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APPLICATIONS OF COMMUTATOR THEORY TO WEIGHTED BMO AND MATRIX ANALOGS OF A_2

BY

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1. Introduction

Our setting will be the unit circle T in the complex plane, although the results in the next section and some of the later results extend easily to $\mathbf{R}^n$. For an interval $I(f)$,

$$I(f) = \frac{1}{|I|}\int_I f.$$

The Hardy-Littlewood maximal operator M^* is defined by

$$M^*(f) = \sup_{X \in I} I(|f|).$$

Throughout this paper C will denote a universal constant, and may change from line to line. A nonnegative weight ν belongs to the Muckenhoupt class A_p for some $1 < p < \infty$ if

$$I(\nu)I(\nu^{-1/p-1})^{p-1} \leq C \quad \text{for each interval } I.$$

A function $b \in \mathrm{BMO}_\nu$ provided

$$I(|b - I(b)|) \leq CI(\nu) \quad \text{for all intervals } I,$$

and $\mathrm{BMO} = \mathrm{BMO}_\nu$ for the function $\nu \equiv 1$. Given b, the commutator of the maximal operator with b is T_b, given by

$$T_b f(x) = \sup_{x \in I} |b(x)I(f) - I(bf)|.$$

Likewise, the commutator of the Hilbert transform H with b is the operator

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$S_b = [H, M_b]$, given by

$$S_b f(x) = |b(x)Hf(x) - H(bf)(x)|.$$

In [1], we derived a two-weighted norm inequality for S_b and used it to give a vector-valued analog of the Hunt, Muckenhoupt, and Wheeden Theorem. In this paper, we will derive similar theorems for T_b and analogs of the Hardy-Littlewood maximal operator. Usually, the Hardy-Littlewood maximal operator is much easier to deal with than the Hilbert transform, yet in this commutator setting, the opposite is apparently so. Even in the unweighted case, although it was known that T_b is bounded on L^2 if and only if $b \in \mathrm{BMO}$, the proofs proceeded via Muckenhoupt's theory of weights, and no direct proof was known for many years.

2. A commutator theorem

We will establish:

THEOREM 2.1. *Let μ and $\lambda \in A_p$. Put $\nu = (\mu\lambda^{-1})^{1/p}$. Then $b \in \mathrm{BMO}_\nu$ if and only if T_b: $L^p(\mu) - L^p(\lambda)$ is a bounded operator.*

An immediate consequence is:

COROLLARY 2.2. $b \in \mathrm{BMO}$ *if and only if T_b is a bounded operator on L^p.*

Corollary 2.2 has an interesting history. It first appeared in [5], where the sufficiency was derived from Muckenhoupt's Theorem by a clever interpolation argument, and for some time no direct proof was known. When we were working on 2.1, we thought we had obtained the first such direct proof. But Peter Jones, after seeing an early draft of this paper, communicated to us an elegant (unpublished) proof by Jones and Stromberg using Carleson measure theory [7]. At about that time, Coifman, Meyer, and Stein presented their Tent Space theory in [5]. Corollary 2.2 can be derived from a slightly modified version of their Theorem 5:

THEOREM 2.3. *Let $1 < p < \infty$. For each $x \in \mathbf{R}$, let $I_{x,t}$ be an interval of length t containing x. For a function $f \in L^p(\mathbf{R})$, put*

$$f(x, t) = I_{x,t}(f)$$

and let μ be a function defined on $\mathbf{R}^2_+$. Put

$$M_\mu f(x) = \sup_t |\mu(x, t) f(x, t)|.$$

Then the operator M_μ is a bounded operator on L^p if and only if

$$\frac{1}{|I|}\int_I\Big(\sup_{t\le|I|}|\mu(x,t)|^p\Big)\,dx < C \tag{2.4}$$

for all intervals I.

Now suppose $b \in \mathrm{BMO}$. Set $\mu(x,t) = b(x) - I_{x,t}(b)$. To bound T_b, it will suffice to bound

$$\tilde{T}_b f(x) = \sup_t |b(x)I_{x,t}(f) - I_{x,t}(bf)|$$

with norm independent of the collection $p\{I_{x,t}\}$. Now

$$|\big(M_\mu - \tilde{T}_b\big)f(x)| \le \sup_t |I_{x,t}(b)I_{x,t}(f) - I_{x,t}(bf)|$$

and it's an easy application of Hölder's inequality to show that the operator

$$f \to \sup_t |I_{x,t}(b)I_{x,t}(f) - I_{x,t}(bf)|$$

is bounded on L^p. Hence T_b is bounded providing (2.4) holds. Let J be the interval concentric with I but of twice the length, and set

$$b_I(x) = [b(x) - J(b)]\chi_J(x).$$

Since, for $x \in I$,

$$\sup_{t\le|I|}|\mu(x,t)| \le 2M^*(b_I)(x),$$

(2.4) follows from Hardy and Littlewood's Theorem.

We would like to thank the reviewer and Prof. Coifman for pointing out the connection between 2.2 and 2.3. It would be interesting to see a weighted version of Tent Space theory that would lead to weighted versions of 2.3.

To prove 2.1, let $1 < q < p$ but near p. We will denote the conjugate exponent with a prime, $1/q + 1/q' = 1$. For $r \ge 1$, define these operators:

$$S_r(b; w, I) = I(|b - I(b)|^r w^r)^{1/r},$$

$$\Lambda_r(f; w, I) = I(|fw|)^{1/r},$$

$$K_r^*(b, f, w)(x) = \sup_{x\in I} S_{rq'}(b; w, I)\Lambda_{rq}(f; w^{-1}, I)$$

and $K^* = K_1^*$.

We use a result from [1].

LEMMA 2.3. *Let* μ *and* $\lambda \in A_p$, $\nu = (\mu\lambda)^{1/p}$, *and* $b \in \mathrm{BMO}_\nu$. *For an appropriate choice of* $q < p$ *and any* r *with* $1 \leq r < p/q$, *there exists a weight* w *depending on* r *such that* $w^{rq'} \in A_{q'}$ *and*

$$\int [K_r^*(b, f, w)(x)]^p \lambda(x)\, dx \leq C \int |f(x)|^p \mu(x)\, dx.$$

Proof of Theorem 2.1. Let $b \in \mathrm{BMO}_\nu$. Fix an r with $1 < r < p/q$ and let w and $\tilde{w}$ be the weights from Lemma 2.3 for 1 and r respectively, so

$$\int [K^*(b, f, w)]^p \lambda \leq C \int |f|^p \mu \tag{1}$$

and

$$\int [K_r^*(b, f, \tilde{w})]^p \lambda \leq C \int |f|^p \mu. \tag{2}$$

Let

$$\varepsilon = \frac{1}{2\pi} \int_T T_b f(x)\, dx.$$

By the Calderon-Zygmund decomposition, for each $\alpha \geq \varepsilon$, there exist disjoint intervals $\{I_n^\alpha\}$ such that:

(3) $T_b f(x) \leq \alpha$ for a.a. x off $\bigcup_n I_n^\alpha$.
(4) $\alpha < I_n^\alpha(T_b f) \leq 2\alpha$ for each n.
(5) Given $\beta \leq \alpha$ and n, there exists a k with $I_n^\beta \subset I_k^\alpha$.

