}_2|S_4 = \tilde{f}_2 + \tilde{q}_{2,S_4} = \tilde{f}_2 + q_{2,a_2c_2^{-1}b_2} + q_{2,a_2^{-1}c_2b_2^{-1}}|T, \tag{1.20}$$

(1.21)
$$\begin{aligned}\tilde{f}_2|T = \tilde{f}_2 + \tilde{q}_{2,T} = {} & \tilde{f}_2 + q_{2,a_2}|TS_4 + q_{2,a_2c_2^{-1}a_2}|TS_4^2 + q_{2,S_4^4c_2^{-1}a_2}|TS_4^3 \\ & + q_{2,a_2^{-1}}|S_4 + q_{2,a_2^{-1}c_2a_2^{-1}}|S_4^2 + q_{2,a_2^{-1}c_2S_4^{-4}}|S_4^3.\end{aligned}$$

Both $\tilde{q}_{2,S_4}$ and $\tilde{q}_{2,T}$ are rational functions. Since any rational function $\tilde{f}_2$ trivially satisfies (1.20), we impose the further restriction that $\tilde{f}_2$ be a periodic function, that is, $\tilde{f}_2(z + \sqrt{2}) = \tilde{f}_2(z)$. This is equivalent to

$$\tilde{q}_{2,S_4} = q_{2,a_2c_2^{-1}b_2} + q_{2,a_2^{-1}c_2b_2^{-1}}|T = 0$$

in (1.20). Then $\tilde{f}_2$ is an automorphic integral with a rational period function on $G(\sqrt{2})$, for $\tilde{q}_{2,T}$ satisfies the two relations in (1.6) if $\tilde{q}_{2,S_4} = 0$.

Applying the consistency condition

$$q_{2,M_1M_2} = q_{2,M_1}|M_2 + q_{2,M_2} \quad \text{for } M_1, M_2 \in G'(\sqrt{2}),$$

we see that, since $S_4^4 = a_2c_2^{-1}b_2$ and $TS_4^4T = a_2^{-1}c_2b_2^{-1}$, $\tilde{q}_{2,S_4} = 0$ is equivalent to

$$(1.22) \qquad q_{2,a_2}|TS_4TS_4^3 - q_{2,a_2}|TS_4^4 + q_{2,b_2} - q_{2,b_2}|S_4^{-1}TS_4 + q_{2,c_2}|S_4^{-1}TS_4 - q_{2,c_2}|TS_4TS_4^3 = 0.$$

Also, with (1.22), $\tilde{q}_{2,T}$ in (1.21) becomes

$$(1.23) \quad \tilde{q}_{2,T} = q_{2,a_2}|TS_4 - q_{2,a_2}|TS_4T + q_{2,a_2}|(TS_4)^2T - q_{2,a_2}|(TS_4)^2 + q_{2,a_2}|TS_4^2 - q_{2,a_2}|TS_4^2T + q_{2,a_2}|TS_4^3 - q_{2,a_2}|(TS_4)^2S_4 + q_{2,c_2}|(TS_4)^2 - q_{2,c_2}|(TS_4)^2T - q_{2,c_2}|S_4^{-1}T + q_{2,c_2}|(TS_4)^2S_4.$$

On the basis of the above construction, we may state the following theorem.

THEOREM 5. *If we have any three rational functions* $q_{2,a_2}, q_{2,b_2}, q_{2,c_2}$ *such that*

$$q_{2,a_2}|TS_4TS_4^3 - q_{2,a_2}|TS_4^4 + q_{2,b_2} - q_{2,b_2}|S_4^{-1}TS_4 + q_{2,c_2}|S_4^{-1}TS_4 - q_{2,c_2}|TS_4TS_4^3 = 0$$

in (1.22), *then* $\tilde{q}_{2,T}$ *in* (1.23) *is a rational period function for* $G(\sqrt{2})$.

Proof. $\tilde{q}_{2,T}$ satisfies the two relations in (1.6) from the construction.

(c) The case $\lambda = \sqrt{3}$. Construction of automorphic integrals with rational period functions on $G(\sqrt{3})$ from those on $G'(\sqrt{3})$.

Note. By Theorem 3, $G'(\sqrt{3})$ is generated by $\langle a_3, b_3, c_3, d_3, e_3 \rangle$, where

$$(1.24) \qquad a_3 = S_6TS_6^{-1}T, \quad b_3 = TS_6^{-1}TS_6, \quad c_3 = (S_6T)^2(S_6^{-1}T)^2,$$
$$d_3 = (TS_6^{-1})^2(TS_6)^2, \quad e_3 = (S_6T)^3(S_6^{-1}T)^3 \quad \text{for } S_6 = \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix}.$$

Construction. Let f_3 be an automorphic integral of weight $2k$, $k \in Z$, with rational period functions on $G'(\sqrt{3})$; f_3 is a meromorphic function in $\mathscr{H}$

satisfying the condition (1.1), that is, $f_3|M = f_3 + q_{3,M}$, where $M \in G'(\sqrt{3})$, and $q_{3,M}$ is a rational function. Further f_3 is meromorphic in the local uniformizing parameter at each cusp of a fundamental region for $G'(\sqrt{3})$. Since $G'(\sqrt{3})$ is generated by a_3, b_3, c_3, d_3, e_3 defined in (1.24), the condition (1.1) is equivalent to

$$\begin{aligned}(1.25)\qquad & f_3|a_3 = f_3 + q_{3,a_3}, \quad f_3|b_3 = f_3 + q_{3,b_3},\\ & f_3|c_3 = f_3 + q_{3,c_3}, \quad f_3|d_3 = f_3 + q_{3,d_3},\\ & f_3|e_3 = f_3 + q_{3,e_3}.\end{aligned}$$

Since $G'(\sqrt{3})$ is a free group, there is no element of finite order.

Now, consider the full group $G(\sqrt{3})$. We know that

$$G(\sqrt{3}) = \left(\bigcup_{j=0}^{5} G'(\sqrt{3})S_6^j\right) \cup \left(\bigcup_{j=0}^{5} G'(\sqrt{3})TS_6^j\right) \quad \text{for } S_6 = \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix}.$$

Let us define the following function $\tilde{f_3}$ such that

$$\tilde{f_3} = \sum_{j=0}^{5} \left(f_3|S_6^j + f_3|TS_6^j\right),$$

where f_3 is an automorphic integral with rational period functions on $G'(\sqrt{3})$. First note that the function is meromorphic in $\mathscr{H}$. At the cusps of a fundamental region for the Hecke group $G(\sqrt{3})$ the behavior of $\tilde{f_3}$ is determined by the behavior of f_3 at the cusps of a fundamental region for the corresponding commutator subgroup $G'(\sqrt{3})$. Furthermore, we have

(i)

$$(1.26)\qquad \tilde{f_3}|S_6 = \sum_{j=0}^{5} \left(f_3|S_6^j + f_3|TS_6^j\right) + q_{3,S_6^6} + q_{3,TS_6^6T}|T$$

since $S_6^6, TS_6^6T \in G'(\sqrt{3})$.

(ii)

$$\tilde{f_3}|T = f_3|T + \sum_{j=1}^{5} f_3|S_6^jTS_6^{-j}T|TS_6^j + f_3 + \sum_{j=1}^{5} f_3|TS_6^jTS_6^{-j}|S_6^j.$$

Since, by (1.24), we know that

$$\begin{gathered} S_6TS_6^{-1}T = a_3, \quad S_6^2TS_6^{-2}T = a_3c_3^{-1}a_3, \\ S_6^3TS_6^{-3}T = a_3c_3^{-1}e_3c_3^{-1}a_3, \quad S_6^4TS_6^{-4}T = S_6^6b_3^{-1}e_3c_3^{-1}a_3, \\ S_6^5TS_6^{-5}T = S_6^6b_3^{-1}TS_6^{-6}T, \end{gathered}$$

the above implies that

$$(1.27)\quad \tilde{f}_3|T = f_3 + q_{3,a_3}|TS_6 + q_{3,a_3c_3^{-1}a_3}|TS_6^2 + q_{3,a_3c_3^{-1}e_3c_3^{-1}a_3}|TS_6^3 + q_{3,S_6^6b_3^{-1}e_3c_3^{-1}a_3}|TS_6^4 + q_{3,S_6^6b_3^{-1}TS_6^{-6}T}|TS_6^5 + q_{3,a_3^{-1}}|S_6 + q_{3,a_3^{-1}c_3a_3^{-1}}|S_6^2 + q_{3,a_3^{-1}c_3e_3^{-1}c_3a_3^{-1}}|S_6^3 + q_{3,a_3^{-1}c_3e_3^{-1}b_3S_6^{-6}}|S_6^4 + q_{3,TS_6^6Tb_3S_6^{-6}}|S_6^5.$$

