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UNIVERSALLY MEASURABLE SETS OF FINITELY ADDITIVE PRODUCT MEASURES

BY

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Introduction

In Problem 11 of the Scottish Book (see Mauldin [3]) Banach and Ulam posed the following problem: Let μ be a finitely additive measure on the power set of the integers which gives each singleton measure zero. Let $\mathbf{Z}^D$ denote some product space where each factor space is a copy of the integers. Let μ^D denote the finitely additive product measure on the algebra generated by the cylinder sets of $\mathbf{Z}^D$ which satisfies

$$\mu^D\Big(\prod_{d\in D} A_d\Big) = \prod_{d\in D} \mu(A_d)$$

for any cylinder ΠA_d ($A_d \neq \mathbf{Z}$ for at most finitely many $d \in D$). A set $E \subseteq \mathbf{Z}^D$ is said to be μ^D-measurable iff there is only one finitely additive probability measure on the algebra generated by E and the cylinder sets which extends μ^D. If a set E is μ^D-measurable for every finitely additive probability measure which gives singletons measure zero, then E is said to be universally product measurable. Banach and Ulam asked the question: are the sets "all pairs of relatively prime integers" or "all sequences of integers converging to infinity" universally product measurable Later, Mauldin asked whether there is characterization of universally product measurable sets. See Mauldin [3]. In this paper it is shown that

(1) $E \subseteq \mathbf{Z}^2$ is universally product measurable iff E is a finite union of rectangles.

(2) For $2 < n < \infty$, $E \subseteq \mathbf{Z}^n$ is universally product measurable iff there exist a finite union of cylinders T and a finite set F of $\mathbf{Z}$ so that

$$T \Delta E \subseteq \bigcup_{j=1}^{n} p_j^{-1}(F)$$

where $p_j: \mathbf{Z}^n \to \mathbf{Z}$ is the jth coordinate projection.

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(3) Let $\mathbf{N}$ denote the positive integers and $\mathbf{Z}^+$ the nonnegative integers. A subset of $\mathbf{Z}^N$ is universally product measurable iff for every $\varepsilon > 0$, there exists a finite subset F of $\mathbf{Z}$ and a finite partion $V = \{A_1, A_2, \dots, A_k\}$ of $\mathbf{Z} \setminus F$, and an $n \in \mathbf{N}$ so that if $i\colon V \to \mathbf{Z}^+$ with $\Sigma i(A_j) = N$, then

$$\frac{\text{cardinality of } \{T \in V(i) | T \cap E \neq \varnothing \neq T \setminus E\}}{(i)} < \epsilon$$

where (i) is the multinomial coefficient

$$\frac{N!}{i(A_1)!i(A_2)!\dots i(A_k)!}$$

and $V(i)$ is the set of all $T = A_{j_1} \times A_{j_2} \times \dots A_{j_N} \times \mathbf{Z} \times \mathbf{Z} \times \dots$ such that for each $s = 1, 2, \dots, k$,

$$\text{cardinality of } \{m | A_{j_m} = A_s\} = i(A_s).$$

In each case a more general theorem is developed and the idea of universally product measurable is compared with the idea of naturally measurable sets in the work of E. Granirer [2] and the work of Dubins and Margolies [1].

1. Preliminary ideas and finite products

Let X be a nonempty set and $\mathbf{B}$ be an algebra of subsets of X. If $E \subseteq X$, then the $\mathbf{B}$-boundary of E is defined to be the set

$$\partial_{\mathbf{B}} E := \{A \in \mathbf{B} : \exists C \in \mathbf{B} \ni C \subseteq E \subseteq C \cup A\}.$$

The set $\partial_{\mathbf{B}} E$ is obviously closed under finite intersections and the empty set $\varnothing$ is in $\partial_{\mathbf{B}} E$ if and only if $E \in \mathbf{B}$.

Let $M(\mathbf{B})$ denote the set of finitely additive probability measures on $\mathbf{B}$. If J is an ideal in $\mathbf{B}$, then set

$$M(\mathbf{B}|J) := \{\nu \in M(\mathbf{B}) : \nu(A) = 0 \text{ if } A \in J\}.$$

It is always assumed that $\varnothing \in J$ and $X \notin J$ wherever J appears. If $K \subseteq M(\mathbf{B})$, then a subset E of X is said to be K-measurable iff for each $\mu \in K$, there is only one extension of μ in $M(\mathbf{B}_E)$ where $\mathbf{B}_E$ is the algebra generated by $\mathbf{B} \cup \{E\}$.

