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# VOLUME AND ENERGY STABILITY FOR ISOMETRIC MINIMAL IMMERSIONS

BY

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It is well known that a minimal isometric immersion from a Riemannian manifold $M$ into a Riemannian manifold $N$ is also a harmonic immersion. An open question is whether the second order behavior of the volume is similar to that of the energy for such immersions. That is, does stability of the energy integral imply stability of the volume integral, or vice versa, for an immersion between Riemannian manifolds that is both minimal and harmonic? We will show that these two types of stability are not necessarily equivalent, and will exhibit several examples of surfaces in flat tori which are stable for the energy, but not for the volume integral.

The results in this paper are from the author's thesis work under the direction of Philippe Tondeur at the University of Illinois.

## 1. Codimension-one minimal immersions and stability

Let $f\colon M^m \to N^{m+1}$ be an isometric immersion from a closed Riemannian manifold $M$ into a Riemannian manifold $N$. We recall the second variation formulas of energy and volume for such an immersion that is minimal, thus also harmonic:

$$(1.1)\qquad \left.\frac{d^2E(f_t)}{dt^2}\right|_{t=0} = \int_M \langle \nabla^n \nu, \nabla^n \nu\rangle - \langle \rho\nu, \nu\rangle + \langle \nabla^t \nu, \nabla^t \nu\rangle \, dV,$$

$$(1.2)\qquad \left.\frac{d^2V(f_t)}{dt^2}\right|_{t=0} = \int_M \langle \nabla^n \nu^n, {}^n\nu^n\rangle - \langle \rho\nu^n, \nu^n\rangle - \langle \mathfrak{B}\nu^n, \nu^n\rangle \, dV.$$

Here, $dV$ denotes the Riemannian measure of $(M, g_M)$, and $f_t, t \in (-a, a)$, denotes a variation of the immersion $f$ with $f_0 = f$. This variation generates a vector field $\nu$ along $f$, with $\nu^n$ denoting the normal component of $\nu$, with

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respect to the metric $g_N$. $\rho$ denotes the Ricci operator of $N$, $\nabla$ the connection of the pull-back bundle $f^*TN$, and $\nabla^t, \nabla^n$ the tangential, normal components of $\nabla$. $\langle \mathfrak{B}\nu^n, \nu^n \rangle$ is the norm-square of the $\nu^n$ component of the second fundamental form of the immersion.

THEOREM 1.1. *Let $f\colon M^m \to N^{m+1}$ where $M$ is a compact, oriented, Riemannian manifold without boundary, and $N$ is a flat, oriented Riemannian manifold. Then, $f$ is minimal and volume stable iff it is totally geodesic.*

*Proof.* Assume that $f$ is totally geodesic. Then $f$ is minimal, and

$$\left.\frac{d^2V(f_t)}{dt^2}\right|_{t=0} = \int_M \langle \nabla^n \nu^n, \nabla^n \nu^n \rangle \, dV \geq 0 \quad \text{for all variations } f_t.$$

Conversely, assume that $f$ is minimal and volume stable. Then,

$$\left.\frac{d^2V(f_t)}{dt^2}\right|_{t=0} = \int_M \langle \nabla^n \nu^n, \nabla^n \nu^n \rangle - \langle \mathfrak{B}\nu^n, \nu^n \rangle \, dV \geq 0$$

for all variations $f_t$. If $f$ is not totally geodesic, then for some variational vector field $\nu$ we have $\int_M \langle \mathfrak{B}\nu^n, \nu^n \rangle \, dV > 0$, as $\langle \mathfrak{B}\nu^n, \nu^n \rangle \geq 0$ on $f(M)$. Since $\nu^n = hF_{m+1}$, where $h$ is some smooth function on $f(M)$ and $F_{m+1}$ is the global unit normal on $f(M)$, we get

$$\int_M \langle \mathfrak{B}\nu^n, \nu^n \rangle \, dV = \int_M h^2 \langle \mathfrak{B}F_{m+1}, F_{m+1} \rangle \, dV > 0.$$

Thus, $\int_M \langle \mathfrak{B}F_{m+1}, F_{m+1} \rangle \, dV > 0$.