If we let

$$q_{3,S_6^6} + q_{3,TS_6^6T}|T \equiv \tilde{q}_3$$

and

$$\tilde{q}_{3,T} = q_{3,a_3}|TS_6 + q_{3,a_3c_3^{-1}a_3}|TS_6^2 + q_{3,a_3c_3^{-1}e_3c_3^{-1}a_3}|TS_6^3 + q_{3,S_6^6b_3^{-1}e_3c_3^{-1}a_3}|TS_6^4 + q_{3,S_6^6b_3^{-1}TS_6^{-6}T}|TS_6^5 + q_{3,a_3^{-1}}|S_6 + q_{3,a_3^{-1}c_3a_3^{-1}}|S_6^2 + q_{3,a_3^{-1}c_3e_3^{-1}c_3a_3^{-1}}|S_6^3 + q_{3,a_3^{-1}c_3e_3^{-1}b_3S_6^{-6}}|S_6^4 + q_{3,TS_6^6Tb_3S_6^{-6}}|S_6^5,$$

then (1.26) and (1.27) become

$$(1.28)\quad \tilde{f}_3|S_6 = \tilde{f}_3 + \tilde{q}_{3,S_6} = \tilde{f}_3 + q_{3,S_6^6} + q_{3,TS_6^6T}|T = \tilde{f}_3 + q_{3,a_3c_3^{-1}e_3d_3^{-1}b_3} + q_{3,a_3^{-1}c_3e_3^{-1}d_3b_3^{-1}}|T,$$

and

$$(1.29)\quad \tilde{f}_3|T = \tilde{f}_3 + \tilde{q}_{3,T} = f_3 + q_{3,a_3}|TS_6 + q_{3,a_3c_3^{-1}a_3}|TS_6^2 + q_{3,a_3c_3^{-1}e_3c_3^{-1}a_3}|TS_6^3 + q_{3,S_6^6b_3^{-1}e_3c_3^{-1}a_3}|TS_6^4 + q_{3,S_6^6b_3^{-1}TS_6^{-6}T}|TS_6^5 + q_{3,a_3^{-1}}|S_6 + q_{3,a_3^{-1}c_3a_3^{-1}}|S_6^2 + q_{3,a_3^{-1}c_3e_3^{-1}c_3a_3^{-1}}|S_6^3 + q_{3,a_3^{-1}c_3e_3^{-1}b_3S_6^{-6}}|S_6^4 + q_{3,TS_6^6Tb_3S_6^{-6}}|S_6^5.$$

Since any rational function $\tilde{f}_3$ trivially satisfies (1.28), we impose the further restriction that $\tilde{f}_3$ be a period function, that is $\tilde{f}_3(z+\sqrt{3}) = \tilde{f}_3(z)$. This is equivalent to $\tilde{q}_{3,S_6} \equiv 0$ in (1.28). Then $\tilde{f}_3$ is an automorphic integral with rational period function on $G(\sqrt{3})$. For $\tilde{q}_{3,T}$ satisfies the two relations in (1.6) if $\tilde{q}_{3,S_6} \equiv 0$.

Applying the consistency condition on $q_{3,M_1M_2} = q_{3,M_1}|M_2 + q_{3,M_2}$ for $M_1, M_2 \in G'(\sqrt{3})$, we see that (with $S_6^6 = a_3c_3^{-1}e_3d_3^{-1}b_3$, $TS_6^6T = a_3^{-1}c_3e_3^{-1}d_3b_3^{-1}$) $\tilde{q}_{3,S_6} \equiv 0$ is equivalent to

$$(1.30)\quad \tilde{q}_{3,S_6} = q_{3,a_3}|TS_6TS_6^5 - q_{3,a_3}|TS_6^6 + q_{3,b_3} - q_{3,b_3}|S_6^{-1}TS_6 + q_{3,c_3}|(TS_6)^3S_6^3 - q_{3,c_3}|TS_6TS_6^5 + q_{3,d_3}|S_6^{-1}TS_6 - q_{3,d_3}|\left(S_6^{-1}T\right)^2S_6^2 + q_{3,e_3}|\left(S_6^{-1}T\right)^2S_6^2 - q_{3,e_3}|(TS_6)^3S_6^3 = 0.$$

Furthermore, $\tilde{q}_{3,T}$ in (1.29) becomes

(1.31)

$$\begin{aligned}\tilde{q}_{3,T} = {} & q_{3,a_3}|TS_6 - q_{3,a_3}|TS_6T + q_{3,a_3}|(TS_6)^2T - q_{3,a_3}|(TS_6)^2\\ & + q_{3,a_3}|TS_6^2 - q_{3,a_3}|TS_6^2T + q_{3,a_3}|TS_6^3 - q_{3,a_3}|TS_6^3T\\ & + q_{3,a_3}|(TS_6)^2S_6T - q_{3,a_3}|(TS_6)^2S_6\\ & + q_{3,b_3}|S_6^{-1} - q_{3,b_3}|S_6^{-1}T - q_{3,b_3}|S_6^{-2}T + q_{3,b_3}|(S_6^{-1}T)^2\\ & + q_{3,b_3}|S_6^{-2} - q_{3,b_3}|S_6^{-1}TS_6^{-1}\\ & + q_{3,c_3}|(TS_6)^2 - q_{3,c_3}|(TS_6)^2T + q_{3,c_3}|(TS_6)^3T - q_{3,c_3}|(TS_6)^3\\ & + q_{3,c_3}|(TS_6)^2S_6 - q_{3,c_3}|(TS_6)^2S_6T\\ & + q_{3,d_3}|S_6^{-1}TS_6^{-1} - q_{3,d_3}|S_6^{-1}TS_6^{-1}T\\ & + q_{3,e_3}|(TS_6)^3 - q_{3,e_3}|(TS_6)^3T.\end{aligned}$$

On the basis of the above construction, we may state the following theorem.

THEOREM 6. *If we have any rational period functions* q_{3,a_3}, q_{3,b_3}, q_{3,c_3}, q_{3,d_3}, q_{3e_3} *satisfying condition* (1.30), *say,*

$$\begin{aligned} & q_{3,a_3}|TS_6TS_6^5 - q_{3,a_3}|TS_6^6 + q_{3,b_3} - q_{3,b_3}|S_6^{-1}TS_6\\ & \quad + q_{3,c_3}|(TS_6)^3S_6^3 - q_{3,c_3}|TS_6TS_6^5 + q_{3,d_3}|S_6^{-1}TS_6\\ & \quad - q_{3,d_3}|(S_6^{-1}T)^2S_6^2 + q_{3,e_3}|(S_6^{-1}T)^2S_6^2\\ & \quad - q_{3,e_3}|(TS_6)^3S_6^3 = 0,\end{aligned}$$

then $\tilde{q}_{3,T}$ *in* (1.31) *is a rational period function on* $G(\sqrt{3})$.

Proof. $\tilde{q}_{3,T}$ in (1.31) satisfies the two relations in (1.6) from the constructions.

We have found a relation between rational period functions of automorphic integrals on $G(\lambda_n)$ and those on $G'(\lambda_n)$, for $n = 3, 4, 6$. Now, by applying an appropriate operator from $G(\lambda_n)$ ($n = 4, 6$) to $\Gamma(1)$, we can get more examples of rational period functions on $\Gamma(1)$. We note that the pairwise commensurability of the Hecke groups $G(\lambda_n)$ ($n = 3, 4, 6$) permits the construction of modular integrals with rational period functions for $G(\sqrt{2})$ and $G(\sqrt{3})$. This is demonstrated by the following lemma which uses a construction introduced by Bogo and Kuyk [1].

LEMMA 7. *Let F be an automorphic integral for $G(\lambda)$ where $\lambda = \sqrt{2}$ or $\sqrt{3}$ of weight $2k$ with generating period function $q_{\lambda, T} = q_\lambda$. Then*

$$F_1(z) = F_\lambda(\lambda z) + \lambda^{-2k} \sum_{t=0}^{\lambda^2 - 1} F_\lambda\left(\frac{z+t}{\lambda}\right)$$

is a modular integral with generating period function $q_{1,T} = q_1$ where

$$q_1 = q_\lambda(\lambda z) + \lambda^{-2k} q_\lambda(z/\lambda) + \lambda^{-2k} q_\lambda\left(\frac{z-1}{\lambda}\right) + (1-z)^{-2k} q_\lambda\left(\frac{\lambda z}{1-z}\right) \quad \text{if } \lambda = \sqrt{2}$$

and

$$q_1 = q_\lambda(\lambda z) + \lambda^{-2k} q_\lambda(z/\lambda) + \lambda^{-2k} q_\lambda\left(\frac{z-1}{\lambda}\right) + \lambda^{-2k} q_\lambda\left(\frac{z+1}{\lambda}\right)$$
$$+ (z+1)^{-2k} q_\lambda\left(\frac{\lambda z}{z+1}\right) + (1-z)^{-2k} q_\lambda\left(\frac{\lambda z}{1-z}\right) \quad \text{if } \lambda = \sqrt{3}.$$

Remark. A. Parson and K. Rosen [18] used this lemma to get the examples of rational period functions with poles in $Q(\sqrt{3})$, $Q(\sqrt{21})$.

Proof of Lemma 7. See [18].

4. An application of Theorem 4: Examples of quadratic fields containing poles of rational period functions (pole-matching method)

Let us go back to Theorem 4. If we have two rational functions $q_{a_1} = q_a$, $q_{b_1} = q_b$ such that

$$q_a - q_a|M = q_b - q_b|N \quad \text{where } M = TS^{-1}TS,\ N = S^{-4}TS,$$

then

$$q_T = q_a - q_a|T + q_a|S^{-1} - q_a|S^{-1}T + q_b|S^{-2}$$
$$- q_b|S^{-2}T + q_b|S^{-3} - q_b|S^{-3}T$$

satisfies the relations in (1.6): Now, if q_a has a pole at the point α, then $q_a|M$ has a pole at $M^{-1}\alpha$. On the other hand, if q_b has the pole β, $q_b|M$ has a pole $N^{-1}\beta$. In order to satisfy the relation,

$$q_a - q_a|M = q_b - q_b|N,$$

these poles should be matched. By applying this idea we get the following theorem.

THEOREM 8. *Given a quadratic field*

$$K = Q\left(\sqrt{\frac{F_{2m+1}}{F_{2m-1}}}\right),$$

where F_n is the Fibonacci sequence, $F_0 = 0$, F_1, $F_2 = 1$, $F_3 = 2, \ldots$, then there exists a nontrivial rational period function q_T of modular integral with weight $2k$, k odd with poles in K. To prove this theorem, we need the following lemmata.