PROPOSITION 1. *Suppose that $K \subseteq M(\mathbf{B})$ is weak *-closed and $E \subseteq X$. Then E is K-measurable iff*

$$\inf_{A \in \partial E} \sup_{\mu \in K} \mu(A) = 0.$$

Proof. By the well known result of Horn-Tarski, for any $\mu \in M(\mathbf{B})$ and $E \subseteq X$, E is μ-measurable iff $\bar{\mu}E = \underline{\mu}E$ where

$$\bar{\mu}E = \sup\{\mu A \colon A \in \mathbf{B} \text{ and } E \subseteq A\}$$

and

$$\underline{\mu}E = \inf\{\mu A \colon A \in \mathbf{B} \text{ and } A \subseteq E\}.$$

Easily,

$$\bar{\mu}E - \underline{\mu}E = \inf_{A \in \partial_{\mathbf{B}} E} \mu A.$$

Suppose that E is K-measurable. Then for each $\varepsilon > 0$ and each $\mu \in K$, there is an $A_\mu \in \partial_{\mathbf{B}} E$ satisfying $\mu(A_\mu) < \varepsilon$. Since the collection of sets

$$\big\{\{\nu \in M(\mathbf{B}) \colon \nu A_\mu < \varepsilon\} \colon \mu \in K\big\}$$

is a weak* open cover of the compact set K, there is a finite sequence $A_1, A_2, \dots, A_n \in \partial_{\mathbf{B}} E$ such that for each $\mu \in K$ there is an A_i satisfying $\mu A_i < \varepsilon$. Thus $\mu(\bigcap_i A_i) < \varepsilon$. For all $\mu \in K$ and $\bigcap_i A_i \in \partial_{\mathbf{B}} E$; i.e.,

$$\inf_{A \in \partial_{\mathbf{B}} E} \sup_{\mu \in K} \mu A = 0.$$

For the converse, it is sufficient to note that

$$\inf_{A \in \partial_{\mathbf{B}} E} \sup_{\mu \in K} \mu A \geq \sup_{\mu \in K} \inf_{A \in \partial_{\mathbf{B}} E} \mu A, \qquad \text{Q.E.D.}$$

Let $\mathbf{B}$ and X be as before and let D denote a nonempty set. Let

$$X^* := \prod_{d \in D} X_d$$

where $X_d = X$ for each $d \in D$. If F is a finite subset of D and for $d \in D$, $A_d \in \mathbf{B}$, then $[A_d]_{d \in F}$ denotes the cylinder $\prod_{d \in D} C_d$ where $C_d = A_d$ if $d \in F$ and $C_d = X$ otherwise. If D is finite, we will often use the standard notation $A_1 \times A_2 \times A_3 \times \cdots$. Let $\mathbf{B}^*$ denote the set algebra on X^* generated by the cylinder sets, and if J is an ideal in $\mathbf{B}$, then let J^* denote the ideal in $\mathbf{B}^*$ generated by the set $\{[A_{d_0}] \colon d_0 \in D, A_{d_0} \in J\}$. Also if $m \in M(\mathbf{B})$, let m^* denote the unique measure on $\mathbf{B}^*$ satisfying

$$m^*([A_d]_{d \in F}) = \prod_{d \in F} m(A_d).$$

As is well known the map $*: M(\mathbf{B}) \to M(\mathbf{B}^*)$ defined by $*(m) = m^*$ is weak* to weak* continuous. Thus the set

$$M(\mathbf{B}|J)^* := \{m^*: m \in M(\mathbf{B}|J)\}$$

is weak*-compact. Note that $M(\mathbf{B}^*|J^*)$ does not equal $M(\mathbf{B}|J)^*$ in general.

The question of universal product measurability will be settled for finite products.

THEOREM 2. *Let n be a positive integer. Then $E \subseteq \mathbf{Z}^n$ is universally product measurable iff there is a $T \in \mathbf{B}^*, T \subseteq E$, such that $E \setminus T$ can be covered by finitely many rectangles each of which has at least one side finite. If $n = 2$, then E is universally measurable iff $E \in \mathbf{B}^*$.*

Proof. Let $X = \mathbf{Z}$, let J be the set of finite subsets of $\mathbf{Z}$, $\mathbf{B}$ the power set of $\mathbf{Z}$, and $K = M(\mathbf{B}|J)^*$. Then E ($\subseteq \mathbf{Z}^n = X^*$) being universally product measurable is the same as being K-measurable.

Suppose E is K-measurable. Since K is weak*-compact, Proposition 1 implies there exists $A \in \partial_{\mathbf{B}} E$ so that $\sup_{\nu \in K} \nu(A) < n^{-n}$. Since $A \in \mathbf{B}^*$, A is the finite union of cylinder sets. We will show that if

$$T = C_1 \times C_2 \times \cdots \times C_n \subseteq A,$$

then one of $C_1, \ldots, C_n$ is a finite set.