Let

$$\tau(\alpha) = \sum_n \lambda(I_n^\alpha) = \sum_n \int_{I_n^\alpha} \lambda(x)\, dx$$

and

$$\sigma(\alpha) = \lambda([x\colon K^*(b, M^*f, w)(x) + K_r^*(b, f, \tilde{w})(x) > \alpha]).$$

Since $\lambda \in A_p$, λ satisfies the A_∞ condition [3]. Thus there exists a $\delta > 0$ so that for any interval I and measurable set $E \subset I$,

$$\frac{\lambda(E)}{\lambda(I)} \leq C\left(\frac{|E|}{|I|}\right)^\delta. \tag{6}$$

We will establish the distribution inequality

(7) $\qquad \tau(3\alpha) \le \sigma(\gamma\alpha) + K\gamma^{\delta}\tau(\alpha)$ for each $\alpha \ge \varepsilon$ and $\gamma > 0$.

For this, fix $I = I_m^{\alpha}$ and let $I_n = I_n^{3\alpha}$. Let F be the set of integers with $n \in F$ if and only if $I_n \subset I$. By (5), as we run through the I_m^{α}'s, this process will exhaust all the $I_n^{3\alpha}$'s.

If

$$I \subset [x\colon K^*(b, M^*f, w)(x) + K_r^*(b, f, \tilde{w})(x) > \gamma\alpha],$$

then

$$\bigcup_F I_n \subset [x\colon K^*(b, M^*f, w)(x) + K_r^*(b, f, \tilde{w})(x) > \gamma\alpha] \cap I.$$

Otherwise, there exists an $x_0 \in I$ with

$$K^*(b, M^*f, w)(x_0) \le \gamma\alpha \quad \text{and} \quad K_r^*(b, f, \tilde{w})(x_0) \le \gamma\alpha.$$

Let $2I$ denote the interval concentric with I but of twice the length. Put $f_1 = f\chi_{2I}$ and $f_2 = f - f_1$. Then by (4),

$$3\alpha \sum_F |I_n| \le \sum_F \int_{I_n} T_b f.$$

So there exist intervals J_x containing x such that

$$\begin{aligned}
3\alpha \sum_F |I_n| &\le \sum_F \int_{I_n} |b(x)J_x(f) - J_x(bf)|\, dx \\
&= \sum_F \int_{I_n} |b - I(b)|J_x(f) - J_x([b - I(b)]f_1) \\
&\qquad -J_x([b - I(b)]f_2)|\, dx \\
&\le \sum_F \int_{I_n} |b - I(b)|J_x(|f|) + \sum_F \int_{I_n} J_x(|b - I(b)|\,|f_1|) \\
&\qquad + \sum_F \left| \int_{I_n} J_x([b - I(b)]f_2) \right| \\
&\le \sum_F \int_{I_n} |b - I(b)|M^*f + \sum_F \int_{I_n} M^*([b - I(b)]f_1) \\
&\qquad + \sum_F \left| \int_{I_n} J_x([b - I(b)]f_2) \right| \\
&= K_1 + K_2 + \sum_F \left| \int_{I_n} J_x([b - I(b)]f_2) \right|
\end{aligned}$$

Let $R_x = I \cup J_x$. For $x \in I_n \subset I$, R_x is an interval, and

$$\begin{aligned} 3\alpha \sum_F |I_n| &\le K_1 + K_2 + \sum_F \left| \int_{I_n} J_x([b - I(b)] f_2) - R_x([b - I(b)] f_2) \right| \\ &\quad + \sum_F \left| \int_{I_n} R_x([b - I(b)] f_2) \right| \\ &= K_1 + K_2 + K_3 + K_4. \end{aligned}$$

We estimate these pieces in turn. First,

$$\begin{aligned} K_1 &\le \int_I |b - I(b)| M^* f \\ &= |I| I\big(|b - I(b)| w w^{-1} M^* f\big) \\ &\le |I| S_{q'}(b; w, I) \Lambda_q\big(M^* f; w^{-1}, I\big) \quad \text{(by Holder's inequality)} \\ &\le |I| K^*(b, M^* f, w)(x_0) \\ &\le \gamma\alpha |I|. \end{aligned}$$

Next, using Holder's inequality again, and the boundedness of the maximal operator on L^r, we have

$$\begin{aligned} K_2 &\le |I|^{1-1/r} \left(\int_I M^*([b - I(b)] f_1)^r \, dx \right)^{1/r} \\ &\le |I|^{1-1/r} \left(\int_T M^*([b - I(b)] f_1)^r \right)^{1/r} \\ &\le C |I|^{1-1/r} \left(\int_T |b - I(b)|^r |f_1|^r \right)^{1/r} \\ &\le C |I| \left(\frac{1}{|2I|} \int_{2I} |b - I(b)|^r |f|^r \right)^{1/r} \\ &\le C |I| \left[\left(\frac{1}{|2I|} \int_{2I} |b - 2I(b)|^r |f|^r \right)^{1/r} + |I(b) - 2I(b) 2I(|f|^r)^{1/r} \right] \\ &= C |I| (A + B). \end{aligned}$$

Of these,

$$\begin{aligned} A &= 2I\big(|b - 2I(b)|^r \tilde{w}^r |f \tilde{w}^{-1}|^r\big)^{1/r} \\ &\le S_{rq'}(b; \tilde{w}, 2I) \Lambda_{rq}\big(f; \tilde{w}^{-1}, 2I\big) \\ &\le K_r^*(b, f, \tilde{w})(x_0) \\ &\le \gamma\alpha, \end{aligned}$$

and

$$\begin{aligned}|I(b) - 2I(b)| &\le \frac{2}{|2I|}\int_{2I}|b - 2I(b)| \\ &\le 2S'_{qf}(b; \tilde{w}, 2I)2I(\tilde{w}^{-q})^{1/q} \\ &\le 2S_{rq'}(b; \tilde{w}, 2I)2I(\tilde{w}^{-rq})^{1/rq}\end{aligned}$$

so that

$$\begin{aligned}B &\le 2S_{rq'}(b; \tilde{w}, 2I)2I(\tilde{w}^{-rq})^{1/rq}\Lambda_{rq}(f; \tilde{w}^{-1}, 2I)2I(\tilde{w}^{rq'})^{1/rq'} \\ &\le CK_r^*(b, f, \tilde{w})(x_0) \quad \left(\text{as } \tilde{w}^{rq'} \in A_q\right) \\ &\le C\gamma\alpha.\end{aligned}$$

Thus $K_2 \le C\gamma\alpha|I|$ also.