LEMMA 9 (James and Knopp [10], [11]). *Let $M \in SL(2, R)$ be hyperbolic. Then a nonconstant meromorphic function $r(z)$ on the complex plane satisfies $r|_{-2k}M = r|M = r$ if and only if*

$$r(z) = A(z - \alpha_1)^{-k}(z - \alpha_2)^{-k}$$

where $A \in C$ and α_1, α_2 are the real fixed points of M or

$$r(z) = A(z - \alpha_1)^{-k}$$

where $A \in C$ and α_1, ∞ are the fixed points of M.

Remark. Let us consider the relation (1.14):

$$q_a - q_a|M = q_b - q_b|N \text{ for } q_{a_1} = q_a,\ q_{b_1} = q_b.$$

In particular, suppose q_a has only one pole α, q_b has only one pole β, and poles match in the following way:

$$\begin{array}{cc} \alpha & \beta \\ \downarrow & \downarrow \\ M^{-1}\alpha & N^{-1}\beta \end{array} \qquad \text{for } M = TS^{-1}TS,\ N = (S^{-4}TS).$$

Then claim that the q_T in (1.15) is a constant multiple of the example in Theorem 1 [13].

For, we have the relation $q_a - q_a|M = q_b - q_b|N$ and, by the pole matching, $q_a = q_a|M$ and $q_b = q_b|N$. Lemma 9 implies that

$$q_a = c_1(z - \alpha)^{-k}(z - \alpha')^{-k},$$
$$q_b = c_2(z - \beta)^{-k}(z - \beta')^{-k},$$

where $c_1, c_2 \in C$, α, α' are the fixed points and β, β' are the fixed points of N. We get the rational function q_T from q_a, q_b by (1.15). A calculation shows that q_T is the same as the rational function in Theorem 1 [13].

LEMMA 10. *Let*

$$M = TS^{-1}TS = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad N = S^{-4}TS = \begin{pmatrix} 4 & 5 \\ -1 & -1 \end{pmatrix};$$

then

$$(1) \qquad M^m = \begin{pmatrix} F_{2m-1} & F_{2m} \\ F_{2m} & F_{2m+1} \end{pmatrix}$$

$$(2) \quad (M^{-1})^{m-1}(N^{-1})^m(M^{-1}) = \begin{pmatrix} F_{2m+1}F_{2m} + F_{2m-1}^2 & -2F_{2m+1}F_{2m} \\ -2F_{2m-1}F_{2m} & F_{2m+1}F_{2m} + F_{2m-1}^2 \end{pmatrix}$$

where F_i *is the Fibonacci sequence,* $F_0 = 0$, $F_1 = 1$, $F_2 = 1, \ldots$. *Thus the fixed points of* $(M^{-1})^{m-1}(N^{-1})^m(M^{-1})$ *are*

$$\left(\pm\sqrt{\frac{F_{2m+1}}{F_{2m-1}}}\right), \quad m \in Z^+.$$

Proof. Since it can be proved by induction on m, we omit it.

LEMMA 11. *For* $m \in Z^+$,

$$\text{(a)} \qquad F_{2i-1}^2F_{2m+1} - F_{2i}^2F_{2m-1} > F_{2m-1} \quad \textit{for } 1 \le i \le m-1,$$
$$F_{2i+1}^2F_{2m+1} - F_{2i+2}^2F_{2m-1} > F_{2m-1} \quad \textit{for } 1 \le i \le m-1.$$
$$\text{(b)} \qquad F_{2i+3}^2F_{2m-1} - F_{2i+2}^2F_{2m+1} > F_{2m-1} \quad \textit{for } 0 \le i \le m-1,$$
$$F_{2i+1}^2F_{2m-1} - F_{2i}^2F_{2m+1} > F_{2m-1} \quad \textit{for } 1 \le i \le m-1.$$

Proof. (a). This becomes

$$F_{2i+1}^2F_{2m+1} - F_{2i+2}^2F_{2m-1} > F_{2m-1} \quad \text{for } 1 \le i \le m-1$$

because $F_1^2F_{2m+1} - F_2^2F_{2m-1} = F_{2m} > F_{2m-1}$ for $m > 1$. Now, since

$$\begin{aligned} &F_{2m+1}F_{2i+1}^2 - F_{2m-1}F_{2i+2}^2 - F_{2m-1} \\ &\quad = F_{2m+1}kF_{2i+1}^2 - F_{2m-1}(F_{2i+3}F_{2i+1} - 1) - F_{2m-1} \\ &\quad = [F_{2m+1}F_{2i+1} - F_{2m-1}F_{2i+3}]F_{2i+1} \end{aligned}$$

it is enough to show that

$$F_{2m+1}F_{2i+1} - F_{2m-1}F_{2i+3} > 0 \quad \text{for } 1 \le i \le m-1.$$

But

$$
\begin{aligned}
F_{2m+1}&F_{2i+1} - F_{2m-1}F_{2i+3} \\
&= [F_{2m} + F_{2m-1}]F_{2i+1} - F_{2m-1}[F_{2i+2} + F_{2i+1}] \\
&= F_{2m}F_{2i+1} - F_{2m-1}F_{2i+2} \\
&= (F_{2m-1} + F_{2m-2})F_{2i+1} - F_{2m-1}(F_{2i+1} + F_{2i}) \\
&= F_{2m-2}F_{2i+1} - F_{2m-1}F_{2i} \\
&= F_{2m-2}(F_{2i} + F_{2i-1}) - (F_{2m-2} + F_{2m-3})F_{2i} \\
&= F_{2m-2}F_{2i-1} - F_{2m-3}F_{2i} \\
&= (F_{2m-3} + F_{2m-4})F_{2i-1} - F_{2m-3}(F_{2i-1} + F_{2i-2}) \\
&= F_{2m-4}F_{2i-1} - F_{2m-3}F_{2i-2} \\
&\ \vdots \\
&= F_{2m-2i}F_{2i-2i+3} - F_{2m-2i+1}F_{2i-2i+2} \\
&= F_{2m-2i}F_3 - (F_{2m-2i} + F_{2m-2i-1})F_2 \\
&= F_{2m-2i} - F_{2m-2i-1} > 0 \quad \text{for } 1 \le i \le m - 1.
\end{aligned}
$$

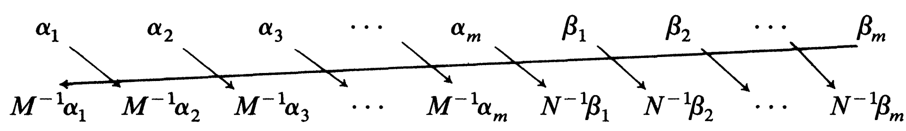
(b) Because it is similar to (a), we omit the proof. Now, we return to the proof of Theorem 8.

Proof of Theorem 8. The relation (1.14),

$$q_a - q_a|M = q_b - q_b|N,$$

suggests the following: If q_a has poles at $\{\alpha_i\}$, $i = 1, \ldots, m$, then $q_a|M$ has poles at $\{M^{-1}\alpha_i\}$, $i = 1, \ldots, m$. And if q_b has poles at $\{\beta_j\}$, $j = 1, \ldots, n$, then $q_b|N$ has poles at $\{N^{-1}\beta_j\}$, $j = 1, \ldots, n$. Because of the relation (1.14) they should match each other. In particular, we consider the following pole matching scheme:

(1.32)



where an arrow indicates that they are the same. Here the $\{\alpha_i\}$ are the poles of q_a and the $\{\beta_j\}$ are the poles of q_b. In particular, let

$$q_a = r_1(z) + r_2(z) + \cdots + r_m(z) \text{ and } q_b = u_1(z) + u_2(z) + \cdots + u_m(z)$$

where $r_i(z), u_i(z), 1 \le i \le m$, are rational functions that have the form $A_i(z - l_i)^{-k}(z - l_i')^{-k}$, for some $l_i \in Q(\sqrt{N})$, l_i' is the algebraic conjugate of

l_i in $Q(\sqrt{N})$, and $A_i \in C$ is a constant. Since q_a has poles at $\{\alpha_i\}$, $1 \le i \le m$, we choose

$$r_i(z) = A_i(z-\alpha_i)^{-k}(z-\alpha_j')^{-k} \quad \text{for } A_i \in C.$$

Also, since q_b has poles at β_j, $1 \le j \le m$, we choose

$$u_j(z) = B_j(z-\beta_j)^{-k}(z-\beta_j')^{-k} \quad \text{for } B_j \in C.$$

Then the relation (1.14),

$$q_a - q_a|M = q_b - q_b|N,$$

becomes

$$\begin{aligned} &r_1 + r_2 + \cdots + r_m - r_1|M - r_2|M - \cdots - r_m|M \\ &\quad = u_1 + u_2 + \cdots + u_m - u_1|N - u_2|N = \cdots - u_m|N \end{aligned}$$

for

$$q_a = \sum_1^m r_i, \quad q_b = \sum_1^m u_i.$$

By the above pole-matching scheme (1.32), we can assume that

$$\begin{aligned} r_1 &= r_2|M,\ r_2 = r_3|M, \ldots, r_{m-1} = r_m|M, \\ r_m &= -u_1|N,\ u_1 = u_2|N, \ldots, u_{m-1}|N, \\ u_m &= -r_1|M. \end{aligned}$$

This implies that

$$r_1 = r_1|(M^{-1})^{m-1}(N^{-1})^m(M^{-1}). \tag{1.33}$$

If we apply Lemma 9 and Lemma 10 (2), then we get the explicit formula $r_1(z)$:

$$r_1(z) = A_1\left(z - \sqrt{\frac{F_{2m+1}}{F_{2m-1}}}\right)^{-k}\left(z + \sqrt{\frac{F_{2m+1}}{F_{2m-1}}}\right)^{-k} \tag{1.34}$$

without loss of generality put $A_1 = 1$. Also we have the equation (1.14),

$$q_a - q_a|M = q_b - q_b|N.$$

By the relation (1.33), q_T in (1.15) is

(1.35)

$$\begin{aligned} q_T = & \sum_{i=0}^{m-1} r_1 | (M^{-1})^i + \sum_{i=0}^{m-1} r_1 | (M^{-1})^i S^{-1} \\ & - \sum_{i=0}^{m-1} r_1 | (M^{-1})^i T - \sum_{i=0}^{m-1} r_1 | (M^{-1})^i S^{-1} T \\ & - \sum_{i=1}^{m} r_1 | (M^{-1})^{m-1} (N^{-1})^i S^{-2} \\ & - \sum_{i=1}^{m} r_1 | (M^{-1})^{m-1} (N^{-1})^i S^{-3} + \sum_{i=1}^{m} r_1 | (M^{-1})^{m-1} (N^{-1})^i S^{-2} T \\ & + \sum_{i=1}^{m} r_1 | (M^{-1})^{m-1} (N^{-1})^i S^{-3} T. \end{aligned}$$

(This q_T satisfies the two relations in (1.6).) It remains to show that q_T in (1.35) is not zero.