Suppose not. Since each of $C_1, \ldots, C_n$ is an infinite, there exist $m_1, \ldots, m_n \in M(\mathbf{B}|J)$ so that $m_i(C_i) = 1$ for each i. Define

$$m = \frac{m_1 + \cdots + m_n}{n}.$$

Then $m \in M(\mathbf{B}|J)$ and

$$m^*(T) = \prod m(A_i) \geq \prod \frac{1}{n} m_i(A_i) = \left(\frac{1}{n}\right)^n$$

which contradicts $m^*(T) < (1/n)^n$.

The converse is obvious. Also the statement of the theorem, concerning $\mathbf{Z}^2$, follows from the fact that an arbitrary subset of a rectangle, which has one side finite, is a finite union of rectangles.

Theorem 2 can obviously be generalized to the following statement: If X is a set, $\mathbf{B}$ an algebra of subsets of X, J an ideal in $\mathbf{B}$, and $n \in \mathbf{N}$, then $E \subseteq X^n$ is $M(\mathbf{B}|J)^*$-measurable iff there exist $T \in \mathbf{B}^*$ and $U \in J^*$ so that $E \Delta T \subseteq U$.

2. Infinite products

Unlike the case of finite product spaces, we gain totally new universally product measurable sets in infinite product spaces—that is, we find sets which differ from **B*** sets by more than a set of universal product measure zero. In order to find a characterization for these sets we shall begin in a more abstract setting.

As before, let X, **B**, and J be a set, an algebra and an ideal respectively. A set of one to one transformations G is said to be a $\mathbf{B}|J$-transformation group whenever the following six conditions hold: Let T, T_1, T_2 be elements of G.

(1) There exist A_1 and $A_2 \in J$ so that

$$\mathrm{Domain}(T) = X \setminus A_1 \quad \text{and} \quad \mathrm{Range}(T) = X \setminus A_2.$$

(2) If $A \in \mathbf{B} \setminus J$, then $T(A) := \{T(x) | x \in \mathrm{Domain}(T) \cap A\} \in \mathbf{B} \setminus J$.
(3) If $A \in J$, then $T(A) \in J$.
(4) $T^{-1} \in G$.
(5) $T_1 T_2 := T_1 \circ T_2|_{T_2^{-1}(\mathrm{Domain}\ T_1)} \in G$.
(6) For each finite **B**-partition V of X and each finite subset H of G, there exists a finite **B**-partition, W, of X which refines V and so that for each $T \in H$, and $A \in W$, there exist $C \in W$ so that for $T(A)$, $C \in J$.

Condition (6) should be compared with the idea of an "adequate tiling" in the setting of amenable groups; see Dubins and Margolies [1]. Note also if each $T \in G$ has finite order, then (6) is satisfied.

For each finite **B**-partition V of X, a subgroup G_V of G is defined by

$$G_V := \{T \in G | \text{ if } A \in V, \text{ then there exists } C \in V \text{ so that } T(A) \Delta C \in J\}.$$

This subgroup induces a natural equivalence class of orbits on V, $[\ \ ]_V$, by

$$[A]_V := \{C \in V | \text{there exists } T \in G_V \text{ so that } T(C) \Delta A \in J\}.$$

A subset $\{A_i\}_{i=1}^k$ of V is said to be a cross section of V if

$$\bigcup [A_i] = V \quad \text{and} \quad [A_i] \cap [A_j] = \varnothing \text{ if } i \neq j.$$

If $A \in \mathbf{B}$ and $E \subseteq X$, then A is said to border E if $A \cap E \neq \varnothing \neq A \setminus E$. Given V, we define the set

$$\mathrm{border}_V(E) := \{A \in V | A \text{ borders } E\}$$

and also given $A \in V$, a partial border is defined by

$$\operatorname{border}_{V,A}(E) := \{C \in [A]_V | C \text{ borders } E\}.$$

Let $M(\mathbf{B}|J|G)$ denote the set

$$\{m \in M(B|J)\colon \text{if } T \in G, \text{ and } A \in \mathbf{B},\ m(T(A)) = (A)\}.$$

THEOREM 3. *Suppose G is a $\mathbf{B}|J$-transformation group. A subset E of X is $M(\mathbf{B}|J|G)$-measurable iff for each $\varepsilon > 0$, there exists a finite $\mathbf{B}$-partition V of X so that*

$$\frac{\#\operatorname{border}_{V,A}(E)}{\#[A]_V} < \varepsilon$$

for each $A \in V \setminus J$, (where $\#C$ is the cardinality of C).

