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ON HURWITZ GENERATION AND GENUS ACTIONS OF SPORADIC GROUPS

BY

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1. Introduction

Let S_G denote an orientable surface of least genus on which the finite group G acts in an effective and orientation-preserving manner. We define the genus $g(G)$ of the group G to be the genus of the surface S_G. By the Neilson Realization Problem [5], no generality is lost if we further assume S_G to be a closed Riemann surface and G to be conformally represented in its action on S_G. By a theorem of Schwarz, if S_G has genus at least 2 then G embeds with finite index in the full automorphism group $\mathrm{Aut}(S_G)$, but little can be said about the nature of the embedding. This is in sharp contrast to the situation when G is assumed to be simple and $(2, s, t)$-generated [13].

The main purpose of this paper is to prove the following result.

THEOREM 4.2. *Let G be a sporadic simple group other than McL or Fi_{24}'. Then* $\mathrm{Aut}(S_G) \cong G$ *where S_G is a surface of least genus for G. Moreover, if G is isomorphic to one of McL, Fi_{24}', then we have either* $\mathrm{Aut}(S_G) \cong G$ *or* $\mathrm{Aut}(S_G) \cong \mathrm{Aut}\, G$.

Pursuant to establishing this result, we are led to consider the following two questions.

(1) Which of the sporadic groups are generated by an involution x and element y of order 3?

(2) Among such groups, which are Hurwitz (i.e., have the additional property that x, y can be chosen to have product of order $t = 7$)?

We are able to settle (1) for all sporadics; only M_{11}, M_{22}, M_{23}, McL fail to have the prescribed generation. The table below summarizes the situation for the remaining sporadics, with the relevant conjugate classes indicated in the appropriate columns. (Descriptions of these classes, as well as the character

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Table 1. Sporadics which are $(2, 3, t)$-generated

G	Out G	2	3	t
M_{12}	2	$2A$	$3A$	$10A$
J_1	1	$2A$	$3A$	$7A$
J_2	2	$2B$	$3B$	$7A$
HS	2	$2B$	$3A$	$11A$
J_3	2	$2A$	$3B$	$10A$
M_{24}	1	$2A$	$3B$	$23A$
He	2	$2B$	$3B$	$7D$
Ru	1	$2B$	$3A$	$7A$
Suz	2	$2B$	$3C$	$11A$
$O'N$	2	$2A$	$3A$	$11A$
Co_3	1	$2B$	$3C$	$7A$
Co_2	1	$2C$	$3A$	$23A$
Fi_{22}	2	$2C$	$3D$	$11A$
HN	2	$2B$	$3B$	$7A$
Ly	1	$2A$	$3B$	$7A$
Th	1	$2A$	$3C$	$19A$
Fi_{23}	1	$2B$	$3D$	$23A$
Co_1	1	$2B$	$3C$	$23A$
J_4	1	$2B$	$3A$	$37A$
Fi_{24}'	2	$2B$	$3D$	$29A$
B	1	$2D$	$3B$	$47A$
M	1	$2A$	$3C$	$47A$

tables and maximal subgroup structures of the underlying groups, can be found in [2].) No claim is made as to the minimality of t subject to such generation, although we do remark that in a number of cases (e.g., when Out G is cyclic of order 2) our intention was to achieve small values for t. As for the status of (2), we have attempted a treatment of only those sporadics for which the maximal subgroup structure has been completely determined (viz. M_{11}, M_{12}, J_1, M_{22}, J_2, M_{23}, HS, J_3, M_{24}, McL, He, Ru, Suz, $O'N$, Co_3, Co_2, HN, Ly, Co_1) [2]. This is because the problem of establishing Hurwitz generation is inexorably linked to that of determining an effective upper bound on the number of $L_2(7)$-classes, the difficulty of which rises dramatically with the order of the group. For the nineteen sporadics indicated above, question (2) is entirely settled. There are precisely seven Hurwitz groups among them, namely those which appear having the prescribed $(2, 3, 7)$-generation in Table 1.

We now introduce the notation we shall use in subsequent sections. Let G be a finite group, K_1, K_2, K_3 conjugate classes of G, and z a fixed representative of K_3. We denote by $\Delta_G(K_1, K_2, K_3)$ the number of distinct ordered pairs (x, y) satisfying

(i) $x \in K_1$, $y \in K_2$,
(ii) $xy = z$.

It is well known that $\Delta_G(K_1, K_2, K_3)$ is a structure constant of the algebra $Z(\mathbf{C}G)$ and can readily be calculated from the character table for G by the formula

$$\Delta_G(K_1, K_2, K_3) = \frac{|K_1|\,|K_2|}{|G|} \sum_{\chi \in \operatorname{Irr} G} \frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1)}.$$

(Obviously $\Delta_G(K_1, K_2, K_3)$ is independent of the representative z of K_3 chosen.)

The number of such pairs which additionally satisfy

(iii) $G = \langle x, y \rangle$

shall be denoted by $\Delta_G^*(K_1, K_2, K_3)$. Clearly a group G admits (r, s, t)-generation if and only if there exist G-conjugate classes K_1, K_2, K_3, whose representatives have respective orders r, s, t for which $\Delta_G^*(K_1, K_2, K_3) > 0$.