First, let us simplify (1.35). We know that

(1.36)

$$\begin{aligned} (N^{-1})^i &= \underbrace{(\underline{S^{-1}TS^4}S^{-1}TS^4 \dots S^{-1}TS^4)}_{i \text{ factors}} = \underbrace{[S^{-2}(\underline{STS^2}\,STS^2 \dots STS^2)S^2]}_{i \text{ factors}} \\ &= S^{-2}(M)^i S^2 \quad (\text{since } M = STS^2 = TS^{-1}TS,\ N = S^{-4}TS). \end{aligned}$$

By (1.33) and (1.36), we know that

$$\begin{aligned} & r_1 | (M^{-1})^{m-1} (N^{-1})^i S^{-2} \\ & \quad = r_1 | MN^m (N^{-1})^i S^{-2} \\ & \quad = r_1 | ST (M^{-1})^{m-1} = r_1 | ST(\underline{S^{-1}TSTS^{-1}}TST \dots S^{-1}TST) \\ & \quad = r_1 | S(M)^{m-i} T. \end{aligned}$$

Therefore

$$\sum_{i=1}^{m} r_1 | (M^{-1})^{m-1} (N^{-1})^i S^{-2} = \sum_{i=1}^{m} r_1 | S(M)^{m-i} T = \sum_{j=0}^{m-1} r_1 | SM^j T.$$

So (1.35) becomes

$$(1.37)\qquad \begin{aligned} q_T = r_1 &+ \sum_{i=1}^{m-1} r_1|(M^{-1})^i - \sum_{i=0}^{m-1} r_1|(M^{-1})^i T \\ &+ \sum_{i=0}^{m-1} r_1|(M^{-1})^i S^{-1} - \sum_{i=0}^{m-1} r_1|(M^{-1})^i S^{-1}T \\ &+ \sum_{i=0}^{m-1} r_1|SM^i - \sum_{i=0}^{m-1} r_1|SM^iT \\ &+ \sum_{i=0}^{m-1} r_1|SM^iTS^{-1}T - \sum_{i=0}^{m-1} r_1|SM^iTS^{-1}. \end{aligned}$$

To prove that q_T is not identically zero, it is enough to show through r_1 is never cancelled in (1.37).

Since

$$r_1(z) = \left(z - \sqrt{\frac{F_{2m+1}}{F_{2_m-1}}}\right)^{-k}\left(z + \sqrt{\frac{F_{2m+1}}{F_{2m-1}}}\right)^{-k}$$

by (1.34), for

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$$

we consider Table (1.38). Now, I claim that r_1 in (1.37) is never cancelled. Since, in (1.34),

$$r_1(z) = \left(z - \sqrt{\frac{F_{2m+1}}{F_{2m-1}}}\right)^{-k}\left(z + \sqrt{\frac{F_{2m+1}}{F_{2m-1}}}\right)^{-k},$$

the coefficient of $r_1(z)$ equals 1. The worst possibility is the following: For s at least one of the coefficients of the terms in (1.37),

$$\begin{aligned} &\{r_1|(M^{-1})^i, r_1|(M^{-1})^iT, r_1|T, r_1|(M^{-1})^iS^{-1}, \\ &r_1|(M^{-1})^iS^{-1}T, r_1|S(M)^i, r_1|S(M)^iT, r_1|SM^iTS^{-1}T, r_1|SM^iTS^{-1}\} \\ &\qquad\qquad \text{for some } 0 \le i \le m-1, \end{aligned}$$

is 1 or -1, and the poles are matched. Table (1.38) shows that this would imply

$$(1.39)\quad F_{2i}^2F_{2m-1} - F_{2i-1}^2F_{2m+1} = \pm F_{2m-1} \quad \text{for some } 1 \le i \le m-1,$$

Table (1.38)

$r_1 \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix}$	Coefficient $\left[a^2 - c^2\left(\frac{F_{2m+1}}{F_{2m-1}}\right)\right]$
$r_1 \mid (M^{-1})^i$ $1 \le i \le m - I$	$\frac{F_{2i+1}^2 F_{2m-1} - F_{2i}^2 F_{2m+1}}{F_{2m-1}}$
$r_1 \mid (M^{-1})^i T$ $1 \le l \le m - 1$	$\frac{F_{2i}^2 F_{2m-1} - F_{2i-1}^2 F_{2m+1}}{F_{2m-1}}$
$r_1 \mid T$	$-\frac{F_{2m+1}}{F_{2m-1}}$
$r_1 \mid (M^{-1})^i S^{-1}$ $0 \le i \le m - 1$	$\frac{F_{2i+1}^2 F_{2m-1} - F_{2i}^2 F_{2m+1}}{F_{2m-1}}$
$r_1 \mid (M^{-1})^i S^{-1} T$ $0 \le i \le m - 1$	$\frac{F_{2i+2}^2 F_{2m-1} - F_{2i+1}^2 F_{2m+1}}{F_{2m-1}}$
$r_1 \mid SM^i$ $0 \le i \le m - 1$	$\frac{F_{2i+1}^2 F_{2m-1} - F_{2i}^2 F_{2m+1}}{F_{2m-1}}$
$r_1 \mid SM^i T$ $0 \le i \le m - 1$	$\frac{F_{2m-1} F_{2i+2}^2 - F_{2m+1} F_{2i+1}^2}{F_{2m-1}}$
$r_1 \mid SM^i TS^{-1} T$ $0 \le i \le m - 1$	$\frac{F_{2i+3}^2 F_{2m-1} - F_{2i+2}^2 F_{2m+1}}{F_{2m-1}}$
$r_1 \mid SM^i TS^{-1}$ $0 \le i \le m - 1$	$\frac{F_{2m-1} F_{2i+2}^2 - F_{2i+1}^2 F_{2m+1}}{F_{2m-1}}$

or

$$F_{2i+2}^2 F_{2m-1} - F_{2i+1}^2 F_{2m+1} = \pm F_{2m-1}$$

or

$$F_{2i+3}^2 F_{2m-1} - F_{2i+2}^2 F_{2m+1} = \pm F_{2m-1}$$

or

$$F_{2i+1}^2 F_{2m-1} - F_{2i}^2 F_{2m+1} = \pm F_{2m-1} \quad \text{for } 0 \le \text{some } i \le m - 1.$$

But, by Lemma 11, we know that

$$(1.40) \qquad F_{2i+2}^2 F_{2m-1} - F_{2i+1}^2 F_{2m-1} = -F_{2m-1}$$

in the case $m = 1$ and $i = 0$, or

$$F_{2i+1}^2 F_{2m-1} - F_{2i}^2 F_{2m+1} = F_{2m-1}$$

in the case $i = 0$ and all m.

Then, from the Table (1.38), (1.40) implies that the coefficients of

$$\left\{r_1|(M^{-1})^0S^{-1}, r_1|SM^0\right\}$$

equal $(1)^{-k}$ for every m, and the coefficient of

$$\left\{r_1|(M^{-1})^0S^{-1}T, r_1|S(M)^0T, r_1|S(M)^0TS^{-1}\right\}$$

equals $(-1)^k$ for $m = 1$. The simple calculation shows that

$$\left\{r_1|S^{-1}, r_1|S, r_1|S^{-1}T, r_1|ST, r_1|STS^{-1}\right\}$$

is different from $r_1(z)$, for k *odd*, by looking at poles. Therefore, r_1 in (1.37) is never cancelled for every m, so q_T is a nontrivial function, $q_T \not\equiv 0$.
The proof is complete.

5. Some generalizations

(A) *A generalization of Theorem 4.*
We generalize Theorem 4 to all of the Hecke groups. If q_a, q_b satisfy (1.14),

$$q_a - q_a|TS^{-1}TS = q_b - q_b|S^{-3}TSTS^2,$$

then

$$\tilde{q}_T = q_a - q_a|T + q_a|S^{-1} - q_a|S^{-1}T + q_b|S^{-2} - q_b|S^{-2}T + q_b|S^{-3} - q_b|S^{-3}T$$

is a rational period function of a modular integral with weight $2k$ on (1).
This can be generalized to all of the Hecke groups.