To estimate K_3, fix $x \in I$ and write $J = J_x$ and $R = R_x$. Then

$$J([b - I(b)]f_2) - R([b - I(b)]f_2) = 0 \quad \text{if } J \subset 2I,$$

so we can assume that $J \oplus 2I$. But then $|J| \ge \frac{1}{2}|I|$, and so $|R| \le 3|J|$. Now,

$$\begin{aligned}&|J([b - I(b)]f_2) - R([b - I(b)]f_2)| \\ &\quad = \left|\frac{1}{w|J|}\int_J [b - I(b)]f_2 - \frac{1}{|R|}\int_{J\cup(R\sim J)}[b - I(b)]f_2\right|.\end{aligned}$$

But $R \sim J \subset I$ and $f_2 = 0$ on I, so this is really

$$\begin{aligned}&\left|\left(\frac{1}{|J|} - \frac{1}{|R|}\right)\int_J [b - I(b)]f_2\right| \\ &\quad \le \frac{|R| - |J|}{|R|\,|J|}\int_J |b - I(b)|\,|f_2| \\ &\quad \le \frac{3|I|}{|R|^2}\int_R |b - I(b)|\,|f| \\ &\quad \le \frac{3|I|}{|R|}[R(|b - R(b)|\,|f|) + |I(b) - R(b)|R(|f|)] \\ &\quad \le \frac{3|I|}{|R|}\left[S_{q'}(b; w, R)\Lambda_q(f; w^{-1}, R) + \frac{1}{|I|}\int_I |b - R(b)|R(|f|)\right] \\ &\quad \le \frac{3|I|}{|R|}K^*(b, f, w)(x_0) + 3R(|b - R(b)|)R(|f|) \\ &\quad \le 3K^*(b, M^*f, w)(x_0) \\ &\qquad + 3S_{q'}(b; w, R)R(w^{-q})^{1/q}R(w^{q'})^{1/q'}\Lambda_q(f; w^{-1}, R) \\ &\quad \le CK^*(b, M^*f, w)(x_0) \quad (\text{by the } A_{q'} \text{ condition}) \\ &\quad \le C\gamma\alpha.\end{aligned}$$

And so $K_3 \le C\gamma\alpha\Sigma_F|I_n| \le C\gamma\alpha|I|$ also.

For K_4, again fix $R = R_x$. Then

$$\begin{aligned}
|R([b - I(b)] f_2)| &= \left| \frac{1}{|I|} \int_I R([b - I(b)] f_2) \right| \\
&= \left| \frac{1}{|I|} \int_I R([b - I(b)] f) - R([b - I(b)] f_1) \right| \\
&\leq \frac{1}{|I|} \int_I R(|b - I(b)| \, |f_1|) \\
&\quad + \left| \frac{1}{|I|} \int_I R([b - I(b)] f) - [b - I(b)] R(f) \right. \\
&\qquad\qquad \left. + [b - I(b)] R(f) \right| \\
&\leq \frac{1}{|I|} \int_I |b - I(b)| R(|f|) + \frac{1}{|I|} \int_I R(|b - I(b)| |f_1|) \\
&\quad + \frac{1}{|I|} \int_I |bR(f) - R(bf)| \\
&\leq I(|b - I(b)| M^* f) + I(M^* \{[b - I(b)] f_1\}) + I(T_b f).
\end{aligned}$$

These first two terms are bounded exactly like K_1 and K_2, while $I(T_b) \leq 2\alpha$ by (4). Hence,

$$|R([b - I(b)] f_2)| \leq C\gamma\alpha + 2\alpha$$

so that

$$K_4 \leq C\gamma\alpha \Sigma_F |I_n| + 2\alpha \Sigma_F |I_n| \leq C\gamma\alpha |I| + 2\alpha \Sigma_F |I_n|.$$

So we have

$$3\alpha \Sigma_F |I_n| \leq C\gamma\alpha |I| + 2\alpha \Sigma_F |I_n|,$$

or

$$\Sigma_F |I_n| \leq C\gamma |I|.$$

By (6),

$$\Sigma_F \lambda(I_n) \leq K\gamma^\delta \lambda(I).$$

Whatever the case with I, we have

$$\begin{aligned}
&\Sigma_F \lambda(I_n) \\
&\quad \leq \lambda([x: K^*(b, M^*f, w)(x) + K_r^*(b, f, \tilde{w})(x) > \gamma\alpha] \cap I) \\
&\qquad + K\gamma^\delta \lambda(I).
\end{aligned}$$

Summing over I gives (7).

Next define $\rho(\alpha) = \lambda([x: M^*(T_b f)(x) > \alpha])$. Suppose $x \notin \bigcup_n 2I_n^\alpha$ and I is any interval containing x. Then

$$\int_I T_b f = \int_{I \cap \bigcup_n I_n^\alpha} T_b f + \int_{I \sim \bigcup_n I_n^\alpha} T_b f \leq \int_{I \cap \bigcup_n I_n^\alpha} T_b f + \alpha|I|$$

by (3). But if $I \cap I_n^\alpha \neq \varnothing$, since $x \in I$ and $x \notin 2I_n^\alpha$, $|I| \geq \frac{1}{2}|I\alpha_n|$, and so $I_n^\alpha \subset 5I$. Hence, by (4),

$$\int_{I \cap \bigcup_n I_n^\alpha} T_b f \leq \sum_{I_n^\alpha \subset 5I} \int_{I_n^\alpha} T_b f \leq 2\alpha \sum_{I_n^\alpha \subset 5I} |I_n^\alpha| \leq 10\alpha|I|.$$

Thus $\int_I T_b f \leq 11\alpha|I|$, so that $M^*(T_b f)(x) \leq 11\alpha$. As a consequence,

$$[x: M^*(T_b f)(x) > 11\alpha] \subset \bigcup_n 2I_n^\alpha.$$

Now $\lambda(2I_n^\alpha) \leq C\lambda(I_n^\alpha)$, an upshot of the A_p condition [3], so

$$\rho(11\alpha) \leq \sum_n \lambda(2I_n^\alpha) \leq C\sum_n \lambda(I_n^\alpha) = C\tau(\alpha), \tag{8}$$

at least for $\alpha \geq \varepsilon$.