Finally, for H a fixed subgroup of G containing z, we denote by $\Sigma_H(K_1, K_2, L)$ the number of distinct pairs (x, y) which satisfy (i), (ii), and

(iv) $\langle x, y \rangle \leq H$.

Here L denotes the H-conjugate class to which z belongs, while K_1, K_2 are, as before, G-classes. The reader should observe the fundamental difference between $\Sigma_H(K_1, K_2, L)$ and $\Delta_H(L_1, L_2, L_3)$ where L_1, L_2, L_3 are H-conjugate classes. That is, each of $H \cap K_1$, $H \cap K_2$ decomposes into a disjoint union of H-classes, and the symbol Σ serves to indicate that the number of pairs in $K_1 \times K_2$ which satisfy (i), (ii), and (iv) is obtained by *summing* over the totality of such classes.

In what follows we rely heavily on the following elementary result which appears as Lemma 3.3 in [14]:

(1.1) Let G be a finite centerless group for which we have

$$\Delta_G^*(K_1, K_2, K_3) < |C_G(z)|, \quad z \in K_3.$$

Then $\Delta_G^*(K_1, K_2, K_3) = 0$ and $\langle x, y \rangle$ is a proper subgroup of G for all $x \in K_1$, $y \in K_2$ with $xy = z$.

Clearly (1.1) gives a useful criterion for non-generation.

2. Sporadics which are $(2, 3, t)$-generated

In this section we verify the data in Table 1. We first show the list of groups appearing there is exhaustive.

LEMMA 2.1. *Let G be isomorphic to M_{11}, M_{22}, M_{23}, or McL. Then G is not $(2, 3, t)$-generated.*

Proof. For G isomorphic to M_{11}, M_{22}, or M_{23} the result follows from [14], so we assume $G = McL$. Clearly only those instances which give rise to positive values for $\Delta_G(K_1, K_2, K_3)$ need be considered. These appear below with appropriate conjugate classes identified.

$\Delta_G(K_1, K_2, K_3)$	K_1	K_2	K_3	$\lvert C_G(z)\rvert$
3	$2A$	$3A$	$12A$	12
6	$2A$	$3A$	$30A$	30
49	$2A$	$3B$	$7A$	14
20	$2A$	$3B$	$8A$	8
27	$2A$	$3B$	$9A$	27
11	$2A$	$3B$	$11A$	11
7	$2A$	$3B$	$14A$	14

By (1.1) we see directly that G is not $(2, 3, 12)$-, $(2, 3, 30)$-, or $(2, 3, 14)$-generated. Moreover $\Sigma_U(2A, 3B, 9A) = 27$ for $U \leq G$ isomorphic to $U_4(3)$ and $\Sigma_K(2A, 3B, 11A) = 11$ for $K \leq G$ isomorphic to M_{11}. This proves non-generation of each of the types $(2, 3, 9)$ and $(2, 3, 11)$. Now choose $M \leq G$ with $M \cong M_{22}$. Then $\Sigma_M(2A, 3B, 8A) = 16$ whence

$$\Delta_G^*(2A, 3B, 8A) \leq \Delta_G(2A, 3B, 8A) - \Sigma_M(2A, 3B, 8A) = 4.$$

Thus, by (1.1), G cannot be $(2, 3, 8)$-generated. Finally note that a fixed element of order 7 is contained in precisely two conjugates U, U^g of $U \cong U_4(3)$ in G. As G is a rank-3 permutation group in its action on the G-conjugates of U, we see that $H = U \cap U^g$ must be isomorphic to either $3^4{:}A_6$ or $L_3(4)$, the two 2-point stablizers under this action. But this implies $H \cong L_3(4)$ as H has order divisible by 7. Thus

$$\Sigma_U(2A, 3B, 7A) = \Sigma_{U^g}(2A, 3B, 7A) = 35,$$
$$\Sigma_H(2A, 3B, 7A) = 21,$$

and we obtain

$$\Sigma_U + \Sigma_{U^g} - \Sigma_H = 49.$$

We now conclude that every $(2, 3, 7)$-subgroup of G is contained in either U or U^g, whence G cannot be $(2, 3, 7)$-generated. The proof of the lemma is now complete.

LEMMA 2.2. *The group M_{12} is $(2,3,10)$-generated.*

Proof. This is given in [14].

LEMMA 2.3. *The group J_1 is $(2,3,7)$-generated.*

Proof. We first observe that, as $2,3,7$ are pairwise co-prime, no $(2,3,7)$-group can have a solvable quotient. Using this fact, along with the maximal subgroup structure of J_1 [2], it is clear that J_1 possesses no proper $(2,3,7)$-subgroup. The lemma follows as $\Delta_{J_1}(2A,3A,7A) = 49$.

LEMMA 2.4. *The group J_2 is $(2,3,7)$-generated.*

Proof. This is given in [4].

LEMMA 2.5. *The group HS is $(2,3,11)$-generated.*

Proof. The only maximal subgroups of HS with order divisible by 11 are, up to isomorphism, M_{22} and M_{11} [2]. But the HS-class $2B$ fails to meet groups of either type, as involutions in each are necessarily squares. This proves HS has no proper subgroup of type $(2B,3A,11A)$; as $\Delta_{HS}(2B,3A,11A) = 33$ the result follows.