THEOREM 12. *If we have any rational functions h, g such that, for*

$$S_n = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix}, \ \lambda_n = 2\cos\frac{\pi}{n},$$

we have

(1.41) $$h - h|TS_n^{-1}TS_n = g - g|S_n^{-3}TS_nTS_n^2$$

then

(1.42) $$q_{n,T} = h - h|T + h|S_n^{-1} - h|S_n^{-1}T + g|S_n^{-2} - g|S_n^{-2}T + g|S_n^{-3} - g|S_n^{-3}T$$

is a rational period function of an automorphic integral with weight $2k$ on the Hecke group

$$G(\lambda_n), \ \lambda_n = 2\cos\frac{\pi}{n}, \quad n \geq 3.$$

Proof. According to §2, it is enough to check the two relations in (1.6). The condition $q_{n,T}|T + q_{n,T} = 0$ is obvious from the form of $q_{n,T}$ in (1.42).
On the other hand,

$$\begin{aligned}
&\sum_0^{n-1} q_{n,T}|(TS_n)^i \quad \left(\text{with } \sum_{i=0}^{n-1} = \sum_0^{n-1}\right)\\
&= \sum_0^{n-1} h|(TS_n)^i - \sum_0^{n-1} h|T(TS_n)^i\\
&\quad + \sum_0^{n-1} h|S_n^{-1}(TS_n)^i - \sum_0^{n-1} h|S_n^{-1}T(TS_n)^i\\
&\quad + \sum_0^{n-1} g|S_n^{-2}(TS_n)^i - \sum_0^{n-1} g|S_n^{-2}T(TS_n)^i\\
&\quad + \sum_0^{n-1} g|S_n^{-3}(TS_n)^i - \sum_0^{n-1} g|S_n^{-3}T(TS_n)^i\\
&\qquad \left(\text{since } \sum_0^{n-1} h|(TS_n)^i = \sum_0^{n-1} h|S_n^{-1}T(TS_n)^i \right.\\
&\qquad \left.\text{and } \sum_0^{n-1} g|S_n^{-2}(TS_n)^i = \sum_0^{n-1} g|S_n^{-3}T(TS_n)^i\right)\\
&= -\sum_0^{n-1}\left[h|T(TS_n)^i + \sum_0^{n-1} h|S_n^{-1}(TS_n)^{i+1}\right.\\
&\qquad \left. - \sum_0^{n-1} g|S_n^{-2}T(TS_n)^{i+2} + \sum_0^{n-1} g|S_n^{-3}(TS_n)^{i+3}\right]\\
&= \sum_0^{n-1}\left[-h|T(TS)^i + h|S_n^{-1}(TS_n)^{i+1}\right.\\
&\qquad \left. - g|S_n^{-2}T(TS_n)^{i+2} + g|S_n^{-3}(TS_n)^{i+3}\right] = 0.
\end{aligned}$$

This holds because (1.44) implies that

$$\begin{aligned}
&\sum_0^{n-1}\left[h|S_n^{-1}(TS_n) - h|T - g|S_n^{-1}(TS_n) + g|S_n^{-3}(TS_n)^3\right]|(TS_n)^i\\
&= \sum_0^{n-1}\left[h|S_n^{-1}(TS_n) - h|T - g|S_n^{-2}T(TS_n)^2 + g|S_n^{-3}(TS_n)^3\right]|(TS_n)^i = 0.
\end{aligned}$$

The proof is complete.

Remark. In particular, if $h|TS_n^{-1}TS_n = h$, $g|S_n^{-3}TS_nTS_n^2 = g$ then $q_{n,T}$ in (1.42) is a constant multiple of the examples in Theorem 2.4 [21], for $\lambda = \sqrt{2}$ $(n = 4)$ and $\lambda = \sqrt{3}$ $(n = 6)$.

Note. We give some explicit examples of $q_{n,T}$ that arise from (1.42).
The relation (1.41) is

$$h - h|TS_n^{-1}TS_n = g - g|S_n^{-3}TS_nTS_n^2.$$

For instance, if we choose $h = -g|S_n^{-3}TS_nTS_n^2$, and $g = -h|TS_n^{-1}TS_n$, then the relation (1.41) is satisfied. Furthermore, since

$$h = -g|S_n^{-3}TS_nTS_n^2 = h|TS_n^{-1}TS_n^{-2}TS_nTS_n^2,$$

Lemma 9 implies that

$$h = h|TS_n^{-1}TS_n^{-2}TS_nTS_n^2 = h|\begin{pmatrix} 1 + 2\lambda^2 & 4\lambda^3 \\ \lambda(2\lambda^2) & 4\lambda^4 - 2\lambda^2 + 1 \end{pmatrix}$$

if and only if

$$h = c\left(z - \frac{1 - \lambda^2 + \sqrt{\lambda^4 + 1}}{\lambda}\right)^{-k}\left(z - \frac{1 - \lambda^2 - \sqrt{\lambda^4 + 1}}{\lambda}\right)^{-k}.$$

Without loss of generality assume $c = 1$.

(a) The case $\lambda = 1$. A simple calculation shows that the q_T of (1.42) is

$$\begin{aligned} q_T = {} & 2(z - \sqrt{2})^{-k}(z + \sqrt{2})^{-k} + 2^{-k+1}\left(z - \frac{1}{\sqrt{2}}\right)^{-k}\left(z + \frac{1}{\sqrt{2}}\right)^{-k} \\ & + 2(z - 1 + \sqrt{2})^{-k}(z - 1 - \sqrt{2})^{-k} \\ & + 2(z + 1 + \sqrt{2})^{-k}(z + 1 - \sqrt{2})^{-k} \end{aligned}$$

for k odd.

Note that $q_T \equiv 0$ for k even.

(b) The case $\lambda = \sqrt{2}$. A simple calculation shows that $q_{4,T}$ in (1.42) is

$$\begin{aligned} q_{4,T} = {} & 2\left(z - \frac{-\sqrt{2} + \sqrt{10}}{2}\right)^{-k}\left(z - \frac{-\sqrt{2} - \sqrt{10}}{2}\right)^{-k} \\ & + 2^{-k+1}\left(z - \frac{-\sqrt{2} + \sqrt{10}}{4}\right)^{-k}\left(z - \frac{-\sqrt{2} - \sqrt{10}}{4}\right)^{-k} \\ & + 2\left(z - \frac{\sqrt{2} + \sqrt{10}}{2}\right)^{-k}\left(z - \frac{\sqrt{2} - \sqrt{10}}{2}\right)^{-k} \\ & + 2^{-k+1}\left(z - \frac{\sqrt{2} + \sqrt{10}}{4}\right)^{-k}\left(z - \frac{\sqrt{2} - \sqrt{10}}{4}\right)^{-k} \end{aligned}$$

for k odd.

Note that $q_{4,T} \equiv 0$ for k even.
(c) The case $\lambda = \sqrt{3}$. A simple calculation shows that $q_{6,T}$ in (1.42) is

$$\begin{aligned} q_{6,T} = &\left(z - \frac{-2\sqrt{3} + \sqrt{30}}{3}\right)^{-k}\left(z - \frac{-2\sqrt{3} - \sqrt{30}}{3}\right)^{-k} \\ &-(-2)^{-k}\left(z - \frac{-2\sqrt{3} + \sqrt{30}}{6}\right)^{-k}\left(z - \frac{-2\sqrt{3} - \sqrt{30}}{6}\right)^{-k} \\ &+\left(z - \frac{\sqrt{3} + \sqrt{30}}{3}\right)^{-k}\left(z - \frac{\sqrt{3} - \sqrt{30}}{3}\right)^{-k} \\ &-(-3)^{-k}\left(z - \frac{\sqrt{3} + \sqrt{30}}{9}\right)^{-k}\left(z - \frac{\sqrt{3} - \sqrt{30}}{9}\right)^{-k} \\ &+3^{-k}\left(z - \frac{-\sqrt{3} + \sqrt{30}}{9}\right)^{-k}\left(z - \frac{-\sqrt{3} - \sqrt{30}}{9}\right)^{-k} \\ &-(-1)^{-k}\left(z - \frac{-\sqrt{3} + \sqrt{30}}{3}\right)^{-k}\left(z - \frac{-\sqrt{3} - \sqrt{30}}{3}\right)^{-k} \\ &+2^{-k}\left(z - \frac{2\sqrt{3} + \sqrt{30}}{6}\right)^{-k}\left(z - \frac{2\sqrt{3} - \sqrt{30}}{6}\right)^{-k} \\ &-(-1)^{-k}\left(z - \frac{2\sqrt{3} + \sqrt{30}}{3}\right)^{-k}\left(z - \frac{2\sqrt{3} - \sqrt{30}}{3}\right)^{-k} \end{aligned}$$

for k even or odd.

Remark. Theorem 5 and Theorem 6 can also be generalized to all of the Hecke groups as Theorem 12.

(B) A generalization of Knopp's construction [13].
M. Knopp [13] initiated the study of rational period functions of modular integrals which differ from the period functions of Eichler integrals.
Here, we generalize M. Knopp's construction of [13].

THEOREM 13. *Let $G(\lambda_n)$, $\lambda_n = 2\cos \pi/n$ with $n \in Z$, $n \geq 3$, be the Hecke group. Let f be a rational function such that*

$$f|[S_n^l, T] = f|S_n^l T S_n^{-l} T = f,$$

where, as before

$$S_n = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix}, T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, l = 1, \ldots, \in Z^+.$$

Then if k is an odd integer, the **nontrivial** *function*

$$(1.43) \qquad q_{n,T} = \sum_{i=0}^{l-1} f|TS_n^iT - \sum_{i=0}^{l-1} f|S_n^iT - \sum_{i=1}^{l} f|TS_n^i + \sum_{i=1}^{l} f|S_n^i$$

is the generating period function of some automorphic integral of weight $2k$ for $G(\lambda_n)$.

Note. When $n = 3$ and $l = 1$, q_T in (1.43) is an example by M. Knopp [13].

Remark. Since

$$S_n^lTS_n^{-l}T = \begin{pmatrix} \lambda_n^2 l^2 + 1 & \lambda_n l \\ \lambda_n l & 1 \end{pmatrix} \quad \text{and} \quad f|S_n^lTS_n^{-l}T = f,$$

f has poles in $Q(\sqrt{\lambda_n^2 l^2 + 4}, \lambda_n)$ by Lemma 9. Furthermore, $q_{n,T}$ is obtained from f by subjecting the variable z to linear fractional transformations, so we conclude that these $\{q_{n,T}\}$ have poles in $Q(\sqrt{\lambda_n^2 l^2 + 4}, \lambda_n)$.