Now let $J_n = \int_\varepsilon^N \alpha^{p-1}\tau(\alpha)\,d\alpha$. Since $\tau(\alpha) \leq \lambda(T) < \infty$, J_N is finite. And using (7),

$$\begin{aligned}
J_N &= p\int_{\varepsilon/3}^{N/3} 3^p\alpha^{p-1}\tau(3\alpha)\,d\alpha \\
&\leq p3^p\int_0^\varepsilon \alpha^{p-1}\tau(3\alpha)\,d\alpha + p3^p\int_\varepsilon^{N/3}\alpha^{p-1}\tau(3\alpha)\,d\alpha \\
&\leq (3\varepsilon)^p\lambda(T) + p3^p\int_\varepsilon^{N/3}\alpha^{p-1}\sigma(\gamma\alpha) + p3^pK\gamma^\delta\int_\varepsilon^{N/3}\alpha^{p-1}\tau(\alpha)\,d\alpha \\
&\leq (3\alpha)^p\lambda(T) + p(3\gamma^{-1})^p\int_0^\infty \alpha^{p-1}\sigma(\alpha)\,d\alpha + 3^pK\gamma^\delta J_N.
\end{aligned}$$

Choose γ so small that $3^pK\gamma^\delta = 1/2$. This gives

$$J_N \leq 2(3\varepsilon)^p\lambda(T) + 2p(3\gamma^{-1})^p\int_0^\infty \alpha^{p-1}\sigma(\alpha)\,d\alpha.$$

This bound is independent of N so we can let $N \to \infty$ to get

$$p\int_\varepsilon^\infty \alpha^{p-1}\tau(\alpha)\,d\alpha \leq 2(3\varepsilon)^p\lambda(T) + 2p(3\gamma^{-1})^p\int_0^\infty \alpha^{p-1}\sigma(\alpha)\,d\alpha.$$

Using (8), we get

$$\begin{aligned}\int M^*(T_bf)^p(x)\lambda(x)\,dx &= p\int_0^\infty \alpha^{p-1}\rho(\alpha)\,d\alpha\\ &= 11^p p\int_0^\infty \alpha^{p-1}\rho(11\alpha)\,d\alpha\\ &\le (11\varepsilon)^p\lambda(T) + Cp\int_\varepsilon^\infty \alpha^{p-1}\tau(\alpha)\,d\alpha\\ &\le c_1\lambda(T)\varepsilon^p + c_2 p\int_0^\infty \alpha^{p-1}\sigma(\alpha)\,d\alpha.\end{aligned}$$

But

$$\begin{aligned}p\int_0^\infty \alpha^{p-1}\sigma(\alpha)\,d\alpha &= \int[K^*(b, M^*f, w) + K_r^*(b, f, \tilde{w})]^p\lambda\\ &\le C\left(\int (M^*f)^p\mu + \int |f|^p\mu\right)\\ &\qquad\qquad \text{(by (1) and (2))}\\ &\le C\int |f|^p\mu \quad \text{(by Muckenhoupt's Theorem [7]).}\end{aligned}$$

So the first half of the theorem will be proven provided we can show

$$\lambda(T)\varepsilon^p \le C\int |f|^p\mu. \tag{9}$$

For this,

$$\begin{aligned}T_bf(x) &= \sup_{x\in I}|[b - T(b)]I(f) - I([b - T(b)]f)|\\ &\le |b - T(b)|M^*f(x) + M^*([b - T(b)]f)(x),\end{aligned}$$

so

$$\varepsilon \le \frac{1}{2\pi}\int_T |b - T(b)|M^*f + \frac{1}{2\pi}\int_T M^*f + \frac{1}{2\pi}\int_T M^*([b - T(b)]f).$$

For the first of these,

$$\begin{aligned}\frac{1}{2\pi}\int_T |b - T(b)|M^*f &\le S_{q'}(b; w, T)\Lambda_q(M^*f, w^{-1}, T)\\ &\le K^*(b, M^*f, w)(x) \quad \text{for any } x \in T,\end{aligned}$$

while the other integral is

$$\begin{aligned}\frac{1}{2\pi}\int_T M^*([b-T(b)]f) &\le \left(\frac{1}{2\pi}\int_T M^*([b-T(b)]f)^r\right)^{1/r}\\ &\le C\left(\frac{1}{2\pi}\int_T |b-T(b)|^r|f|^r\right)^{1/r}\\ &\le CC_{rq'}(b;\tilde{w},T)\Lambda_{rq}(f;\tilde{w}^{-1},T)\\ &\le CK_r^*(b,f,\tilde{w})(x)\quad\text{for any } x\in T.\end{aligned}$$

So,

$$\begin{aligned}\lambda(T)\varepsilon^p &\le \int_T [K^*(b,M^*f,w)+CK_r^*(b,f,\tilde{w})]^p\lambda\\ &\le C\int |f|^p\mu,\end{aligned}$$

by (1), (2), and Muckenhoupt's Theorem again, and we have (9).

Conversely, if T_b: $L^p(\mu)\to L^p(\lambda)$ is bounded, then fix I and let $f=\chi_I$. We have

$$\begin{aligned}CI(\mu) &= C\frac{1}{|I|}\int_T |f|^p\mu\\ &\ge \frac{1}{|I|}\int_T (T_bf)^p\lambda\\ &\ge \frac{1}{|I|}\int_I (T_bf)^p\lambda\\ &\ge \frac{1}{|I|}\int_I |bI(f)-I(bf)|^p\lambda\\ &= \frac{1}{|I|}\int_I |b-I(b)|^p\lambda.\end{aligned}$$

Therefore, using Holder's inequality a couple of times,

$$\begin{aligned}I(|b-I(b)|) &= I\left(|b-I(b)|\lambda^{1/p}\,l^{-1/p}\right)\\ &\le I\left(|b-I(b)|^p\lambda\right)^{1/p}I\left(\lambda^{-p'/p}\right)^{1/p'}\\ &\le CI(\mu)^{1/p}I\left(\lambda^{-p'/p}\right)^{1/p'}I\left(\nu^{1/2}\nu^{-1/2}\right)^2\\ &\le CI(\mu)^{1/p}I\left(\lambda^{-p'/p}\right)^{1/p'}I(\nu)I\left(\lambda^{1/p}\mu^{-1/p}\right)\\ &\le CI(\mu)^{1/p}I\left(\mu^{-p/p}\right)^{1/p'}I\left(\lambda^{-p'/p}\right)^{1/p'}I(\lambda)^{1/p}I(\nu)\\ &\le CI(\nu)\quad\text{by the } A_p \text{ conditions.}\end{aligned}$$

So $b\in \mathrm{BMO}_\nu$.

3. Characterizations of weighted BMO

In this section we will apply a technique developed by De Francias in his proof of the Jones' Factorization Theorem [8] to the commutator T_b.

A weight $\nu \in A_1$ if $M^*\nu(x) \leq C\nu(x)$ almost everywhere.

THEOREM 3.1. *Let $w \in A_2$. Then $b \in \mathrm{BMO}_w$ if and only if there exists a $u \in L^1$ with $uw \in A_1$ and with*

$$|b(x) - I(b)| \leq Cu(x)w(x)I(u)^{-1}$$

for almost all x and every interval I containing x.