For the variation of $f$ given by $f_t(x) = \exp_{f(x)}(t \cdot F_{m+1})$, we have

$$\left.\frac{d^2V(f_t)}{dt^2}\right|_{t=0} = \int_M \langle \nabla^n F_{m+1}, \nabla^n F_{m+1} \rangle \, dV - \int_M \langle \mathfrak{B}F_{m+1}, F_{m+1} \rangle \, dV.$$

Now,

$$\int_M \langle \nabla^n F_{m+1}, \nabla^n F_{m+1} \rangle \, dV = \int_M \sum_{k=1}^m g_N\big(\nabla^n_{E_k} F_{m+1}, \nabla^n_{E_k} F_{m+1}\big) \, dV$$

where $\{E_k\}$ $(k = 1, \ldots, m)$ is an orthonormal frame for $TM$ (locally) with respect to the metric $g_M$. Also,

$$\begin{aligned}\nabla^n_{E_k} F_{m+1} &= g_N\big(\nabla_{E_k} F_{m+1}, F_{m+1}\big) F_{m+1} \\ &= 1/2\big[E_k\big(g_N(F_{m+1}, F_{m+1}) \circ f\big)\big] F_{m+1} \\ &= 0.\end{aligned}$$

Thus,

$$\left.\frac{d^2V(f_t)}{dt^2}\right|_{t=0} = -\int_M \langle \mathfrak{B} F_{m+1}, F_{m+1}\rangle\, dV < 0.$$

This contradicts the assumed stability of $f$, and $f$ must be totally geodesic.

PROPOSITION 1.2. *Let $f\colon M^m \to N^{m+1}$ where $M$ is a compact, oriented, Riemannian manifold without boundary, and $N$ is a flat Riemannian manifold. Then, if $f$ is a harmonic immersion, it must be energy stable.*

*Proof.* This is clear from (1.1).

From these two results it follows that if we can find a minimal isometric immersion $f\colon M^m \to N^{m+1}$ that is not totally geodesic, with $M$ and $N$ as above, then $f$ will be a minimal immersion that is *unstable* for the volume integral. On the other hand, it will be a harmonic immersion that is *stable* for the energy integral. To find such an immersion we will look at the case where $M$ is of dimension 2. We will use the following result of Chen and Nagano found in [3].

PROPOSITION 1.3. *Let $f\colon M^2 \to N$ be a harmonic immersion from a compact two-dimensional Riemannian manifold into a Riemannian manifold $N$ of non-positive sectional curvature. Then, the Euler characteristic of $M$ satisfies $\chi(M) \leq 0$. If $\chi(M) = 0$, then $f$ must be totally geodesic.*

If $N$ is flat, then $\chi(M) = 0$ iff $f$ is totally geodesic. This result together with Theorem 1.1 gives the following result.

COROLLARY 1.4. *Let $f\colon M^2 \to T^3$ be a minimal Riemannian immersion with $M$ a compact, oriented surface without boundary, and $T^3$ the flat three-torus. Then, $M$ has genus $g \geq 2$ iff $f$ is unstable.*

We now discuss several surfaces in $T^3$ which exhibit opposite energy and stability behavior. The first such surface is Schwarz's surface.

Let $P$ be a quadrilateral in $R^3$ given by four line segments of equal length such that the angles between the edges are all $\pi/3$. $P$ will bound a unique smooth minimal surface $M(P)$ with $P$ as its boundary. By Schwarz's Reflection Theorem we can reflect this minimal surface through an edge $e$ via the reflection $S_e$ to get a new minimal surface $M(P) \cup S_e(M(P))$. We continue reflecting this new surface across the remaining edges, and then reflect again through any new edges. Continuing this process indefinitely, we get a complete minimal surface $M'$ in $R^3$ that is smooth and also triply-periodic. Dividing out by the lattice-group action we get a minimal surface in $T^3$. A sketch of this surface can be found in Schwarz's original paper [8, p. 4]. However, this surface is not orientable. We can easily generate an orientable covering of this surface in $R^3$, yielding a minimal surface $M_T \subset T^3$ of genus 9.

By Corollary 1.4, $M_T$ is *unstable* for the area integral, but *stable* for the energy integral. This result is surprising because the original minimal surface with boundary, $M(P)$, minimizes both area and energy.

The curve $P$ that generates Schwartz's surface can also be generated from the fundamental root vectors for the root system $A_3$ of the simple Lie group $SL(6, C)$. Re-normalizing the root vectors to have length $\sqrt{2}$ we can let $\alpha_1 = (0,1,-1)$, $\alpha_2 = (1,-1,0)$, $\alpha_3 = (0,1,1)$. Then, $P$ is given by the edge vectors $e_1 = \alpha_1$, $e_2 = \alpha_2$, $e_3 = \alpha_3$, and $e_4 = -(\alpha_1 + \alpha_2 + \alpha_3)$.

The relationship between minimal surfaces in tori and root systems was discovered by T. Nagano and B. Smyth, in [5]. They show that the root systems $C_3$ and $B_3$ also generate minimal surfaces in $T^3$. Other examples can be found in [6] and [7].

## 2. Volume unstable minimal surfaces in higher dimensional manifolds

In [5] it is shown that to every irreducible root system in $R^n$ there is a nonsingular minimal surface $M^2$ in $T^n$. We will apply the following result to these surfaces.