Proof of Theorem 13. Define $r = f|T - f$, again a rational function. Since $T^2 = I$, we have $r|T + r = 0$. Now,

$$q_{n,T} = \sum_0^{l-1} r|S_n^iT - \sum_1^{l} r|S_n^i \quad \text{with} \sum_{i=0}^{l} = \sum_0^{l},$$

and since $(TS_n)^n = I$, it follows that

$$\begin{aligned} \sum_0^{n-1} q_{n,T}|(TS_n)^i &= \sum_0^{l-1} r|S_n^{i+1}(TS_n)^{n-2} - \sum_1^{l} r|S_n^i(TS_n)^{n-1} \\ &\quad + \sum_0^{l-1} r|S_n^{i+1}(TS_n)^{n-3} - \sum_1^{l} r|S_n^i(TS_n)^{n-2} \\ &\quad + \cdots + \sum_0^{l-1} r|S_n^iT - \sum_1^{l} r|S_n^i \\ &= 0 \text{ (since the sum telescapes).} \end{aligned}$$

On the other hand,

$$q_{n,T}|T + q_{n,T} = f|T - f|TS_n^l + f|S_n^l - f|TS_n^lT + f - f + f|S_n^lT - f|T \equiv 0,$$

since $f|TS_n^l = f|S_n^lT$. So this $q_{n,T}$ satisfies the two relations in (2.6).

Now, I claim that $q_{n,T}$ is nontrivial, for k odd, i.e., $q_{n,T} \not\equiv 0$. To show this, let us consider the following. From (1.43),

$$q_{n,T} = \sum_0^{l-1} r|S_n^iT - \sum_1^l r|S_n^i \quad \text{where } r = f|T - f.$$

Since $f = f|S_n^lTS_n^{-l}T$, Lemma 9 implies that

$$r = -c\left(z - \frac{\lambda_n l + \sqrt{(\lambda_n l)^2 + 4}}{2}\right)^{-k}\left(z - \frac{\lambda_n l - \sqrt{(\lambda_n l)^2 + 4}}{2}\right)^{-k}$$

for k odd. Without loss of generality assume $c = 1$. Note that $r(z) \equiv 0$ for k even. So,

(1.44)

$$\begin{aligned} q_{n,T} &= \sum_0^{l-1} r|S_n^iT - \sum_1^l r|S_n^i = r + \sum_1^{l-1} r|S_n^iT - \sum_1^l r|S_n^i \\ &= -\sum_0^{l-1}\left(i^2\lambda_n^2 - i\lambda_n^2 l - 1\right)^{-k}\left(z - \frac{-2i\lambda_n + \lambda_n l + \sqrt{(\lambda_n l)^2 + 4}}{2\left(1 + i\lambda_n^2 l - i^2\lambda_n^2\right)}\right)^{-k} \\ &\quad \times\left(z - \frac{-2i\lambda_n + \lambda_n l - \sqrt{(\lambda_n l)^2 + 4}}{2\left(1 + i\lambda_n^2 l - i^2\lambda_n^2\right)}\right)^{-k} \\ &\quad + \sum_1^l\left(z - \frac{-2i\lambda_n + \lambda_n l + \sqrt{(\lambda_n l)^2 + 4}}{2}\right)^{-k} \\ &\quad \times\left(z - \frac{-2i\lambda_n + \lambda_n l - \sqrt{(\lambda_n l)^2 + 4}}{2}\right)^{-k} \end{aligned}$$

Now, we claim that $r(z)$ in (1.44) is never cancelled. Since the coefficients of $(-\sum_1^l r|S_n^i)$ in $q_{n,T}$ equal 1 from (1.44) the worst case is that the coefficients of $(+\sum_0^{l-1} r|S_n^iT)$, $(i\lambda_n^2 - i\lambda_n^2 l - 1)^{-k}$, equal $1^{-k} = 1$ for k odd, and the poles of $\sum_0^{l-1} r|S_n^iT$ are matched to those of $\sum_1^l r|S_n^i$.

Table A
Examples $q_{n,T}$ in Theorem 13 for odd k
(a) The case $\lambda = 1$

l	Poles of q_T
$l = 1$	$\pm\left\{\frac{1 \pm \sqrt{5}}{2}\right\}$
$l = 2$	$\pm\left\{1 \pm \sqrt{2}, \pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}\right\}$
$l = 3$	$\pm\left\{\frac{3 \pm \sqrt{13}}{2}\right\}, \pm\left\{\frac{1 \pm \sqrt{13}}{6}\right\}, \pm\left\{\frac{1 \pm \sqrt{13}}{2}\right\}$
$l = 4$	$\pm\{2 \pm \sqrt{5}\}, \pm\{1 \pm \sqrt{5}\}, \pm\left\{\frac{1 \pm \sqrt{5}}{4}\right\}, \pm\frac{1}{\sqrt{5}}, \pm\sqrt{5}$

(b) The case $\lambda = \sqrt{2}$

l	poles of $q_{n,T}$
$l = 1$	$\pm\left\{\frac{\sqrt{2} \pm \sqrt{6}}{2}\right\}$
$l = 2$	$\pm\{\sqrt{2} \pm \sqrt{3}, \pm\sqrt{3}, \pm\frac{1}{\sqrt{3}}$
$l = 3$	$\pm\left\{\frac{3\sqrt{2} \pm \sqrt{22}}{2}\right\}, \pm\left\{\frac{\sqrt{2} \pm \sqrt{22}}{10}\right\}, \pm\left\{\frac{\sqrt{2} \pm \sqrt{22}}{2}\right\}$
$l = 4$	$\pm\{2\sqrt{2} \pm 3\}, \pm\left\{\frac{\sqrt{2} \pm 3}{7}\right\}, \pm\frac{1}{3}, \pm 3,$ $\pm(\sqrt{2} \pm 3\}$

(c) The case $\lambda = \sqrt{3}$

l	Poles of $q_{n,T}$
$l = 1$	$\pm\left\{\frac{-\sqrt{3} \pm \sqrt{7}}{2}\right\}$
$l = 2$	$\pm\{\sqrt{3} \pm 4\}, \pm\frac{1}{2}, \pm 2$
$l = 3$	$\pm\left\{\frac{3\sqrt{3} \pm \sqrt{31}}{2}\right\}, \pm\left\{\frac{\sqrt{3} \pm \sqrt{31}}{7}\right\}, \pm\left\{\frac{\sqrt{3} \pm \sqrt{31}}{2}\right\}$
$l = 4$	$\pm\{2\sqrt{3} \pm \sqrt{13}\}, \pm\frac{1}{\sqrt{13}}, \pm\sqrt{13}, \pm\left\{\frac{\sqrt{3} \pm \sqrt{13}}{10}\right\}$ $\pm\{\sqrt{3} \pm \sqrt{13}\}$

Now, compare coefficients. Suppose $i^2\lambda_n^2 - i\lambda_n^2 l - 1 = 1$ for some $0 \le i \le l-1$ so that $i\lambda_n^2(i-l) = 2$, for some $0 \le i \le l-1$. This implies that $i(i-l) = 2$ and $\lambda_n^2 = 1$ or $\lambda_n^2 = 2$ and $i(i-l) = 1$. These are impossible since $0 \le i \le l-1$. Thus $q_{n,T}$ is a nontrivial rational period function of an automorphic integral with weight $2k$ (k odd).

The proof is complete.

In Table A, I gave a few explicit examples, obtained from Theorem 13, of rational period functions on $G(\lambda)$.

6. Rational period functions with poles in arbitrary real quadratic fields

1. *Existence of rational period functions for modular integrals of weight* $2k$, k *even or odd, and with poles in an arbitrary real quadratic field* $Q(\sqrt{N})$, $N \in Z^+$.

(a) Construction of the rational period function.

In this section we once again construct rational functions satisfying the two relations in (1.6); an argument involving Pell's equation shows that these rational period functions can be constructed with poles in *arbitrarily chosen* real quadratic fields $Q(\sqrt{N})$, $N \in Z^+$, and with *k even or odd.*

THEOREM 14. *Let g be a nonconstant meromorphic function such that*

$$g|_{-2k}S_n^l T = g|S_n^l T = g,$$

where

$$S_n = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $l \ge 4$. *If k is an integer* **(odd or even)**,

$$q_{n,T} = \sum_{i=1}^{l-1} g|S_n^i - \sum_{i=1}^{l-1} g|S_n^i T \tag{1.45}$$

is a **nontrivial** *rational period function of an automorphic integral of weight* $2k$ *for* $G(\lambda_n)$.

Remark. By Lemma 9,

$$g(z) = g|S_n^l T = c \cdot \left(z - \frac{\lambda_n l + \sqrt{\lambda_n^2 l^2 - 4}}{2}\right)^{-k} \times \left(z - \frac{\lambda_n l - \sqrt{\lambda_n^2 l^2 - 4}}{2}\right)^{-k}. \tag{1.46}$$

Without loss of generality assume $c = 1$. Therefore, $q_{n,1}(z)$ has poles in

$$Q\left(\lambda_n, \sqrt{(\lambda_n l)^2 - 4}\right).$$

Proof. The proof is given in two steps.

Step 1. To demonstrate that $q_{n,T}$ generates a period function for an automorphic integral it suffices to show that $q_{n,T}$ satisfies the two relations in (1.6).

(i) $q_{n,T}|T + q_{n,T} = 0$ is trivial by a construction of $q_{n,T}$.