Proof. Suppose $b \in \mathrm{BMO}_w$. Both w and w^{-1} are A_2 weights, so by Theorem 2.1, the commutator

$$T_b\colon L^2(w) \to L^2(w^{-1})$$

is a bounded operator. Notice that T_b and M^* are sublinear operators. Consider the operator $w^{-1/2}T_b w^{-1/2}$. We have

$$\int |w^{-1/2}T_b w^{-1/2}f|^2 = \int T_b\left(w^{-1/2}f\right)^2 w^{-1} \leq C\int |f|^2,$$

so this operator is bounded on L^2. By Muckenhoupt's Theorem, M^* is bounded on $L^2(w^{-1})$, or $w^{-1/2}M^*w^{1/2}$ is bounded on L^2. Let

$$S^* = w^{-1/2}M^*w^{1/2} + w^{-1/2}T_b w^{-1/2}.$$

Then S^* is a bounded, positive, sublinear operator on L^2. $K \geq \|S^*\|$. Take $f \in L^2$ with $f \geq 0$. Define S^{*n} inductively by $S^{*n}f = S^*(S^{*n-1}f)$, and let

$$g = \sum_{n=0}^{\infty} K^{-n}S^{*n}f.$$

So $g \in L^2$ and

$$S^*g \leq \sum_{n=0}^{\infty} K^{-n}S^{*(n+1)}f = K(g - f) \leq Kg.$$

Thus

$$w^{-1/2}M^*w^{1/2}g \leq Kg \quad \text{and} \quad w^{-1/2}T_b w^{-1/2}g \leq Kg.$$

Let $u = w^{-1/2}g$. Since $w^{-1/2}$ and g are in L^2, $u \in L^1$. Also,

$$M^*(wu) = M^*(w^{1/2}g) \le Kw^{1/2}g = K(wu)$$

so that $wu \in A_1$.

Next, let $x \in I$. Then

$$\begin{aligned} |b(x) - I(b)|I(u) &\le |b(x)I(u) - I(bu)| + |I(bu) - I(b)I(u)| \\ &\le T_b u(x) + I(|bI(u) - I(bu)|) \\ &\le T_b u(x) + I(T_b u) \\ &= T_b(w^{-1/2}g)(x) = I(T_b w^{-1/2}g) \\ &\le Kw(x)u(x) + KI(wu) \\ &\le Kw(x)u(x) + KM^*(wu)(x) \\ &\le (K + K^2)w(x)u(x) \quad (\text{as } wu \in A_1). \end{aligned}$$

Conversely, fix an interval I. Then

$$I(|b - I(b)|) \le CI(u)^{-1}I(uw) \le CI(u^{-1})I(uw)$$

by Cauchy-Schwartz. But $uw \in A_1$, so for almost any $x \in I$,

$$I(uw) \le M^*(uw)(x) \le Cu(x)w(x),$$

and so $I(u^{-1})I(uw) \le CI(u^{-1}uw) = CI(w)$, and $b \in \mathrm{BMO}_w$.

In the first direction of the proof above, we could include the operator $w^{1/2}M^*w^{-1/2}$ in S^* also. This would given an additional condition, $w^{1/2}M^*w^{-1/2}g \le Kg$, or $u \in A_1$. So we would have both u and wu in A_1. Of course $w = (w \cdot u)/u$, so this is a Jones' Factorization of w. We have:

COROLLARY 3.2. *Let $w \in A_2$. Then $b \in \mathrm{BMO}_w$ if and only if there exists a Jones' Factorization of w, $w = u/v$ for u and v in A_1, for which*

$$|b(x) - I(b)| \le Cu(x)I(v)^{-1}$$

for almost all x and for every interval I containing x.

There is a Hilbert transform version of Theorem 2.1. Let $\tilde{f}$ denote the conjugate analytic function for f. Define the commutator S_b by

$$S_b g(x) = |b(x)\tilde{f}(x) - (bf)^{\sim}(x)|.$$

Then the result of [1] is:

THEOREM 3.3. *Let $w \in A_2$. Then $b \in \mathrm{BMO}_w$ if and only if S_b: $L^2(w) \to L^2(w^{-1})$ is a bounded operator.*

We will also make use of two simple lemmas which the reader can verify.

LEMMA 3.4. $\int_T g\tilde{h} = -\int_T \tilde{g}h$.

LEMMA 3.5. *Let* $w \in A_2$. *Then* $b \in \mathrm{BMO}_w$ *if and only if*

$$\left|\int_T b\tilde{f}\right| \leq C\int_T (|f| + |\tilde{f}|)w$$

for all f *with* f *and* $\tilde{f} \in L^1(w)$.

The corresponding Hilbert transform version of 2.1 is:

THEOREM 3.6. *Let* $w \in A_2$. *Then* $b \in \mathrm{BMO}_w$ *if and only if there exists a* $u \in A_1$ *for which*

$$\frac{S_b u}{wu} \in L^\infty.$$

Proof. Let $b \in \mathrm{BMO}_w$. By 3.3, S_b: $L^2(w) \to L^2(w^{-1})$ is a bounded operator. Since S_b is also sublinear, we can mimic the proof of 3.1. Let

$$T = w^{1/2}M^*w^{-1/2} + w^{-1/2}S_b w^{-1/2}.$$

Then we can find a nonnegative $g \in L^2$ with $Tg \leq Cg$. Put $u = w^{-1/2}g$. Then we have $M^*u \leq Cu$ and $S_b u \leq Cwu$. So $u \in A_1$ and $(S_b u)/wu \in L^\infty$.

Conversely, if $(S_b u)/wu \in L^\infty$ for some $u \in A_1$, then in particular, u is bounded below, so that $(u + i\tilde{u})^{-1}$ is analytic. Fix f with f and $\tilde{f}$ in $L^1(w)$. Let

$$g + i\tilde{g} = (f + i\tilde{f})(u + i\tilde{u})^{-1}.$$

Then

$$\begin{aligned}\left|\int b\tilde{f}\right| &= \left|\int b\,\mathrm{Im}[(g - i\tilde{g})(u + i\tilde{u})]\right| \\ &= \left|\int b(g\tilde{u} + \tilde{g}u)\right| \\ &= \left|\int b\tilde{u}g - (bu)^{\sim} g\right| \quad \text{(by 3.4)} \\ &\leq \int (S_b u)|g| \\ &\leq C\int |g|uw.\end{aligned}$$

But

$$g = \operatorname{Re}\frac{f + i\tilde{f}}{u + i\tilde{u}} = \frac{fu + \tilde{f}\tilde{u}}{u^2 + \tilde{u}^2},$$

and so

$$\left|\int b\tilde{f}\right| \leq C\int \frac{u^2}{u^2 + \tilde{u}^2}|f|w + \frac{|u\tilde{u}|}{u^2 + \tilde{u}^2}|\tilde{f}|w$$
$$\leq C\int (|f| + |\tilde{f}|)w.$$

By Lemma 3.5, $b \in \mathrm{BMO}_w$.