LEMMA 2.6. *The group J_3 is $(2,3,10)$-generated.*

Proof. The isomorphism types of maximal subgroups of J_3 which contain an element of order 10 are $L_2(16){:}2$, $L_2(19)$, $(3 \times A_6){:}2$, and $2_-^{1+4}{:}A_5$ [2]. We claim the J_3-class $3B$ fails to meet $L_2(16){:}2$ and $2_-^{1+4}{:}A_5$ subgroups. Indeed each element of order 3 in $L_2(16){:}2$ centralizes an element of order 5, while those in $2_-^{1+4}{:}A_5$ commute with the involution in its center. Now for $L \leq J_3$ isomorphic to $L_2(19)$, we calculate $\Sigma_L(2A,3B,10A) = 20$. As a fixed element of order 10 is contained in precisely two copies of $L_2(19)$ (one from each J_3-class) we get a net contribution of 40. Finally we establish an upper bound on $\Sigma_K(2A,3B,10A)$, where K is the unique copy of $(3 \times A_6){:}2$ in J_3 which contains a fixed element of order 10. First observe that $K/O_3(K) \cong PGL_2(9)$ [2]. As $\Delta_P(2D,3A,10A) = 10$ for $P \cong PGL_2(9)$, and every such pair is accountable as the image of at most three such pairs in K, we have $\Sigma_K(2A,3B,10A) \leq 30$. Thus

$$\Delta^*_{J_3}(2A,3B,10A) \geq \Delta_{J_3}(2A,3B,10A) - 70 = 120 - 70 = 50,$$

i.e., J_3 is $(2A,3B,10A)$-generated.

LEMMA 2.7. *The group M_{24} is $(2,3,23)$-generated.*

Proof. This follows trivially as $\Delta_{M_{24}}(2A, 3B, 23A) = 23$ and involutions in copies of $L_2(23)$ are of type $2B$ while elements of order three in copies of M_{23} are of type $3A$.

LEMMA 2.8. *The group He is $(2,3,7)$-generated.*

Proof. We show He is a $(2B, 3B, 7D)$-group. The isomorphism types of maximal subgroups of He with order divisible by 42 are as follows:

$$2^2{}^{\cdot}L_3(4).S_3,\quad 2_+^{1+6}.L_3(2),\quad 7^2{:}2L_2(7),\quad 3^{\cdot}S_7,\quad 7_+^{1+2}{:}(S_3\times 3),$$
$$S_4\times L_3(2)\quad\text{and}\quad 7{:}3\times L_3(2)\ [2].$$

For

$$H\cong 7^2{:}2L_2(7),\qquad K\cong 7_+^{1+2}{:}(S_3\times 3),$$

any $(2,3,7)$-subgroup of H or K must map onto a $(2,3,7)$-subgroup of

$$H/O_7(H)\cong SL(2,7)\quad\text{and}\quad K/O_7(K)\cong S_3\times 3,$$

respectively. As $SL(2,7)$ has a unique (central) involution, this is clearly not possible in either case. From [2] we see that elements of order 7 in $2^2{}^{\cdot}L_3(4).S_3$ and $S_4\times L_3(2)$ subgroups are of He-type $7A$ or $7B$. As $3^{\cdot}S_7$ is the centralizer of a $3A$ element in He, its elements of order 7 must be of type $7C$. Finally, any $(2,3,7)$-subgroup of $N\cong 7{:}3\times L_3(2)$ must clearly be contained in the unique complement $L\cong L_3(2)$ in N, whence its 7-elements are of He-type $7C$. This proves any proper $(2B,3B,7D)$-subgroup of He is contained in a copy of $2_+^{1+6}{:}L_3(2)$. But from [1], we see that

$$\Sigma_R(2B,3B,7A)=21\quad\text{for}\quad R\cong 2_+^{1+6}{:}L_3(2).$$

(Indeed there are three pairs of R-classes $(2X,3Y)$ for which $\Delta_R(2X,3Y,7A) = 7$; for remaining pairs $\Delta_R = 0$.) We also see from [1] that elements of order 7 in R are of He-type $7D$. By comparison of centralizer orders, it is easy to show that a fixed element of order 7 lies in precisely seven conjugates of R in He. Thus

$$\Delta_{He}^*(2B,3B,7D)\ge \Delta_{He}(2B,3B,7D)-7(21)=441-147=294,$$

whence He is $(2B,3B,7D)$-generated.

LEMMA 2.9. *The group Ru is* $(2,3,7)$*-generated.*

Proof. The only maximal subgroups of Ru with order divisible by 42 are (up to isomorphism)

$$2^6{:}U_3(3){:}2, \quad 2^{3+8}{:}L_3(2), \quad U_3(5){:}2, \quad A_8, \quad L_2(29), \quad L_2(13){:}2 \quad [2].$$

Let H be a proper $(2B,3A,7A)$-subgroup of Ru. As H cannot have a solvable quotient, we see immediately that H embeds in one of

$$2^6{:}U_3(3), \quad 2^{3+8}{:}L_3(2), \quad U_3(5), \quad A_8, \quad L_2(29), \quad L_2(13).$$