THEOREM 2.1. *Let $M^m \subset N^n$ be a compact, oriented submanifold, without boundary, in a parallelizable Riemannian manifold. Then, $M$ is minimal and volume stable iff it is a totally geodesic submanifold.*

Since the root vectors of a root system are linearly independent then the minimal surface generated in $R^n$, whose boundary is the polygon constructed from the root vectors, cannot be planar. Thus, the minimal surface in $T^n$ generated from this minimal surface in $R^n$ cannot be totally geodesic. So, the surfaces of Nagano-Smyth must be unstable for the volume integral by Theorem 2.1. They are also energy stable by Prop. 1.2. Thus, they form an infinite family of minimal surfaces in tori that exhibit contrary stability behavior for the energy and volume.

*Proof of Theorem* 2.1. If $M$ is totally geodesic, then it is minimal. Since $N$ is parallelizable, $\rho \equiv 0$, and

$$\left.\frac{d^2V(M_t)}{dt^2}\right|_{t=0} = \int_M \langle \Delta^n \nu^n, \nabla^n \nu^n \rangle \, dV \geq 0$$

for all variations $M_t$.

Assume conversely that $M$ is minimal and volume stable. Let $V$ be a parallel vector field on $N$, and $\{E_k\}_{k=1}^m$ be an orthonormal basis for $T_xM$, $x \in M$. For the variation of $M$ given by $M_t(x) = \exp_x(t \cdot V)$ we have

$$
\begin{aligned}
(2.1)\quad \left.\frac{d^2V(f_t)}{dt^2}\right|_{t=0} &= \int_M \langle \nabla^n V^n, \nabla^n V^n\rangle - \langle \mathfrak{B} V^n, V^n\rangle \, dV \\
&= \int_M \langle \nabla^n V, \nabla^n V\rangle - \sum_{k,l=1}^m \{\langle \nabla_{E_k} V, E_l\rangle\langle \nabla_{E_l} V, E_k\rangle\} \\
&\quad + \left\{\sum_{k=1}^m \langle \nabla_{E_k} V, E_k\rangle\right\}^2 dV \quad \text{(see [4])} \\
&= 0, \text{ as } V \text{ is parallel.}
\end{aligned}
$$

Define a symmetric, bilinear operator $I$ on normal vector fields on $M$ by

$$I(X, Y) = \int_M \langle \nabla^n X, \nabla^n Y\rangle - \langle \mathfrak{B} X, Y\rangle \, dV$$

for $X, Y$ sections in the normal bundle. Then, $I(V^n, V^n) = 0$. Since $M$ is stable, $I(Z, Z) \geq 0$ for all normal sections $Z$. So,

$$d/ds|_{s=0} I(V^n + sW, V^n + sW) = 2I(V^n, W) = 0$$

for all normal sections $W$ and

$$(2.2)\qquad \int_M \langle \nabla^n V^n, \nabla^n W\rangle \, dV = \int_M \langle \mathfrak{B} V^n, W\rangle \, dV \quad \text{for all } W.$$

Let $W = hV^n$ for $h$ a smooth function on $M$. Then, equation (2.2) yields

$$\int_M \sum_{k=1}^m E_k(h)\langle \nabla^n_{E_k} V^n, V^n\rangle \, dV + \int_M h[\langle \nabla^n V^n, \nabla^n V^n\rangle - \langle \mathfrak{B} V^n, V^n\rangle] \, dV = 0.$$

Since $\langle \nabla^n_{E_k} V^n, V^n\rangle = \frac{1}{2} E_k(\langle V^n, V^n\rangle)$, we get

$$\int_M \tfrac{1}{2} g_M(d_M h, d_M(\langle V^n\, V^n\rangle)) + h[\langle \nabla^n V^n, \nabla^n V^n\rangle - \langle \mathfrak{B} V^n, V^n\rangle] \, dV = 0$$

and

$$\int_M \Delta_M(h) \cdot \langle V^n, V^n\rangle + 2h[\langle \nabla^n V^n, \nabla^n V^n\rangle - \langle \mathfrak{B} V^n, V^n\rangle] \, dV = 0.$$

Let $q = 2[\langle \nabla^n V^n, \nabla^n V^n \rangle - \langle \mathfrak{B} V^n, V^n \rangle]$. Then, $q\colon M \to \mathbf{R}$, and we get

$$\int_M \{\Delta_M(h) \cdot \langle V^n, V^n \rangle + h \cdot q\}\, dV = 0. \tag{2.3}$$