Proof. First we show sufficiency. Suppose $m \in M(\mathbf{B}|J|G)$, $\varepsilon > 0$, and V is a $\mathbf{B}$-partition so that for all $A \in V \setminus J$,

$$\#\operatorname{border}_{V,A}(E) < \varepsilon \cdot \#[A]_V.$$

Now, $\bigcup \operatorname{border}_V E \in \partial_{\mathbf{B}} E$. Thus we only need show $m(\bigcup \operatorname{border}_V E) < \varepsilon$. Let $A_1, A_2, \ldots, A_k$ be a cross section of V. Since $m(A) = 0$ for any $A \in J$, we have

$$\begin{aligned} m\left(\bigcup \operatorname{border}_V E\right) &= \sum_i \frac{\#\operatorname{border}_{V,A_i} E}{\#[A_i]_V} \cdot \#[A_i]_V \cdot m(A_i) \\ &< \sum \varepsilon \cdot \#[A_i]_V \cdot m(A_i) \\ &= \varepsilon. \end{aligned}$$

Now, in order to show necessity, suppose E is $M(\mathbf{B}|J|G)$-measurable yet there is an $\varepsilon > 0$ so that

$$\sup_{A \in V \setminus J} \frac{\#\operatorname{border}_{V,A} E}{\#[A]_V} \geq \varepsilon \quad \text{for all } V$$

By Proposition 1, there is an $L \in \partial_{\mathbf{B}} E$ so that

$$\operatorname*{supremum}_{m \in M(\mathbf{B}|J|G)} m(L) < \varepsilon.$$

Let Q be the set of all finite $\mathbf{B}$-partitions of X and let R be the set of all finite subsets of G. Then $D := Q \times R$ is a directed set when given the direction

$(V, F) \le (W, K)$ iff W refines V and F is a subset of K. Since each element (V, F) of D has only finitely many predecessors, using condition (6) on G, one may choose a net $\{W_{(V,F)}\}_{(V,F) \in D}$ so that:

(i) $W_{(V,F)} \in Q$ for each (V, F).
(ii) $W_{(V,F)}$ refines both V and $\{L, X \setminus L\}$.
(iii) $W_{(V,F)}$ refines $W_{(V_1, F_1)}$ if $(V_1, F_1) \le (V, F)$.
(iv) If $A \in W_{(V,F)}$ and $T \in F$, then there exist $C \in W_{(V,F)}$ so that $C \Delta T(A) \in J$.

Now for each $(V, F) \in D$ choose $m \in M(\mathbf{B}|J)$ so that

(†) $m(C) = m(A)$ if $A \in W_{(V,F)}$ and $C \in [A]_{W(V,F)}$, and
(††) $m(\bigcup\{A \in W_{(V,F)} | \#\, \text{border}_{W,A} E \ge \varepsilon \cdot \#[A]_W\}) = 1$.

Denote this choice by $m_{(V,F)}$. Since $M(\mathbf{B}|J)$ is compact, one can choose a subnet $\{m_{(V,F)_d}\}_{d \in D'}$ which converges to some element, say μ, of $M(\mathbf{B}|J)$.

We need to show that μ is invariant under G. Let $T \in G$ and $A \in \mathbf{B}$. Abbreviate $m_{(V,F)_d}$ by m_d, and $W_{(V,F)_d}$ by W_d. There exist a $b \in D'$ so that if $d > b$, then $(V, F)_d$ refines $(\{A, X \setminus A\}, \{T\})$. Further, by (iv) and (†),

$$m_d(T(A)) = \sum_{\substack{C \subseteq A,\\ C \in W_d}} m_d(T(C)) = \sum_{\substack{C \subseteq A,\\ C \in W_d}} m_d(A) = m_d(A)$$

By taking limits, $\mu(T(A)) = \mu(A)$, i.e., $\mu \in M(\mathbf{B}|J|G)$. This allows us to conclude $\mu(L) < \varepsilon$.

We shall now calculate $\mu(L)$ directly. Temporarily fix $d \in D'$ and let $A_1, \ldots, A_k$ be a cross section of W_d. Using the fact that $\bigcup \text{border}_{W_d} E \subseteq L$, we have

$$\begin{aligned} m_d(L) &\ge \sum_{A_i \notin J} \frac{\#\, \text{border}_{W_d, A_i} E}{\#[A_i]_{W_d}} \cdot \#[A_i]_{W_d} \cdot m_d(A_i) \\ &\ge \varepsilon \cdot m_d\Big(\bigcup\{A \in W_d | \#\, \text{border}_{W_d, A} E \ge \varepsilon \cdot \#[A]_{W_d}\}\Big) \\ &= \varepsilon \end{aligned}$$

by (††). Thus $\mu(L) \ge \varepsilon$, a contradiction. Q.E.D.