But elements of Ru-type $2B$ are necessarily non-squares, whence involutions in $U_3(5)$, A_8, and $L_2(13)$ are of Ru-type $2A$. Thus H embeds in one of $2^6{:}U_3(3), 2^{3+8}{:}L_3(2)$, $L_2(29)$. Let $K \leq Ru$ be isomorphic to $2^6{:}U_3(3)$, so that $K = O_2(K){:}U$ with $U \cong U_3(3)$. From [2] we see that $O_2(K)$ is $2A$-pure; moreover involutions in U are squares so are of type $2A$ as well. As $\Delta_{Ru}(2A,2A,2B) = 0$, all involutions in K must be of Ru-type $2A$. In [9] it is shown that all involutions of Ru-type $2B$ in $J \cong 2^{3+8}{:}L_3(2)$ must lie in $O_2(J)$, whence $\Sigma_J(2B,3A,7A) = 0$ easily follows. Thus H embeds in $L_2(29)$. It is easy to establish that a fixed element of order 7 lies in precisely six conjugates of $L \cong L_2(29)$ and that $L_2(29)$ contains no proper $(2,3,7)$-subgroup. As each involution in $L_2(29)$ commutes with an element of order 7 there, we see that involutions in $L_2(29)$ are of Ru-type $2B$. Thus $\Sigma_L(2B,3A,7A) = 28$ and it follows that

$$\Delta^*_{Ru}(2B,3A,7A) = \Delta_{Ru}(2B,3A,7A) - 6(28) = 560 - 168 = 392.$$

Thus Ru is $(2,3,7)$-generated as claimed.

LEMMA 2.10. *The group Suz is* $(2,3,11)$*-generated.*

Proof. Up to isomorphism, the only maximal subgroups of Suz with order divisible by 11 are $U_5(2)$, $M_{12}{:}2$, and $3^5{:}M_{11}$. From [2] the Suz-class $3C$ fails to meet $U_5(2)$. Moreover, involutions in M_{11} and M_{12} cannot be of Suz-type $2B$, as involutions in M_{11} are 4th powers and those in M_{12} are 6th powers (in $M_{12}{:}2$). As any $(2,3,11)$-subgroup of $M_{12}{:}2$ must clearly lie in M_{12}, we conclude that Suz contains no proper $(2B, 3C, 11A)$-subgroup. As $\Delta_{Suz}(2B,3C,11A) = 715$, the lemma follows.

LEMMA 2.11. *The group $O'N$ is* $(2,3,11)$*-generated.*

Proof. Up to ismorphism, the only maximal subgroups of $O'N$ with order divisible by 11 are J_1 and M_{11} [2]. By comparing centralizer orders, one easily sees that a fixed element of order 11 is contained in a unique copy J of J_1 and in precisely four copies N_1, N_2, N_3, N_4 of M_{11} (two conjugates from each class) in $O'N$. As $\Sigma_{N_j}(2A, 3A, 11A) = 55$ and $\Sigma_{N_i}(2A, 3A, 11A) = 11$ ($i = 1, 2, 3, 4$), we conclude that

$$\Delta^*_{O'N}(2A, 3A, 11A) \geq \Delta_{O'N}(2A, 3A, 11A) - 55 - 4(11) = 715 - 99 = 616$$

and the proof is complete.

LEMMA 2.12. *The group Co_3 is $(2,3,7)$-generated.*

Proof. We show Co_3 has no proper $(2B, 3C, 7A)$-subgroup; as

$$\Delta_{Co_3}(2B, 3C, 7A) = 504$$

the result will immediately follow. Let then H be a proper $(2B, 3C, 7A)$-subgroup of Co_3. Clearly H must be contained in a maximal subgroup with order divisible by 42. We list these by isomorphism types as follows:

$$McL{:}2,\ HS,\ U_4(3){:}2^2,\ M_{23},\ 2^{\cdot}S_6(2),\ U_3(5){:}S_3,\ 2^{4\cdot}A_8,$$
$$L_3(4){:}D_{12},\ S_3 \times L_2(8){:}3 \quad [2].$$

But H cannot possess a solvable quotient. Thus H embeds in one of the following appropriate subgroups:

$$McL,\ HS,\ U_4(3),\ M_{23},\ 2^{\cdot}S_6(2),\ U_3(5),\ 2^{4\cdot}A_8,\ L_3(4),\ L_2(8).$$

But involutions in each of McL, $U_4(3)$, M_{23}, $U_3(5)$, and $L_3(4)$ are necessarily squares, so are of Co_3-type $2A$. Thus H embeds in one of HS, $2^{\cdot}S_6(2)$, $2^{4\cdot}A_8$, or $L_2(8)$. We now consider elements of order 3. Such elements are cubes in $L_2(8)$, so of Co_3-type $3A$. As $2^{\cdot}S_6(2)$ is the centralizer of an involution of type $2A$, elements of order 3 in $2^{\cdot}S_6(2)$ must be of Co_3-type $3A$ as well. Let t be an element of order 3 lying in a copy of HS. Then t centralizes an element of order 5 there, so has Co_3-type $3A$ or $3B$. Finally, let $E \leq M$ with $E \cong 2^4$, $M \cong McL$. M has two classes of such subgroups which fuse in $\operatorname{Aut} M \cong McL{:}2$, in any case $N_M(E) \cong 2^{4\cdot}A_7$. As Co_3 has a unique class of 2^4 subgroups [3], we have $N_M(E) \leq N(E) \cong 2^{4\cdot}A_8$. Thus every element of order 3 in $N(E)$ lies in a conjugate of M. As such elements are cubes, they are necessarily of type $3A$ or $3B$ and we have reached a contradiction.