Suppose that $q \neq 0$ somewhere on $M$. Then, since $\int_M q\, dV = 0$, by (2.2), we have $q\colon M \to [-a, b]$. We can choose a regular value $C > 0$ in $[-a, b]$ such that $W^\wedge = q^{-1}\{[C, b]\}$ will be a smooth $m$-manifold in $M$ with a smooth boundary $(m-1)$-manifold. For such a manifold the eigenvalues of the Laplacian $\Delta_{W^\wedge}$ corresponding to solutions of the Dirichlet problem on $W^\wedge$ are positive, and the eigenfunctions corresponding to the first eigenvalue $\lambda_1$ are strictly negative or strictly positive on $W = \mathrm{Int}(W^\wedge)$ [1, pp. 102–103]. Let $h_1$ be an eigenfunction for $\lambda_1$ on $W^\wedge$. As $h_1$ vanishes on the boundary of $W^\wedge$, we can extend $h_1$ by zero to the rest of $M$, getting a smooth function on $M$. Let $h = h_1$ in (2.3). Then,

$$\int_W \{\Delta_M(h_1) \cdot \langle V^n, V^n \rangle + h_1 \cdot q\}\, dV = 0,$$

Locally, on $W$, $\Delta_M = \Delta_W$. So, we get

$$\int_W h_1[\lambda_1 \cdot \langle V^n, V^n \rangle + q]\, dV = 0.$$

But, $\{\lambda_1 \cdot \langle V^n, V^n \rangle + q\} > 0$, and $h_1 > 0$ or $< 0$ on all of $W$. Thus, $q = 0$ on $M$.

Suppose that $M$ is not totally geodesic. Then, there is some $x \in M$ and some unit normal vector $V_x$ such that $\langle \mathfrak{B} V_x, V_x \rangle > 0$. By parallel translation of $V_x$ we get a parallel vector field $V$ such that $\int_M \langle \mathfrak{B} V^n, V^n \rangle\, dV > 0$. Let $W = hV^n$ in equation (3.3) for $h$ a smooth function on $M$. We get

$$\int_M \Delta_M(h) \cdot \langle V^n, V^n \rangle\, dV = 0 \quad \text{or} \quad \int_M h \cdot \Delta_M(\langle V^n, V^n \rangle)\, dV = 0.$$

By appropriate choice of $h$ we get $\Delta_M(\langle V^n, V^n \rangle) = 0$, and thus $\langle V^n, V^n \rangle =$ constant $= 1$, since it is of unit length at $x$.

Since $\langle V, V \rangle = \langle V^n, V^n \rangle + \langle V^t, V^t \rangle$, then $V^t = 0$-vector field, and $V = V^n$ on $M$. Now,

$$\int_M \langle \Delta^n V^n, \nabla^n V^n \rangle\, dV = \int_M \langle \mathfrak{B} V^n, V^n \rangle\, dV \neq 0.$$

However,

$$\int_M (\Delta^n V^n, \nabla^n V^n)\, dV = \int_M (\Delta^n V, \nabla^n V)\, dV = 0.$$

This is clearly impossible, and $M$ must be totally geodesic.

## 3. The Gauss map and minimal surfaces in $T^3$

Barbosa and Do Carmo in [2] investigated the stability of a minimal surface in $R^3$ by means of the Gauss map of the surface. They showed that if the spherical area of the Gauss map of a minimal surface in $R^3$ is less than $2\pi$, then the surface is stable.

We can define a Gauss map on a minimal surface in $T^3$ in terms of the standard Gauss map of the covering surface in $R^3$. For this new Gauss map we get the following result.

PROPOSITION 3.1. *Let $f$: $M^2 \to T^3$ be a minimal isometric immersion with $M$ a compact, oriented Riemannian manifold without boundary. Then*:

(i) *$f$ is volume stable iff the Gauss map of $M$, $g'$: $M^2 \to S^2$, is a constant.*

(ii) *$f$ is volume unstable iff the Gauss map has maximal spherical area ($= 4\pi$). In fact, $g'$ is* **onto** *$S^2$.*

*Proof.* (i) By Theorem 1.1, $f$ is volume stable iff it is totally geodesic. This is true iff $f(M)$ is a sub-torus in $T^3$. The result is then clear.

(ii) By Theorem 1.1, $f$ is volume unstable iff $f(M)$ is not a sub-torus in $T^3$. Let $M'$ be the covering of $M$ in $R^3$. In [9], F. Xavier showed that if $M'$ is not flat, then the Gauss map of $M'$ cannot omit more than 6 points of the sphere. Thus, $g'$ cannot omit more than 6 points of $S^2$. Since $M$ is compact, $g'$ must be onto. Similarly, if $g'$ is onto, then $f(M)$ cannot be a sub-torus in $T^3$.

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