Of course, both theorems are valid when $w \equiv 1$, the unweighted BMO case. These are rather suprising characterizations of BMO.

COROLLARY 3.7. *The following conditions are all equivalent*:

(a) $b \in \mathrm{BMO}$.

(b) *There exists an A_1 weight u for which*

$$|b(x) - I(b)| \leq Cu(x)I(u)^{-1}$$

for almost all x and every interval I containing x.

(c) *There exists an A_1 weight w for which $(S_b w)/w \in L^\infty$.*

4. The Matrix Classes $\mathcal{A}_2$ and $\mathcal{M}_2$

Let W be a positive definite symmetric $n \times n$ matrix-valued function on the unit circle T. $W(x)$ induces a pointwise inner product on $\mathbf{C}^n$ given by $(f, g)_{W(x)} = (W(x)f, g)$, where the latter if the usual $\mathbf{C}^n$ dot product. We extend this to vector-valued functions:

$$(f, g)_W = \frac{1}{2\pi}\int_T (W(x)f(x), g(x))\,dx.$$

This inner product induces a Hilbert space $L^2(W)$.

The moving average operator A_h is given by

$$A_h f(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt.$$

The matrix weight W is in $\mathcal{A}_2$ if

$$\|A_h f\|_{L^2(W)} \leq C\|f\|_{L^2(W)}$$

with C independent of $h > 0$.

$\mathscr{M}_2$ is the matrix analog of Muckenhoupt's A_2 class. The maximal function is

$$\mathscr{M}_W: L^2(W) \to L^2(\mathbf{R})$$

given by

$$\mathscr{M}_W f(x) = \sup_{x \in I} (W(x) I(f), I(f))^{1/2}.$$

So the average is maximal with respect to the $L^2(W)$ norm. We say $W \in \mathscr{M}_2$ provided

$$\|\mathscr{M}_W f\|_{L^2} \leq C \|f\|_{L^2(W)}$$

for all $f \in L^2(W)$. Notice that in one dimension, $\mathscr{M}_2 = \mathscr{A}_2 = A_2$. For a further discussion of these classes see [2] where some of the material that follows has already appeared.

THEOREM 4.1. *Let $W = U^* \Lambda U$, where U is unitary, U^* is adjoint, Λ diagonal, and the diagonal entries of Λ, $\lambda_{kk} \in A_2$. If for each r and j,*

$$u_{rj} \in \mathrm{BMO}_{\sqrt{\lambda_{rr}\lambda_{kk}^{-1}}} \quad \text{for } k = 1, 2, \ldots, n,$$

then $W \in \mathscr{M}_2$.

This is an application of Theorem 2.1, with the proof virtually identical to the proof of Theorem 5.1 in [1], so we omit the proof.

Let's examine the converse of this theorem. In one sense, this depends on the diagonalization of W. For each x, $W(x)$ can be diagonalized, in a way that is unique only up to the order in which the eigenvalues appear. By mixing up that order as we vary x, we lose all control over the entries of U and Λ. We deal with that problem by restricting in turn to diagonalizations in which first U is nice, and then Λ. If we assume that W has a diagonalization $U^* \Lambda U$ in which U is continuous, then the λ_{kk} all belong to A_2.

THEOREM 4.2. *Let Λ be diagonal, U continuous and unitary. If $W = U^* \Lambda U$ belongs to $\mathscr{A}_2$, then the diagonal entries of Λ, $\lambda_{kk} \in A_2$.*

Since $\mathscr{A}_2$ is trivially contained in $\mathscr{M}_2$, this theorem applies to $\mathscr{M}_2$ as well. For the analysis of U, we will restrict our attention to two dimensions.

THEOREM 4.3. *Let U be a 2×2 unitary matrix and let*

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

Then if $W = U^*\Lambda U \in \mathscr{A}_2$,

$$|u_{ij}| \in \mathrm{BMO}_{\sqrt{\mu/\lambda}} \cap \mathrm{BMO}_{\sqrt{\lambda/\mu}} \quad \textit{for each } u_{ij}.$$

Before proving these, we need some preliminary results.

LEMMA 4.4. *Suppose the moving average* A_{2h} *is a bounded operator on* $L^2(W)$, *with norm* $\|A_{2h}\| = K$. *Then for any* $f \in L^2(W)$ *and* $x \in T$,

$$(A_h W(x) A_h f(x), A_h f(x)) \le 4K^2 A_h(Wf, f)(x).$$

Proof. Let χ be the characteristic function of $(x-h, x+h)$. Since

$$A_h f(x) = A_h(f\chi)(x)$$

and

$$A_h(Wf, f)(x) = A_h(Wf\chi, f\chi)(x),$$

we lose no generality in assuming that f is supported in $(x-h, x+h)$. Using that, we obtain

$$\begin{aligned}
&(A_h W(x) A_h f(x), A_h f(x)) \\
&\quad = \frac{1}{2h}\int_{x-h}^{x+h}\left(W(y)\frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt, \frac{1}{2h}\int_{x-h}^{x+h} f(s)\,ds\right) dy \\
&\quad = \frac{4}{2h}\int_{x-h}^{x+h}\left(W(y)\frac{1}{4h}\int_{y-2h}^{y+2h} f, \frac{1}{4h}\int_{y-2h}^{y+2h} f\right) dy \\
&\quad \le \frac{8\pi}{2h}\frac{1}{2\pi}\int_0^{2\pi}(W(y)A_{2h}f(y), A_{2h}f(y))\,dy \\
&\quad = \frac{8\pi}{2h}\|A_{2h}f\|^2_{L^2(W)} \\
&\quad \le 4K^2\frac{1}{2h}\int_{x-h}^{x+h}(Wf, f),
\end{aligned}$$

as asserted.

LEMMA 4.5. *If* $W \in \mathscr{A}_2$, *then so is* W^{-1}.

Proof. Since A_h is bounded on $L^2(W)$, so is its adjoint A_h^*, given by

$$A_h^* f = W^{-1}A_h(Wf).$$

Hence

$$\begin{aligned}\|A_h f\|^2_{L^2(W^{-1})} &= \frac{1}{2\pi}\int\left(W^{-1}A_h f, A_h f\right)\\ &= \frac{1}{2\pi}\int\left(WW^{-1}A_h WW^{-1}f, W^{-1}A_h WW^{-1}f\right)\\ &= \|A_h^* W^{-1}f\|^2_{L^2(W)}\\ &\le K^2\|W^{-1}f\|^2_{L^2(W)}\\ &= K^2\|f\|^2_{L^2(W^{-1})}.\end{aligned}$$

So indeed, $W^{-1} \in \mathcal{A}_2$.