LEMMA 2.13. *The group Co_2 is $(2,3,23)$-generated.*

Proof. Any proper $(2,3,23)$-subgroup of Co_2 must lie in a copy of M_{23} therein [2]. But involutions in M_{23}, being 4th powers, must be of Co_2-type $2A$ or $2B$. As $\Delta_{Co_2}(2C, 3A, 23A) = 69$, the lemma follows.

LEMMA 2.14. *The group Fi_{22} is $(2,3,11)$-generated.*

Proof. The maximal subgroups of Fi_{22} with order divisible by 11 are, up to isomorphism, $2^{\cdot}U_6(2)$, $2^{10}{:}M_{22}$, and M_{12}. By examination of the relevant permutation characters [2], it is easily shown that the class $3D$ does not meet any copy of $2^{\cdot}U_6(2)$ or $2^{10}{:}M_{22}$ in Fi_{22}. We now consider $M \leq Fi_{22}$ isomorphic to M_{12}. Under the (possibly erroneous) assumption that each of $2C$ and $3D$ meets M, we see that $2C \cap M$ must constitute the M-class of central involutions and $3D \cap M$ that of Sylow-central elements of order 3. (Indeed, the non-central involutions in M are 4th powers, so necessarily of Fi_{22}-type $2A$ or $2B$; while each non-Sylow-central element of order 3 in M commutes with a central involution there, so is not of type $3D$.) Thus

$$\Sigma_M(2C, 3D, 11A) \leq \Delta_{M_{11}}(2B, 3A, 11A) = 11.$$

As a fixed element of order 11 is in precisely two conjugates of M in Fi_{22}, we conclude that

$$\Delta^*_{Fi_{22}}(2C, 3D, 11A) \geq \Delta_{Fi_{22}}(2C, 3D, 11A) - 22 = 1980 - 22 = 1958$$

whence Fi_{22} is $(2C, 3D, 11A)$-generated.

LEMMA 2.15. *The group HN is $(2,3,7)$-generated.*

Proof. From the maximal subgroup structure of HN [2], any proper $(2,3,7)$-subgroup of HN must embed in one of

$$A_{12},\quad 2^{\cdot}HS.2,\quad U_3(8){:}3,\quad (D_{10} \times U_3(5))^{\cdot}2,\quad 2^3.2^2.2^6.(3 \times L_3(2)).$$

Let H be a proper $(2B, 3B, 7A)$-subgroup of HN. As $2^{\cdot}HS.2$ is the centralizer of an involution of HN-type $2A$, we see from [2] that elements of order 3 in $2^{\cdot}HS.2$ are of type $3A$. Similarly, elements of order 3 in $(D_{10} \times U_3(5))^{\cdot}2$ are of type $3A$, as they centralize the $5A$ elements lying in D_{10}. As H cannot have a solvable quotient, we therefore see that H embeds in A_{12}, $U_3(8)$, or $2^3.2^2.2^6.L_3(2)$. But every element of order 3 in $U_3(8)$ commutes with an element of order 7, so is of HN-type $3A$. We now show such elements in $2^3.2^2.2^6.L_3(2)$ are of HN-type $3A$ as well. Let $M \leq HN$ be isomorphic to

$$2^3.2^2.2^6.(L_3(2) \times 3).$$

Choose $x \in M$ of order 3 such that $Z(\overline{M}) = \langle \bar{x} \rangle$ where $\overline{M} = M/O_2(M)$. As $O_2(M)\langle x \rangle \lhd M$, a Frattini argument yields

$$M = O_2(M)N_M(\langle x \rangle) = O_2(M)C_M(x).$$

In particular $C_{\overline{M}}(\bar{x}) = \overline{C_M(x\ )}$, whence 7 divides the order of $C_M(x)$ and x is of HN-type $3A$. As $C_{HN}(x) \cong 3 \times A_9$, we immediately have $C_M(x) = \langle x \rangle \times K$ where K embeds in A_9. Thus $\overline{K} \cong L_3(2)$, and checking the maximal subgroup structure of A_9 [2] we conclude that K embeds in A_8. But $C_{HN}(x) \leq A$ for some subgroup A of HN isomorphic to A_{12}. The restriction $\psi \downarrow A$, where ψ is an irreducible character of HN of degree 133, enables us to identify the HN-type of elements of order 3 in A, so also in K. As only those elements of A with cycle structure $[3^3]$ have HN-type $3B$, we conclude that 3-elements of K are of type $3A$. Thus, as K embeds in $2^3.2^2.2^6.L_3(2)$, elements of order 3 in the latter group are of type $3A$ as claimed. We have only to consider A_{12}. Clearly, from centralizer orders, a fixed element of order 7 lies in a unique copy A of A_{12} in HN. The restriction $133 \downarrow A$ determines, once more, the HN-type of elements of A. We easily obtain

$$\Sigma_A(2B, 3B, 7A) = \Delta_{A_{12}}(2C, 3D, 7A) = 140.$$

Therefore

$$\Delta^*_{HN}(2B, 3B, 7A) = \Delta_{HN}(2B, 3B, 7A) - 140 = 2660 - 140 = 2520,$$

and HN is $(2B, 3B, 7A)$-generated. The proof of the lemma is now complete.