At this point we should compare this with results on naturally measurable subsets of amenable groups. If H is an amenable group, then $E \subseteq H$ is said to be naturally measurable if $\nu(E) = \mu(E)$ for any $\nu, \mu \in M(P(H), \{\varnothing\}, H')$ where $P(H)$ is the power set of H, and H' is the group of left translates on H. The idea is to restrict elements of $M(P(H), \{\varnothing\}, H')$ down to the naturally measurable sets in order to gain a measure totally defined by the group's action on itself. In general, the set of naturally measurable sets is not an algebra, but rather a linear space. See Dubins and Margolies [1]. In the setting of Theorem 3, X need not be a group and the problem is one of extension not restriction. However, if for example, X is an accessible group, $G = X'$, and $\mathbf{B}$ is an algebra contained in the naturally measurable sets, then $\mathbf{B}'$ defined as the set of $M(\mathbf{B}, \{\varnothing\}, X')$-measurable sets will also be a subset of the naturally measurable sets. However, since $\mathbf{B}'$ will be an algebra, it can never be all of the naturally measurable sets.

We can now use Theorem 3 to establish our main characterization of universally product measurable sets. Let D be an infinite set, and recall that X^* denotes X^D. If F is a finite subset of D and $\psi\colon F \to \mathbf{B}$, then $\langle \psi \rangle$ denotes the cylinder set $\prod_{j \in D} A_j$ where $A_j = \psi(j)$ if $j \in F$ and $A_j = X$ otherwise.

THEOREM 4. *Let $X, \mathbf{B}, J$ be as before and let D denote an infinite set. A subset E of X^* is $M(\mathbf{B}|J)^*$-measurable iff for each $\varepsilon > 0$ there is a finite subset F of D, a set $A \in J$, and a finite partition V of $X \setminus A$, so that for every function $i\colon V \to \mathbf{Z}^+$ satisfying $\Sigma i(C) = \#F$, we have*

$$\frac{\#\{\psi\colon F \to V\colon \langle\psi\rangle \text{ borders } E \text{ and } \#\psi^{-1}(C) = i(C) \text{ for all } C \in V\}}{(i)} < \varepsilon$$

where (i) is the multinomial coefficient $(\#F)!/\prod[i(C)!]$.

Proof. For any finite set F of D, and any permutation $\pi\colon F \to F$, we define $T_\pi\colon X^* \to X^*$ by $T_\pi(\langle x_d \rangle_{d \in D}) = \langle y_d \rangle_{d \in D}$ where $y_d = x_{\pi(d)}$ if $d \in F$ and $y_d = x_d$ if $d \notin F$. Let G be the set of all such T_π. Then G is a $\mathbf{B}^*|J^*$-transformation group on X^*. But it is a well known result that the set of extreme points of the set $M(\mathbf{B}^*|J^*|G)$ is $M(\mathbf{B}|J)^*$, and that for each $\mu \in M(\mathbf{B}^*|J^*|G)$, there is a countably additive probability measure ξ on the weak* Baire sets of $M(\mathbf{B}|J)$ so that

$$\mu(A) = \int m^*(A)\, d\xi(m) \quad \text{for all } A \in \mathbf{B}^*$$

See Phelps [4]. This integral representation allows us to note that a set E is $M(\mathbf{B}|J)^*$-measurable iff it is $M(\mathbf{B}^*|J^*|G)$-measurable. We only need to check that the characterization of $M(\mathbf{B}^*|J^*|G)$-measurability given in Theorem 3 is the same as the characterization stated in Theorem 4.

First let's consider a boundary defined as follows: Given a finite subset F of D, $A \in J$, and a finite partition V of $X \setminus A$ into sets in $\mathbf{B} \setminus J$, then

$$W = W(F, V) := \{\langle\psi\rangle | \psi\colon F \to V \cup \{A\}\}$$

is a $\mathbf{B}^*$-partition of X^*. Further if $\psi: F \to V$ (i.e., $\langle\psi\rangle \notin J^*$), then

$$[\langle\psi\rangle]_W = \{\langle\phi\rangle \mid \phi: F \to V \text{ and } \#\phi^{-1}(C) = \#\psi^{-1}(C) \text{ for } C \in V\}.$$

Defining $i: V \to \mathbf{Z}^+$ by $i(C) = \#\psi^{-1}(C)$, we see that $\#[\langle\psi\rangle]_W = (i)$. Thus

$$\frac{\#\{\langle\phi\rangle \in [\langle\psi\rangle]_W \mid \langle\phi\rangle \text{ borders } E\}}{\#[\langle\psi\rangle]_W}$$

is the same number as

$$\frac{\#\{\phi: F \to V \mid \langle\phi\rangle \text{ borders } E \text{ and } \#\phi^{-1}(C) = i(C) \text{ for all } C\}}{(i)}.$$

This easily establishes sufficiency.