Proof of Theorem 4.2. Let g be a scalar function and e_r a standard basis element. Then

$$\|A_h g\|_{L^2(w_{rr})} = \|A_h(ge_r)\|_{L^2(W)} \le K\|ge_r\|_{L^2(W)} = \|g\|_{L^2(w_{rr})}.$$

So the moving average operators are bounded on the scalar $L^2(w_{rr})$'s, and hence $w_{rr} \in A_2$, as is their sum, $\operatorname{tr} W = \operatorname{tr} \Lambda$. In particular, this trace is in L^1, and since each $\lambda_{kk} \ge 0$, each $\lambda_{kk} \in L^1$ also. By Lemma 4.5, so is each λ_{kk}^{-1}. We will show that $\lambda = \lambda_{11} \in A_2$, i.e., that

$$I(\lambda)I(\lambda^{-1}) \le C \quad \text{for all intervals } I.$$

Since λ and $\lambda^{-1} \in L^1$, this is trivial if I is large. We will restrict our attention to $I = [0, h]$ with h small. Transformation by a constant unitary matrix does not affect $\mathcal{A}_2$ so $U(0)WU(0)^* \in \mathcal{A}_2$ with the same norm-constant K. So we may assume that our unitary matrix U had $U(0) = I_n$, the identity matrix (δ_{ij}). By the continuity assumption, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

(1) $$|u_{ij}(x) - \delta_{ij}| < \varepsilon \quad \text{whenever } |x| < \delta.$$

Moreover, U is uniformly continuous, so this δ is independent of our normalization (setting $U(0) = I_n$).

Let A be the operator

$$Af = A_{h/2}f(h/2) = \frac{1}{h}\int_0^h f(t)\,dt.$$

By Lemma 4.4,

$$(AWAf, Af) \le 4K^2A(Wf, f) \quad \text{for all } f \in L^2(W).$$

Taking $f = \lambda^{-1}U^*e_1$, $(Wf, f) = \lambda^{-1}$, so

$$(AWAf, Af) \le 4K^2A\lambda^{-1}.$$

Let $\tilde{\Lambda}$ be Λ but with $0 = \tilde{\lambda}_{11}$ in place of λ, and put $\tilde{W} = U^*\tilde{\Lambda}U$. Since $\tilde{W}$ is positive definite, $(A\tilde{W}Af, Af) \ge 0$, and so

$$(AWAf, Af) - \left(A\tilde{W}Af, Af\right) \le 4K^2A\lambda^{-1}. \tag{2}$$

Now we express w_{ij} in terms of λ and $\tilde{w}_{ij}$ by

$$w_{ij} = \sum_{k=1}^{n} u_{kj}\bar{u}_{ki}\lambda_{kk} = \lambda u_{1j}\bar{u}_{1i} + \tilde{w}_{ij},$$

so that

$$\begin{aligned}(AWAf, Af) &= \sum_{r,s} A(w_{rs})A\bar{f}_rAf_s \\ &= \sum_{r,s} A(\lambda\bar{u}_{1r}u_{1s})A\bar{f}_rAf_s + \left(A\tilde{W}Af, Af\right),\end{aligned}$$

and (2) becomes

$$\sum_{r,s} A(\lambda\bar{u}_{1r}u_{1s})A\bar{f}_rAf_s \le 4K^2A\lambda^{-1}. \tag{3}$$

Notice that $f_r = \lambda^{-1}\bar{u}_{1r}$.

We now take $h \le \delta$. The terms in the sum of (3) are of four types:

Case 1. $r = s = 1$. Here

$$A\left(\lambda|u_{11}|^2\right)|A\left(\lambda^{-1}u_{11}\right)|^2 \ge (1-\varepsilon)^4 A(\lambda)A(\lambda^{-1})^2 \quad \text{by (1).}$$

Case 2. $r = 1, s \ne 1$. Now

$$|A(\lambda\bar{u}_{11}u_{1s})A\left(u_{11}\lambda^{-1}\right)A\left(\lambda^{-1}\bar{u}_{1s}\right)| \le \varepsilon^2 A(\lambda)A(\lambda^{-1})^2 \quad \text{also by (1).}$$

Case 3. $r \ne 1, s = 1$. This is identical to case 2.

Case 4. $r, s \ne 1$. The terms in this case are all bounded by $\varepsilon^4 A(\lambda)A(\lambda^{-1})^2$. So (3) gives

$$A(\lambda)A(\lambda^{-1})\left[(1-\varepsilon)^4 - 2(n-1)\varepsilon^2 - (n-1)^2\varepsilon^4\right] \le 4K^2.$$

By taking ε sufficiently small, we have

$$A(\lambda)A(\lambda^{-1}) \leq 8K^2,$$

which is the A_2 condition for $h = |I| \leq \delta$.

The proof of Theorem 4.3 proceeds by reducing the study of W to the study of the matrix

$$U^* \begin{pmatrix} \sqrt{\lambda/\mu} & 0 \\ f0 & \sqrt{\mu/\lambda} \end{pmatrix} U$$

where the weights are reciprocals. It is interesting that these reciprocal weight pairs are so fundamental to the problem. In current work in conjunction with Ron Kerman, we are finding that reciprocal weight pairs are fundamental in the study of wide ranges of operators.

LEMMA 4.6. *Let $W_i = U^*\Lambda_i U \in \mathscr{A}_2$ for $i = 1$ and 2, where U is unitary, Λ_i diagonal. Then*

$$W = U^*(\Lambda_1\Lambda_2)^{1/2}U \in \mathscr{A}_2 \quad \text{also.}$$

Proof. Let B_i be the logarithms of W_i and let T_z be the analytic family of operators

$$T_z = \exp\left[\frac{1}{2}zB_1 + \frac{1}{2}(1-z)B_2\right]A_h \exp\left[-\frac{1}{2}zB_1 - \frac{1}{2}(1-z)B_2\right].$$

By hypothesis, T_0 and T_1 are bounded operators on $L^2(I_n)$. One easily verifies that the conditions needed for complex interpolation hold, and so $T_{1/2}$ is a bounded operator. Since the B_i's commute

$$T_{1/2} = W^{1/2}A_h W^{-1/2},$$

and the boundedness of this operator on $L^2(I_n)$ is equivalent to $W \in \mathscr{A}_2$.

LEMMA 4.7. *If*

$$W = U^* \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} U \in \mathscr{A}_2,$$

then so is

$$\tilde{W} = U^* \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} U.$$

Proof. Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since J is constant and unitary, $J^*WJ \in \mathscr{A}_2$. But $J^*WJ = {}^t\tilde{W}$, the transpose of $\tilde{W}$. Since $(\tilde{W}f, f) = ({}^t\tilde{W}\bar{f}, \bar{f})$, $\tilde{W}$ must be in $\mathscr{A}_2$ also.