LEMMA 2.16. *The group* Ly *is* $(2, 3, 7)$-*generated.*

Proof. The maximal subgroups of Ly with order divisible by 42 are $G_2(5)$, $3^{\cdot}McL{:}2$, and $2^{\cdot}A_{11}$ [2]. From centralizer orders, one easily determines that a fixed element of order 7 is contained in precisely eight conjugates of $H \cong G_2(5)$, four conjugates of $M \cong 3^{\cdot}McL{:}2$, and a unique conjugate of $A \cong 2^{\cdot}A_{11}$. It is an easy matter to identify the appropriate Ly-classes with those of H, M, and A and to determine

$$\Sigma_H(2A, 3B, 7A) = 546, \quad \Sigma_M(2A, 3B, 7A) = 49 \quad \text{and} \quad \Sigma_A(2A, 3B, 7A) = 56.$$

As $\Delta_{Ly}(2A, 3B, 7A) = 8680$, we have

$$\Delta^*_{Ly}(2A, 3B, 7A) \geq 8680 - 8(546) - 4(49) - 56 = 4060$$

so that Ly is $(2A, 3B, 7A)$-generated.

LEMMA 2.17. *The group* Th *is* $(2, 3, 19)$-*generated.*

Proof. Observe from [2] that all p-local subgroups of Th have been determined. Checking the list, we see that any proper $(2, 3, 19)$-subgroup must be simple (as 2, 3, and 19 are pairwise co-prime); by Lagrange's Theorem and the classification of finite simple groups, $L_2(19)$ and $U_3(8)$ are the only possibilities. But elements of order 3 in $L_2(19)$ are cubes, so not of Th-type $3C$. Similarly, $3C$ cannot meet any copy of $U_3(8)$ in Th (should one exist) as the only classes of elements of order 3 in $U_3(8)$ which are non-cubes have centralizer order 1512, which fails to divide $|C_{Th}(x)|$ for $x \in 3C$. Thus Th has no proper $(2A, 3C, 19A)$-subgroup. As $\Delta_{Th}(2A, 3C, 19A) = 6194$, the lemma follows.

LEMMA 2.18. *The group Fi_{23} is $(2,3,23)$-generated.*

Proof. Once again, all p-locals are determined [2]. From the list (and the classification theorem), the only possibilities for proper $(2,3,23)$-subgroups of Fi_{23} are (up to isomorphism) $L_2(23)$, M_{23}, M_{24}, or a subgroup of $2^{11}{}^{\cdot}M_{23}$. Now in $L_2(23)$ there exist commuting representatives of the (unique) classes of elements of order 2 and 3. As no such representatives are to be found in the Fi_{23}-classes $2B, 3D$, we see that at most one of $2B, 3D$ meets any copy of $L_2(23)$ in Fi_{23}. The same argument holds for any copy of M_{23} in Fi_{23}, should one exist. Now suppose Fi_{23} contains a copy M of M_{24}. As central involutions in M are 4th powers, they are necessarily of Fi_{23}-type $2C$. Suppose the remaining M-class of involutions lie in $2B$. As each representative of this class commutes with a representative of the M-class $3B$, the above argument implies that elements of M-type $3B$ are not of Fi_{23}-type $3D$. Thus if $3D$ meets M, we must have $3D \cap M$ equal to the M-class $3A$. This cannot occur, however, as elements of M-type $3A$ centralize elements of order 5, while those of Fi_{23}-type $3D$ do not. We conclude that at most one of the two Fi_{23}-classes $2B, 3D$ meets M. Finally, we consider the group $2^{11}{}^{\cdot}M_{23}$ (which occurs as a subgroup in Fi_{23}). Clearly, an element of order 23 in M_{23} acts irreducibly on 2^{11} (regarded as an 11-dimensional vector space); thus the action of M_{23} on 2^{11} is irreducible as well. We now observe that $\varphi(t) = 2$ where φ is the irreducible 2-modular character for M_{23} of degree 11 and $t \in M_{23}$ is an arbitrary element of order 3. From this, it is easily checked that t must have a 5-dimensional fixed point space in 2^{11}, whence 2^5 divides $|C_{Fi_{23}}(t)|$. This proves t is not of Fi_{23}-type $3D$. We have therefore proved that Fi_{23} can possess no proper $(2B, 3D, 23A)$-subgroup. As $\Delta_{Fi_{23}}(2B, 3D, 23A) = 11592$, we conclude that Fi_{23} is $(2,3,23)$-generated as claimed.

LEMMA 2.19. *The group Co_1 is $(2,3,23)$-generated.*

Proof. From the maximal subgroup structure of Co_1 [2], we see that the only such groups with order divisible by 23 are Co_2, $2^{11}{:}M_{24}$, and Co_3. But restriction of the irreducible character of degree 276 to the appropriate subgroups shows that $2B$ fails to meet either Conway group, while $3C$ fails to meet M_{24}. Thus Co_1 contains no proper $(2B, 3C, 23A)$-subgroup. As $\Delta_{Co_1}(2B, 3C, 23A) = 138$, the lemma follows.