For necessity we only need to show that for each finite $\mathbf{B}^*$-partition U of X^*, there is a finite subset F of D, an $A \in J$, and a finite partition V of $X \setminus A$ into $\mathbf{B} \setminus J$ sets so that if $W = W(F, V)$, then

$$(*) \qquad \sup_{\substack{C \in W \\ C \notin J^*}} \frac{\#\ \text{border}_{W,C}\, E}{\#[C]_W} \le \sup_{\substack{C \in U \\ C \notin J^*}} \frac{\#\ \text{border}_{U,C}\, E}{\#[C]_U}$$

Since elements of U are finite unions of cylinders, one can find a finite subset F of D and a finite $\mathbf{B}$-partition V' of X so that $W(V', F)$ refines U. Let

$$V = \{C \in V' \mid C \notin J\} \quad \text{and} \quad A = \bigcup\{C \in V' \mid C \in J\}.$$

Then $W = W(V, F)$ is our candidate. In order to establish the inequality (*) for this choice, we first make the following observations:

(a) Let $G(F)$ be the subgroup of G consisting of all T_π, where $\pi: F \to F$. Then $G(F)$ is a subgroup of G_W and $G(F)$ generates the same orbit structure on W as G_W.

(b) For every $T \in G_U$, there exists a $\pi: F \to F$ so that T_π has the same action on U as T has.

Let $A_1, A_2, \dots, A_k$ be a cross section of the elements of U which are not in J^* and let $C = \langle\psi\rangle$ for some $\psi: F \to V$. Since $C \notin J^*$, we have

$$\bigcup[C]_W \subseteq \bigcup_i [A_i]_U.$$

Now,

$$\text{(c)} \qquad \#[C]_W = \sum_i \#\{D \in [C]_W \mid D \subseteq A_i\}\#[A_i]$$

since the number of elements of $[C]_W$ contained in D_1 is the same as the number in D_2 if $D_1 \in [D_2]_U$. However if $B \in W$, $B \subseteq C \in U$, and B borders E, then C must also border E. Thus

$$\#\,\mathrm{border}_{W,C}(E) \leq \sum_i \#\{D \in [C]_W \,|\, D \subseteq A_i\}\#\,\mathrm{border}_{U,A_i}(E)$$

which implies

$$\frac{\#\,\mathrm{border}_{W,C}(E)}{\#[C]_W} \leq \sum_i \frac{\#\{D \in [C]_W \,|\, D \subseteq A_i\}\#[A_i]_U}{\#[C]_W} \frac{\#\,\mathrm{border}_{U,A_i}(E)}{\#[A_i]_U}.$$

Now using (c), we see that

$$\frac{\#\,\mathrm{border}_{W,C}(E)}{\#[C]_W}$$

is less than or equal to a convex combination of

$$\frac{\#\,\mathrm{border}_{U,A_i}(E)}{\#[A_i]_U}, \qquad i = 1, \ldots, k,$$

which implies (*). Q.E.D.

3. Examples

We will now develop some examples concerning universal product measurability in the case of infinite products. If V is a finite collection of disjoint infinite subsets of $\mathbf{Z}$ so that $\mathbf{Z} \setminus \cup V$ is a finite set and if F is a finite subset of $\mathbf{N}$, then the pair (F, V) will be called a partition generator. We make the following abbreviation: if $E \subseteq \mathbf{Z}^{\mathbf{N}}$, then

$$(E, F, V, i) := \frac{\#\{\psi\colon F \to V \,|\, \langle\psi\rangle \text{ borders } E \text{ and } \#\psi^{-1}(A) = i(A) \text{ for all } A \in V\}}{(i)}$$

and

$$(E, F, V) := \sup(E, F, V, i)$$

where the sup is taken over all $i\colon F \to \mathbf{Z}^+$ satisfying $\Sigma i(A) = \#F$. Using the ideas at the end of the proof of Theorem 4, it is not hard to show that if (F', V') is a partition generator refining (F, V) (i.e., $F \subseteq F'$ and V' refines V

modulo a set in J) then for any $E \subseteq \mathbf{Z}^{\mathbf{N}}$,

$$(E, F', V') \leq (E, F, V).$$

From Theorem 4 and the above fact, we derive the characterization of universally product measurable promised in the introduction: $E \subseteq \mathbf{Z}^{\mathbf{N}}$ is universally product measurable if for each $\varepsilon > 0$, there is a finite set G, a finite partition V of $\mathbf{Z} \setminus G$, and $M \in \mathbf{N}$, so that $(E, \{1, \dots, M\}, V) < \varepsilon$.