(ii) Since $(S_nT)^n = I$, and $g|S_n^lT = g$,

$$\sum_0^{n-1} q_{n,T}|(S_nT)^i \quad \left(\text{with } \sum_{i=0}^{n-1} = \sum_0^{n-1}\right)$$

$$= \sum_1^{l-1} g|S_n^i(S_nT)^{n-1} - \sum_1^{l-1} g|S_n^iT(S_nT)^{n-1}$$

$$+ \sum_1^{l-1} g|S_n^i(S_nT)^{n-2} - \sum_1^{l-1} g|S_n^iT(S_nT)^{n-2}$$

$$+ \cdots + \sum_1^{l-1} g|S_n^i(S_nT) - \sum_1^{l-1} g|S_n^iT(S_nT)$$

$$+ \sum_1^{l-1} g|S_n^i - \sum_1^{l-1} g|S_n^iT$$

$$= g|S_n^{l-1}TS_n^{-1} - g|TS_n^{-1} + g|S_n^{l-1}\left(TS_n^{-1}\right)^2$$

$$-g|\left(TS_n^{-1}\right)^2 + \cdots + g|S_n^lT - g|S_nT + g|S_n^{l-1} - g$$

$$= \left(g|S_n^{l-1}TS_n^{-1} - g|\left(TS_n^{-1}\right)^2\right) + \left(g|S_n^{l-1}\left(TS_n^{-1}\right)^2 - g|\left(TS_n^{-1}\right)^3\right) + \cdots + \left(g|\left(S_n^lT\right) - g\right)$$

$$+ \left(g|S_n^{l-1} - g|TS_n^{-1}\right) \equiv 0.$$

Therefore, $q_{n,T}$ is a rational period function of an automorphic integral of weight $2k$.

Step 2. It remains to show that $q_{n,T}$ is nontrivial, that is, that $q_{n,T} \not\equiv 0$. To show this, let us consider the following: From (1.45) and (1.46), we have

(1.47)

$$\begin{aligned} q_{n,T}(z) &= \sum_{i=1}^{l-1} g|S_n^i - \sum_{i=1}^{l-1} g|S_n^iT \\ &= \sum_{i=1}^{l-1}\left(z - \frac{\lambda_n l - 2i\lambda_n + \sqrt{\lambda_n^2 l^2 - 4}}{2}\right)^{-k} \\ &\quad\times\left(z - \frac{\lambda_n l - 2i\lambda_n - \sqrt{\lambda_n^2 l^2 - 4}}{2}\right)^{-k} \\ &\quad - \sum_{i=1}^{l-1}\left(i^2\lambda_n^2 + 1 - i\lambda_n^2 l\right)^{-k}\cdot\left(z - \frac{\lambda_n l - 2i\lambda_n + \sqrt{\lambda_n^2 l^2 - 4}}{2(-i^2\lambda_n^2 + i\lambda_n^2 l - 1)}\right)^{-k} \\ &\quad\times\left(z - \frac{\lambda_n l - 2i\lambda_n - \sqrt{\lambda_n^2 l^2 - 4}}{2(-i^2\lambda_n^2 + i\lambda_n^2 l - 1)}\right)^{-k}. \end{aligned}$$

Since the coefficients of $(\Sigma_{i=1}^{l-1} g|S_n^i)$ in $q_{n,T}$ equal 1 from (1.47), the worst case is that for some $1 \le i \le -1$, the coefficients of $(\Sigma_{i=1}^{l-1} g|S_n^iT)$, $(i^2\lambda_n^2 + 1 - i\lambda_n^2 l)^{-k}$, equal 1 and the poles of $(\Sigma_{i=1}^{l-1} g|S_n^iT)$ are matched to those of $(\Sigma_{i=1}^{l-1} g|S_n^i)$.

Now compare coefficients. Suppose $(i^2\lambda_n^2 + 1 - i\lambda_n^2 l)^{-k} = 1$. There are two possibilities:

(1) $i^2\lambda_n^2 = i\lambda_n^2 l$ for k odd or even. This is impossible for $1 \le i \le l - 1$.

(2) $i\lambda_n^2 l - \lambda_n^2 i^2 = i\lambda_n^2(l - i) = 2$ for k even. This implies either $i(l - 1) = 2$, $\lambda_n^2 = 1$ or $i(l - 1) = 1$, $\lambda_n^2 = 2$ for $i, l \in Z^+$. These hold only for $l = 3$, $i = 1$, $\lambda_n = 1$; $l = 3$, $i = 2$, $\lambda_n = 1$; and $l = 2$, $i = 1$, $\lambda_n = \sqrt{2}$.

Thus, $q_{n,T}$ is a nontrivial rational period function of an automorphic integral with weight $2k$ (k odd or even), whenever $l \ge 4$.

The proof is complete.

Remark. In Theorem 14 $q_{n,T}$ is nontrivial if k is odd for $l = 3$, $\lambda = 1$. Furthermore, this is the same example as given in Knopp's Theorem 1 [13].

COROLLARY 15. *Let g be a nonconstant meromorphic function such that $g|S_n^{2m}T = g$, where $m \in Z^+$. If k is an integer (odd or even), then*

$$q_{n,T} = \sum_{i=1}^{2m-1} g|S_n^i - \sum_{i=1}^{2m-1} g|S_n^iT$$

is a nontrivial rational period function of an automorphic integral of weight $2k$ *for* $G(\lambda)$. *The poles of this* $q_{n,T}$ *are in* $Q(\sqrt{\lambda^2 m^2 - 1}, \lambda)$.

Proof. This is the case $l = 2m$ of Theorem 14.

(b) Theorem on Pell's equation and corollary.

Now, we state the following well-known theorem without proof. With this theorem and Theorem 14 we show the existence of a rational period function of modular integral with weight $2k$ (k even or odd), that has poles in an arbitrary quadratic field $Q(\sqrt{N})$, $N \in Z^+$.

THEOREM 16. (1) *Let* D *be a positive integer which is not a perfect square. Then the equation* $x^2 - Dy^2 = 1$ *has an infinity of integer solutions* (x, y). *Furthermore, if* (x_1, y_1) *is a minimal integer solution of* $x^2 - Dy^2 = 1$, *then* (x_n, y_n) *is also a solution of* $x^2 - Dy^2 = 1$ *where* $(x_n + y_n\sqrt{D}) = (x_1 + y_1\sqrt{D})^n$, $n \in Z^+$.

(2) *Let* D *be a square free positive integer. If* (x_1, y_1) *is the minimal positive integer solution of the equation*

$$x^2 - Dy^2 = 4$$

then every integer solution (x_n, y_n) *satisfies the equation*

$$\frac{x_n + y_n\sqrt{D}}{2} = \left(\frac{x_1 + y_1\sqrt{D}}{2}\right)^n, \quad n \in Z^+$$

and every (x_n, y_n) *of the type*

$$\frac{x_n + y_n\sqrt{D}}{2} = \left(\frac{x_1 + y_1\sqrt{D}}{2}\right)^n$$

satisfies the equation $x^2 - Dy^2 = 4$.

(3) *Let* $\beta = x_i + y_i\sqrt{D}$, $x_i, y_i \in Z^+$.

If the norm of β, $(N\beta)$ *is* 1, *then* $N(2\beta) = 4$. *This implies the existence of a solution of the equation* $x^2 - y^2D = 4$.

We may state the following results.

COROLLARY 17. *There exist* **nontrivial** *rational period functions of modular integrals of weight* $2k$, k **odd or even**, *with poles in an* **arbitrary real quadratic** *field* $Q(\sqrt{N})$, $N \in Z^+$.

Proof. This is an immediate result of Corollary 15 and Theorem 16.

COROLLARY 18. *The collection* $\{q_{T,2k\,N}\}$ *of rational period functions with poles in the real quadratic field* $Q(\sqrt{N})$ *is infinite dimensional over* C.

Proof. In Theorem 14 with $\lambda = 1$ we construct $q_T \not\equiv 0$ such that

$$q_T = \sum_{i=1}^{l-1} g|S^i - \sum_{i=1}^{l-1} g|S^iT$$

with

$$g = g|S^lT = c \cdot \left(z - \frac{l + \sqrt{l^2 - 4}}{2}\right)^{-k} \left(z - \frac{l - \sqrt{l^2 - 4}}{2}\right)^{-k}.$$

Let N be square free such that $l^2 - 4 = Nm^2$. Theorem 16 implies that this has infinitely many integer solutions (l, m). The location of the poles of q_T implies that for each N we get infinitely many linearly independent q_T.

Note. If l, m are even integers, then $(l')^2 - 1 = Nm'$ with $l = 2l'$, $m = 2m'$.

2. *Existence of a rational period function for an automorphic integral of weight* $2k$, *k even or odd, on the Hecke groups* $G(\lambda)$ *with poles in* $Q(\sqrt{p}, \lambda)$, *where p is square free*, $\lambda = \sqrt{2}, \sqrt{3}$.

Now, we shall show the existence of rational period functions with poles at $Q(\sqrt{p}, \lambda)$, p is a positive square free, on the Hecke group $G(\sqrt{2})$, $G(\sqrt{3})$.

Leutbecher [16] has shown that of the Hecke groups only those for $n = 4$ and 6 are commensurable with the modular group. The commensurability of these groups permits construction of automorphic integrals for $G(\sqrt{2})$ and $G(\sqrt{3})$ from modular integrals. This construction is described in the following theorem [8], [18].

THEOREM 19 [18]. *Let* F_1 *be a modular integral of weight* $2k$, $k \in Z$, *with generating period function* $q_T = q$. *Then*

$$F_\lambda(z) = \lambda^{2k}F_1(\lambda z) + F_1(z/\lambda)$$

is an automorphic integral for $G(\lambda)$ *where* $\lambda = \sqrt{2}$ *or* $\sqrt{3}$. *The corresponding generating period function* $q_{\lambda,T} = q_\lambda$ *is*

$$q_\lambda = \lambda^{2k}q(\lambda z) + q(z/\lambda).$$

Remark. It should be pointed out that the definition of $F_\lambda(z)$ is precisely that used by Hecke in his original construction of automorphic forms on $G(\sqrt{2})$ and $G(\sqrt{3})$ from modular forms [8].