LEMMA 4.8. *If*

$$U^*\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}U \in \mathscr{A}_2,$$

then so is

$$U^*\begin{pmatrix} \sqrt{\lambda/\mu} & 0 \\ 0 & \sqrt{\mu/\lambda} \end{pmatrix}U.$$

Proof. By 4.7,

$$U^*\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}U \in \mathscr{A}_2.$$

By 4.5, its inverse

$$U^*\begin{pmatrix} \mu^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix}U \in \mathscr{A}_2.$$

This lemma now follows from 4.6.

Proof of Theorem 4.3. By the previous lemma, it suffices to study $\mathscr{A}_2$ matrices of the form

$$U^*\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}U.$$

Let

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $|a| = |d|$ and $|b| = |c|$, we must show $|a|, |b| \in \mathrm{BMO}_{\lambda^{\pm 1}}$.

Fix an interval I and let A be the operator

$$Af = \frac{1}{|I|}\int_I f.$$

By Lemma 4.4,

$$(AWAf, Af) \leq C(Wf, f)$$

for some constant C independent of f and I. Set $f = W^{-1}e_1$. Since $(Wf, f) = (e_1, f) = f_1$, we have

$$(1) \qquad Aw_{11}(Af_1)^2 + Aw_{22}|Af_2|^2 + 2\,\mathrm{Re}\,Aw_{12}Af_1Af_2 \leq CAf_1.$$

Now $f_2 = a\bar{b}(\lambda^{-1} - \lambda) = -\bar{w}_{12}$ and $w_{22} = f_1$, so (1) is

$$Aw_{11}(Af_1)^2 - Af_1|Af_2|^2 \leq CAf_1,$$

or $Aw_{11}Af_1 - |Af_2|^2 \leq C$. Since

$$f_1 = |a|^2\lambda^{-1} + |b|^2\lambda \quad \text{and} \quad w_{11} = |a|^2\lambda + |b|^2\lambda^{-1},$$

this says

$$L_1 + L_2 + L_3 \leq C$$

where

$$L_1 = A(|a|^2\lambda)A(|b|^2\lambda) - |A(a\bar{b}\lambda)|^2,$$
$$L_2 = A(|a|^2\lambda^{-1})A(|b|^2\lambda^{-1}) - |A(a\bar{b}\lambda^{-1})|^2,$$
$$L_3 = A(|a|^2\lambda)A(|a|^2\lambda^{-1}) + A(|b|^2\lambda)A(|b|^2\lambda^{-1}) - 2\,\mathrm{Re}\,A(a\bar{b}\lambda^{-1})A(\bar{a}b\lambda).$$

By Cauchy-Schwartz, L_1 and $L_2 \geq 0$. For L_3,

$$|2\,\mathrm{Re}\,A(a\bar{b}\lambda^{-1})A(\bar{a}b\lambda)| \leq 2\left[A(|a|^2\lambda^{-1})A(|b|^2\lambda^{-1})A(|a|^2\lambda)A(|b|^2\lambda)\right]^{1/2}$$

so that

$$L_3 \geq \left[A(|a|^2\lambda)^{1/2}A(|a|^2\lambda^{-1})^{1/2} - A(|b|^2\lambda)^{1/2}A(|b|^2\lambda^{-1})^{1/2}\right]^2 \geq 0.$$

So each $L_i \leq C$. We will use L_1 to show that $|b| \in \mathrm{BMO}_{\lambda^{-1}}$. A similar argument with L_2 would give $|b| \in \mathrm{BMO}_\lambda$, and since L_1 and L_2 are symmetric in a and b, the same holds for $|a|$.

Using Cauchy-Schwartz again,

$$|A(a\bar{b}\lambda)|^2 \leq A(|a|^2|b|\lambda)A(|b|\lambda)$$

so $L_1 \le C$ gives

$$A(|a|^2\lambda)A(|b|^2\lambda) - A(|a|^2|b|\lambda)A(|b|\lambda) \le C. \tag{2}$$

Also,

$$A(|b|^3\lambda)A(|b|) - A(|b|^2\lambda)^2 \ge 0.$$

Adding this to (2) and using the fact that $|a|^2 + |b|^2 = 1$ yields

$$A(\lambda)A(|b|^2\lambda) - A(|b|^2\lambda) \le C. \tag{3}$$

Let $u = |b|$ and introduce the inner product

$$(f, g) = \frac{1}{|I|}\int_I f\bar{g}\lambda,$$

with the corresponding norm $\|\cdot\|$. In this notation, (3) is

$$\begin{aligned} C &\ge \|1\|^2\|u\|^2 - (1, u)^2 \\ &= (\|1\| \cdot \|u\| + (1, u))(\|1\| \cdot \|u\| - (1, u)) \\ &\ge \|1\| \cdot \|u\|(\|1\| \cdot \|u\| - (1, u)) \\ &= \frac{1}{2}\big\| \|1\|u - \|u\| \big\|^2 \\ &= \frac{1}{2}\|1\|^2\left\| u - \frac{\|u\|}{\|1\|}\right\|^2. \end{aligned}$$

Let $c_I = \|u\|/\|1\|$. We have shown that

$$I(\lambda)\frac{1}{|I|}\int_I |u - c_I|^2\lambda \le 2C.$$

Finally,

$$\begin{aligned} \frac{1}{|I|}\int |u - c_I| &\le \left(\frac{1}{|I|}\int_I |u - c_I|^2\lambda\right)^{1/2}\left(\frac{1}{|I|}\int_I \lambda^{-1}\right)^{1/2}\left(\frac{1}{|I|}\int_I \lambda^{1/2}\lambda^{-1/2}\right) \\ &\le \left(\frac{1}{|I|}\int_I |u - c_I|^2\lambda\right)^{1/2} I(\lambda)^{1/2} I(\lambda^{-1}) \\ &\le \sqrt{2C}\, I(\lambda^{-1}), \end{aligned}$$

by (4), and hence $u = |b| \in \mathrm{BMO}_{\lambda^{-1}}$.

References

1. S. Bloom, *A commutator theorem and weighted BMO*, Trans. Amer. Math. Soc., vol. 292 (1985), pp. 103–122.
2. ______, *Weighted norm inequalities for vector-valued functions*, Ph.D. dissertation, Washington University, St. Louis, Missouri, 1981.
3. R.R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia. Math., vol. 51 (1974), pp. 241–250.
4. R.R. Coifman, Y. Meyer and E.M. Stein, *Some new function spaces and their applications to harmonic analysis*, J. Funct. Anal., vol. 62 (1985), pp. 304–335.
5. R.R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., vol. 103 (1976), 611–635.
6. P. Jones, private communication.
7. B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., vol. 165 (1972), 207–226.
8. G. Weiss, *Various remarks concerning Rubio de Francias' proof of Peter Jones' Factorization theorem and some applications of the ideas in this proof*, preprint.

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