In actual calculations we do not wish to check every structural number (E, F, V). The next proposition states that we only need to check certain such structural numbers in special cases. For any finite subset F of $\mathbf{N}$, let $\mathrm{Int}_F E$ denote the set

$$\left\{\langle x_i \rangle_{i \in \mathbf{N}} \in \mathbf{Z}^{\mathbf{N}} \mid \text{if } \langle y_i \rangle_{i \in \mathbf{N}} \in \mathbf{Z}^{\mathbf{N}} \text{ and } y_i = x_i \text{ for each } i \in F, \text{ then } \langle y_i \rangle_{i \in \mathbf{N}} \in E\right\}.$$

If (F, V) is a partition generator, then let $\mathrm{Int}_{(F, V)} E$ denote the set

$$\bigcup \{\langle \psi \rangle \mid \psi \colon F \to V, \langle \psi \rangle \in E\}.$$

Proposition 5. *If $E \subseteq \mathbf{Z}^{\mathbf{N}}$, $\{F_i\}_{i=1}^{\infty}$ is an increasing sequence of subsets of $\mathbf{N}$ with $\bigcup_i F_i = \mathbf{N}$, and for each $j \in \mathbf{N}$, (F_j, V_j) is a partition generator with V_{j+1} refining V_j and there is $C_j \in J^*$ so that*

$$\mathrm{Int}_{F_j} E \setminus \mathrm{Int}_{(F_j, V_j)} E \subseteq C_j$$

and

$$\mathrm{Int}_{F_j}(\mathbf{Z}^{\mathbf{N}} \setminus E) \setminus \mathrm{Int}_{(F_j, V_j)}(\mathbf{Z}^{\mathbf{N}} \setminus E) \subseteq C_j,$$

then

$$\inf_{(V, F)} (E, V, F) = \lim_{j \to \infty} (E, F_j, V_j) = \inf_j (E, F_j, V_j).$$

Proof. We begin by showing that for each $j \in \mathbf{N}$,

$$(E, F_j, F_j) = \inf_W (E, F_j, W)$$

We may suppose that W refines V_j. Let $h \colon W \to V_j$ be defined by $A \subseteq h(A)$. Fix $i \colon V_j \to \mathbf{Z}^+$ so that $\Sigma_{A \in W} I(A) = \# F_j$. For each $A \in V_j$, choose $A' \in h^{-1}(A)$. Now define $g \colon V_j \to W$ by $g(A) = A'$, and define $i' \colon W \to \mathbf{Z}^+$ by $i'(C) = i(A)$ if $g(A) = C$ and $i'(C) = 0$ if $g(A) \neq C$ for all $A \in V_j$. Now if

$\psi\colon F_j \to V_j$ and $\langle \psi \rangle$ borders E then $\langle g \circ \psi \rangle$ must also border E (otherwise

$$\langle g \circ \psi \rangle \subseteq \operatorname{Int}_{F_j} E \quad \text{or} \quad \langle g \circ \psi \rangle \subseteq \operatorname{Int}_{F_j} \mathbf{Z}^{\mathbf{N}} \setminus E.$$

Since $\langle g \circ \psi \rangle \notin J^*$, this will imply that either $\langle g \circ \psi \rangle \subseteq \langle \psi \rangle \subseteq \operatorname{Int}_{(F_j, V_j)} E$ or $\subseteq \operatorname{Int}_{(F_j, V_j)} \mathbf{Z}^{\mathbf{N}} \setminus E$ which contradicts that $\langle \psi \rangle$ borders E). Thus

$$(E, F_j, V_j, i) = (E, F_j, W, i')$$

from which it follows that $(E, F_j, V_j) = (E, F_j, W)$. Now for any partition generator (F, V), there is a $j \in \mathbf{N}$ so that $F \subseteq F_j$. Thus

$$(E, F_{j+1}, V_{j+1}) \leq (E, F_j, V_j) \leq (E, F_j, V) \leq (E, F, V). \quad \text{Q.E.D.}$$

Example 1. We will begin with an easy example showing that for infinite product spaces we gain totally new universally product measurable sets, i.e., sets which differ from $\mathbf{B}^*$ sets by more than a set of universal product measure zero.

Let A_1 be an infinite subset of $\mathbf{Z}$ so that $A_2 := \mathbf{Z} \setminus A_1$ is also infinite. Let $E = \bigcup_k \langle \psi_k \rangle$ where $\psi_k\colon \{1, 2, \dots, 2k) \to \{A_1, A_2\}$ is defined by $\psi_k(i) = A_1$ if i is odd or is $2k$ and $= A_2$ if $i = 2, 4, \dots, 2k - 2$.