LEMMA 2.20. *The group* J_4 *is* $(2,3,37)$-*generated.*

Proof. Observe that all p-local subgroups of J_4 have been determined [2]. From this (and the fact that 2, 3, and 37 are pairwise co-prime) we see that any $(2,3,37)$-subgroup of J_4 must be simple. Let H be a proper $(2,3,37)$-subgroup of J_4. By the classification of finite simple groups, the only possibility is $H \cong U_3(11)$. But involutions in H are then 4th powers, so of J_4-type $2A$. As $\Delta_{J_4}(2B, 3C, 37A) = 15577$, the lemma follows.

LEMMA 2.21. *The group* Fi_{24}' *is* $(2,3,29)$-*generated.*

Proof. For $p \neq 2$, the p-local structure of Fi_{24}' has been determined [2]. It is now easy to show that any proper $(2,3,29)$-subgroup of Fi_{24}' must be of the form $K = O_2(K).L_2(29)$ (the extension not necessarily splitting). But the smallest non-trivial $\langle t \rangle$-module (over $\mathbf{Z}_2$) for $t \in K$ of order 29 is 28-dimensional. Thus clearly $O_2(K) = 1$; i.e., $K \cong L_2(29)$. As every element of order 3 in $L_2(29)$ centralizes an element of order 5, the former cannot be of Fi_{24}'-type $3D$. As $\Delta_{Fi_{24}'}(2B, 3D, 29A) = 47096$, the lemma is proved.

LEMMA 2.22. *The group* B *is* $(2,3,47)$-*generated.*

Proof. We first observe that $\Delta_B(2D, 3B, 47A) = 5048364$. Once again, the p-local structure of B is completely determined for $p \neq 2$ [2]. Letting H be a $(2,3,47)$-subgroup of B, we see from this that $\overline{H} = H/O_2(H)$ must be a simple $(2,3,47)$-group, which (as $2^{\cdot}B \leq M$) must be involved in the Fischer–Griess Monster M. We now consult the list of all prospective simple sections of M [2], from which we conclude $\overline{H} = B$. Thus $H = B$ and B is $(2,3,47)$-generated as claimed.

LEMMA 2.23. *The group* M *is* $(2,3,47)$-*generated.*

Proof. Arguing as in the previous lemma, any proper $(2,3,47)$-subgroup K of M must satisfy $\overline{K} = K/O_2(K) \cong B$. As a Sylow 47-subgroup P of B has centralizer of order 94, and as the minimal dimension of a non-trivial irreducible P-module over $\mathbf{Z}_2$ clearly exceeds the 2-rank of M, we see that $K \cong B$ or $2^{\cdot}B$. In either case, we see that the M-class $3C$ fails to meet K by comparison of centralizer orders. Thus M has no proper $(2A, 3C, 47A)$-subgroup. As $\Delta_M(2A, 3C, 47A) = 470$, the lemma follows.

THEOREM 2.24. *A sporadic simple group is* $(2,3,t)$-*generated if and only if it appears in Table* 1.

Proof. This is an immediate consequence of Lemmas 2.1 through 2.23.

3. Hurwitz generation of sporadic groups

In this section we investigate those sporadic groups for which the maximal subgroup structure has been completely determined, namely

$$M_{11},\ M_{12},\ J_1,\ M_{22},\ J_2,\ M_{23},\ HS,\ J_3,\ M_{24},$$
$$McL,\ He,\ Ru,\ Suz,\ O'N,\ Co_3,\ Co_2,\ HN,\ Ly,\ Co_1.$$

From the Riemann–Hurwitz equation, one easily obtains the following lower bound for the genus g of any finite hyperbolic group G:

$$g \geq 1 + \frac{|G|}{84}.$$

(A hyperbolic group is one which admits no effective orientation-preserving action on the 2-sphere or torus.) Within this context, the Hurwitz groups can be described as the family of hyperbolic groups for which this lower bound is achieved. Hurwitz groups are further characterized by their $(2,3,7)$-generation.

Now, among all finite non-abelian simple groups, only A_5 is non-hyperbolic (by virtue of its action on the icosahedron, whose barycentric subdivision embeds in the 2-sphere). Thus, in particular, all sporadic groups are hyperbolic, and those with order divisible by 42 are candidates for Hurwitz groups.

We wish to point out that the results of this section, although central to the study of genus actions, are in no way required for the proof of Theorem 4.2. Thus the reader may view Section 3 as a slight departure of sorts, and can safely proceed to Section 4 without disrupting logical sequence.

LEMMA 3.1. *The groups M_{11}, M_{12}, M_{22}, M_{23}, J_3 and McL are non-Hurwitz.*

Proof. This is a trivial consequence of Lemma 2.1 and Lagrange's Theorem.