Table B
Explicit form of $q_{n,T}$ in Theorem 14, k odd, even
(a) The case $\lambda = 1$

l	Poles of q_T
$l = 4$	$\pm\{1 \pm \sqrt{3}\}, \pm\sqrt{3}, \pm\frac{1}{\sqrt{3}}, \pm\left\{\frac{1 \pm \sqrt{3}}{2}\right\}$
$l = 5$	$\pm\left\{\frac{3 \pm \sqrt{21}}{2}\right\}, \pm\left\{\frac{\pm\sqrt{21}}{2}\right\}, \pm\left\{\frac{3 \pm \sqrt{21}}{6}\right\}, \pm\left\{\frac{1 \pm \sqrt{21}}{10}\right\}$
$l = 6$	$\pm(2 \pm 2\sqrt{2}\}, \pm\{1 \pm 2\sqrt{2}\}, \pm 2\sqrt{2}, \pm\frac{\sqrt{2}}{4},$ $\pm\left\{\frac{1 \pm 2\sqrt{2}}{7}\right\}, \pm\left\{\frac{1 \pm \sqrt{2}}{2}\right\}$

(b) The case $\lambda = \sqrt{2}$

l	Poles of $q_{n,T}$
$l = 4$	$\pm\{\sqrt{2} \pm \sqrt{7}\}, \pm\sqrt{7}, \pm\frac{1}{\sqrt{7}}, \pm\left\{\frac{-\sqrt{2} \pm \sqrt{7}}{5}\right\}$
$l = 5$	$\pm\left\{\frac{3\sqrt{2} \pm \sqrt{46}}{2}\right\}, \pm\left\{\frac{\sqrt{2} \pm \sqrt{46}}{2}\right\}, \pm\left\{\frac{\sqrt{2} \pm \sqrt{46}}{14}\right\}$ $\pm\left\{\frac{\sqrt{2} \pm \sqrt{46}}{22}\right\}$
$l = 6$	$\pm\{2\sqrt{2} \pm \sqrt{17}\}, \pm\{\sqrt{2} \pm \sqrt{17}\}, \pm\sqrt{17}), \pm\frac{1}{\sqrt{17}},$ $\pm\left\{\frac{2\sqrt{2} \pm \sqrt{17}}{9}\right\}, \pm\left\{\frac{\sqrt{2} \pm \sqrt{17}}{15}\right\}.$

(c) The case $\lambda = \sqrt{3}$

l	Poles of $q_{n,T}$
$l = 4$	$\pm\{\sqrt{3} \pm \sqrt{11}\}, \pm\sqrt{11}, \pm\frac{1}{\sqrt{11}}, \pm\left\{\frac{\sqrt{3} \pm \sqrt{11}}{8}\right\}$
$l = 5$	$\pm\left\{\frac{3\sqrt{4} \pm \sqrt{71}}{2}\right\}, \pm\left\{\frac{\sqrt{3} \pm \sqrt{71}}{2}\right\}, \pm\left\{\frac{3\sqrt{3} \pm \sqrt{71}}{22}\right\},$ $\pm\left\{\frac{\sqrt{3} \pm \sqrt{71}}{34}\right\}$
$l = 6$	$\pm\{2\sqrt{3} \pm \sqrt{26}\}, \pm\{\sqrt{3} \pm \sqrt{26}\}, \pm\sqrt{26}, \pm\frac{1}{\sqrt{26}}$ $\pm\left\{\frac{2\sqrt{3} \pm \sqrt{26}}{14}\right\}, \pm\left\{\frac{\sqrt{3} \pm \sqrt{26}}{23}\right\}$

(d) The case $\lambda = \frac{1+\sqrt{5}}{2}$

l	Poles of $q_{n,T}$
$l=4$	$\pm\left\{\frac{1+\sqrt{5}}{2} \pm \sqrt{7+2\sqrt{5}}\right\}$, $\pm\{\sqrt{7}+2\sqrt{5}\}$, $\pm\frac{1}{\sqrt{7+2\sqrt{5}}}$
	$\pm\left\{\frac{1+\sqrt{5} \pm \sqrt{7+2\sqrt{5}}}{11+3\sqrt{5}}\right\}$
$l=5$	$\pm\left\{\frac{3+3\sqrt{5} \pm \sqrt{134+50\sqrt{5}}}{4}\right\}$, $\pm\left\{\frac{1+5 \pm \sqrt{134+50\sqrt{5}}}{4}\right\}$
	$\pm\left\{\frac{3+3\sqrt{5} \pm \sqrt{134+50\sqrt{5}}}{20+8\sqrt{5}}\right\}$, $\pm\left\{\frac{1+\sqrt{5} \pm \sqrt{134+50\sqrt{5}}}{32+12\sqrt{5}}\right\}$
$l=6$	$\pm\left\{1 \pm \sqrt{5} \pm \frac{\sqrt{50+18\sqrt{5}}}{2}\right\}$, $\pm\left\{\frac{\sqrt{50+18\sqrt{5}}}{2}\right\}$
	$\pm\left\{\frac{1+\sqrt{5} \pm \sqrt{50+18\sqrt{5}}}{2}\right\}$, $\pm\left\{\frac{2}{\sqrt{50+18\sqrt{5}}}\right\}$
	$\pm\left\{\frac{2+2\sqrt{5} \pm \sqrt{50+18\sqrt{5}}}{13+5\sqrt{5}}\right\}$, $\pm\left\{\frac{1+\sqrt{5} \pm \sqrt{50+18\sqrt{5}}}{22+8\sqrt{5}}\right\}$

COROLLARY 20. *There exist nontrivial rational period functions of integrals of weight $2k$, k odd or even, with poles in $Q(\sqrt{p}, \lambda)$ on the Hecke groups $G(\lambda)$, where p is a square free positive integer, $\lambda = \sqrt{2}, \sqrt{3}$.*

Proof. This is an immediate result of Corollary 18 and Theorem 19.

7. Conclusion

The number theoretical significance of the Eichler integral of negative integer weight with a polynomial period is well known. For instance, the coefficients of the period polynomial are closely related with the values of an L-function at a certain integer points (see [16], [18]).

In this article, we construct a rational period function of a modular integral of weight $2k$ (*any integer* k) with poles in an arbitrary real quadratic field $Q(\sqrt{N})$, $N \in Z^+$. Since all the examples that we construct are closely related to the kth power of binary quadratic forms, we may expect to obtain additional interesting number theoretical results. For instance, the existence of rational period functions is connected with the class number problem for real quadratic fields (see [3]).

In Table B, we give a few explicit examples, obtained from Theorem 14, of a rational period function on $G(\lambda)$.

Acknowledgement. The author truly thanks professor Marvin Knopp for his help, endless encouragement as well as for introducing this problem to her.

References

1. J. BOGO and W. KUYK, *The Hecke correspondence for $G(q^{1/2})$, q prime: Eisenstein series and modular invariant*, J. Algebra, vol. 43 (1976), pp. 585–605.
2. G. BOL, *Invarianten linear Differentialgleichunger*, Abh. Math. Sem. Univ. Hamburg, vol. 16 (1949), pp. 115–120.
3. Y.J. CHOIE, *Rational period functions, class numbers and Diophantine equations*, J. Number Theory, to appear.
4. H. COHN and M. KNOPP, *Note on automorphic forms with real period polynomials*, Duke Math J., vol. 32 (1965), pp. 115–120.
5. M. EICHLER, *Eine Verallgemeinerung der Abelschen Intergrale*, Math Zeitschr., vol. 67 (1957), pp. 267–298.
6. R.C. GUNNING, *The Eichler cohomology groups and automorphic forms*, Trans. Amer. Math. Soc., vol. 100 (1961), pp. 44–62.
7. HOLGER MEIER and G. ROSENBERGER, *Rationale periodische funktionen fur die Hecke-Gruppen und Dirichlet-Reihen mit Funktionalgleichung*, Resultate Math, vol. 7 (1984), pp. 209–233.
8. E. HECKE, *Lecture note on modular functions and Dirichlet series*, Princeton University Press, Princeton, N.J., 1938.
9. S.Y. HUSSEINI and M. KNOPP, *Eichler cohomology and automorphic forms*, Illinois J. Math., vol. 15 (1971), 00. 565–577.
10. D. JAMES, *Functions automorphic on large domains*, Transs. Amer. Math Soc., vol. 181 (1978), pp. 385–400.
11. M. KNOPP, *Polynomial automorphic forms and nondiscontinuous groups*, Tran. Amer. Math Soc., vol. 123 (1966), pp. 506–520.
12. ______, *Some new results on the Eichler cohomology of automorphic forms*, Bull. Amer. Math. Soc., vol. 80 (1974), pp. 607–632.
13. ______, *Rational period functions of the modular group*, Duke Math. J., vol. 45 (1978), pp. 47–62.
14. ______, *Rational period functions of modular group, II*, Glasgow Math. J., vol. 22 (1981), pp. 185–197.
15. J. LEHNER, *The Eichler cohomology of a Kleinian group*, Math. Ann., vol. 192 (1971), pp. 125–143.
16. A. LEUTBECHER, *Uber die Heckesche Gruppen $G(\lambda)$*, Abh. Math. Sem. Hamburg., vol. 31 (1967), pp. 199–205.
17. JU. MANIN, *Periods of parabolic forms and p-adic Hecke series*, Math. USSR. Sbornik, vol. 21 (1973), pp. 371–393.
18. L. PARSON and K. ROSEN, *Automorphic integrals and rational period functions for the Hecke groups*, Illinois J. Math., vol. 28 (1984), pp. 383–396.
19. G. SHIMURA, *Sur les integrals attachees aux forms automorphes*, J. Math. Soc. Japan, vol. 11 (1959), pp. 291–311.

OHIO STATE UNIVERSITY
COLUMBUS, OHIO

UNIVERSITY OF MARYLAND
COLLEGE PARK, MARYLAND