For $V = \{A_1, A_2\}$ and $F_k = \{1, \dots, 2k\}$ and $\phi\colon F_k \to V$, $\langle \phi \rangle$ borders E iff $\phi(j) = A_1$ if j is odd and $= A_2$ if j is even. Thus

$$(E, F_k, V) = \frac{(k!)^2}{(2k)!} \to 0 \text{ as } k \to \infty,$$

which implies that E is universally product measurable.

We need to show that E differs from each finite union of cylinders by more than a null set. If $T \in \mathbf{B}^*$, then there is a finite partition W of $\mathbf{Z}$, a $k \in \mathbf{N}$, and a $\mathbf{C} \subseteq \{\phi\colon F_k \to W\}$ so that W refines V and

$$T = \bigcup \{\langle \phi \rangle \mid \phi \in \mathbf{C}\}.$$

Choose infinite sets $B_1, B_2 \in W$ so that $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$. Define $\phi_0\colon F_k \to W$ as follows: $\phi_0(j)$ is B_1 if j is odd and is B_2 if j is even. For $i = 1, 2$ define $\phi_i\colon F_{k+1} \to W$ by

$$\phi_i|_{F_k} = \phi_0 \quad \text{and} \quad \phi_i(2k+1) = \phi_i(2k+2) = B_i.$$

Then $\langle \phi_1 \rangle \subseteq E$ and $\langle \phi_2 \rangle \subseteq \mathbf{Z} \setminus E$, $\langle \phi_1 \rangle \cup \langle \phi_2 \rangle \subseteq \langle \phi_0 \rangle$, and both $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$ are not null sets. If $\phi_0 \in \mathbf{C}$, then $\langle \phi_2 \rangle \subseteq \mathrm{T} \Delta E$ and if $\phi_0 \notin \mathbf{C}$, then $\langle \phi_1 \rangle \subseteq T \Delta E$. In either case we are done.

Example 2. We will discuss two necessary conditions for universal product measurability which often give quick ways of showing that a set E is not universally product measurable. The first one is as follows: Give $\mathbf{Z}^{\mathbf{N}}$ its usual product topology. If $E \subseteq \mathbf{Z}^{\mathbf{N}}$ is universally product measurable, then if $A_1 \times A_2 \times \cdots \times A_n \times \mathbf{Z} \times \mathbf{Z} \times \cdots$ is a subset of the topological boundary of E then at least one of the A_i's is finite. This statement follows from the fact that for any $C \in \partial^*_{\mathbf{B}} E$, the topological boundary of E is a subset of C. This criterion allows us to conclude that any set like

$$\left\{ \langle x_n \rangle \in \mathbf{Z}^{\mathbf{N}} \middle| \lim_{n \to \infty} x_n = \infty \right\}$$

is not universally product measurable. We may also conclude that if E is universally product measurable, then the topological boundary of E is a null set. However, we shall see that there are clopen sets of $\mathbf{Z}^{\mathbf{N}}$ which are not universally product measurable.

This brings us to the second necessary condition: If E is universally product measurable, then there exist a partition generator (F, V) so that for each $A \in V$ if $\psi_A \colon F \to \{A\}$, then either $\langle \psi_A \rangle \subseteq E$ or $\langle \psi_A \rangle \subseteq \mathbf{Z}^{\mathbf{N}} \setminus E$.

Here is the proof. Let (F, V) be such that $(E, F, V) < 1$. Now ψ_A is the only ψ satisfying $\psi^{-1}(A) = \#F$. Thus $(E, F, V, i_A) \in \{0, 1\}$ where i_A is defined by $i_A(A) = \#F$. Thus $\langle \psi_A \rangle \subseteq E$ or $\langle \psi_A \rangle \subseteq \mathbf{Z}^{\mathbf{N}} \setminus E$ since $(E, F, V, i_A) = 0$. Q.E.D.

We now give an easy example of a clopen set of $\mathbf{Z}^{\mathbf{N}}$ which is not universally product measurable. Let $A_1, A_2, \ldots$ be a partition of $\mathbf{Z}$ into infinitely many infinite sets. Let E be the set

$$\bigcup \{ \langle \psi_i \rangle | i = 1, 2, \ldots \} \quad \text{where } \psi_i \colon \{1, 2, \ldots, i\} \to \{A_i\}.$$

Clearly, E is a clopen set. In order to see that E is not universally product measurable, let $B_1, B_2, \ldots, B_n$ be disjoint infinite sets so that $\mathbf{Z} \setminus \bigcup B_i$ is finite and let M be in $\mathbf{N}$. There is a $k \leq n$ so that $B_k \cap A_{M+1}$ is infinite. If $\psi \colon \{1, 2, \ldots, M\} \to \{B_k\}$, then $\langle \psi \rangle$ borders E which implies E is not universally measurable by our second condition.

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