LEMMA 3.2. *The group M_{24} is non-Hurwitz.*

Proof. We first observe that

$$\Delta_{M_{24}}(2A,3A,7Z) = \Delta_{M_{24}}(2B,3B,7Z) = 42,$$
$$\Delta_{M_{24}}(2A,3B,7Z) = 0 \quad \text{and} \quad \Delta_{M_{24}}(2B,3A,7Z) = 7.$$

Choose any subgroups M and L of M_{24} isomorphic to M_{23} and $L_2(7)$, respectively. Then an easy calculation reveals

$$\Sigma_M(2A,3A,7Z) = 35 \quad \text{and} \quad \Sigma_L(2B,3B,7Z) = 7.$$

Thus for all possible combinations for X, Y we have

$$\Delta^*_{M_{24}}(2X, 3Y, 7Z) < 42 = |C_{M_{24}}(s)|, \quad s \in M_{24} \text{ of order } 7,$$

so that non-generation follows from (1.1).

LEMMA 3.3. *The group HS in non-Hurwitz.*

Proof. One easily calculates

$$\Delta_{HS}(2A, 3A, 7A) = 35 \quad \text{and} \quad \Delta_{HS}(2B, 3A, 7A) = 28.$$

Now a fixed element s of order 7 stabilizes an edge (pointwise) in the action of HS on the rank-3 graph of valence 22 on 100 vertices [2]. Thus $s \in M \cap M^x$ where $M \cong M^x \cong M_{22}$ and $M \cap M^x \cong M_{21}$. From [2] it is immediate that involutions in M are of HS-type $2A$. As

$$\Sigma_M(2A, 3A, 7A) = 28 \quad \text{and} \quad \Sigma_{M \cap M^x}(2A, 3A, 7A) = 21,$$

we obtain

$$\Delta^*_{HS}(2A, 3A, 7A) = \Delta_{HS}(2A, 3A, 7A) - 2(28) + 21 = 0,$$

proving that HS is not $(2A, 3A, 7A)$-generated.

We next claim that $K \leq HS$ isomorphic to $4^3{:}L_3(2)$ contains a subgroup of HS-type $(2B, 3A, 7A)$. Indeed, as K contains a Sylow 2-subgroup of HS, we can choose $x \in K$ of HS-type $2B$. Write $x = ab$ with $a \in O_2(K)$ and $b \in L$, where L is a fixed complement of $L_3(2)$ in K. As the involutions in $O_2(K)$ are squares (so of HS-type $2A$) we have $b \neq 1$. Thus b clearly has order 2 as $b^2 \in L \cap O_2(K)$. We can therefore choose $y \in L$ of order 3 such that the element by has order 7. But then xy has order 7 as well, as $\overline{xy} = \overline{by}$ (mod $O_2(K)$) and HS has no element of order $7n$ with $n > 1$. We conclude that $\langle x, y \rangle$ is a Hurwitz subgroup of K of HS-type $(2B, 3A, 7A)$.

Consider now the sporadic group Co_3 which is known to contain (as maximal subgroup) a copy G of HS. From the restriction $\psi \downarrow G$ of the irreducible character ψ of degree 23 for Co_3, we are able to identify conjugate classes of G with those of Co_3. In particular, we discover

$$\begin{array}{cccc} G & 2B & 3A & 7A \\ Co_3 & 2B & 3B & 7A. \end{array}$$

As $\Delta_{Co_3}(2B, 3B, 7A) = 84$ and s of order 7 is in precisely six conjugates of G, we see that $\Delta^*_G(2B, 3A, 7A) \leq 14$. If $\Delta^*_G(2B, 3A, 7A) \neq 0$, it now follows that $\Delta^*_G(2B, 3A, 7A) = 14$ as $C_{\mathrm{Aut}\,G}(xy)$ of order 14 normalizes each of the G-classes

$2B, 3A, 7A$. But this implies

$$\Delta_{Co_3}(2B, 3B, 7A) = 6\,\Delta_G^*(2B, 3A, 7A),$$

whence Co_3 can have no Hurwitz subgroup of type $(2B, 3B, 7A)$ other than the six aforementioned conjugates of G. This of course contradicts the earlier established existence of $\langle x, y\rangle$. Thus $\Delta_G^*(2B, 3A, 7A) = 0$, whence HS is not $(2B, 3A, 7A)$-generated. The proof of the lemma is now complete.

LEMMA 3.4. *The group Suz is non-Hurwitz.*

Proof. First observe that a fixed element of order 7 lies in precisely four conjugates of $K \cong G_2(4)$, which is the stabilizer of a vertex of the Suzuki graph [2]. Moreover any two conjugates must clearly intersect in a two-point stabilizer having order divisible by 7, i.e., in a conjugate of $J \cong J_2$. It is routine to identify Suz-classes with those of K and J, and we consequently obtain

$$\begin{aligned}\Delta_{Suz}^*(2B, 3C, 7A) &\le \Delta_{Suz}(2B, 3C, 7A) - 4\,\Sigma_K(2B, 3C, 7A)\\ &\quad + 6\,\Sigma_J(2B, 3C, 7A)\\ &= 1260 - 4(336) + 6(70)\\ &= 336.\end{aligned}$$

Now from [10] we see that Suz has two classes of self-normalizing $L_2(7)$ subgroups of type $(2B, 3C, 7A)$. As those involutions which are squares in K have Suz-type $2A$, it is clear that no $L_2(7)$ of type $(2B, 3C, 7A)$ can lie in K, or in any conjugate thereof. Thus each class of $(2B, 3C, 7A)$-type $L_2(7)$ subgroups gives rise to $168 = 7{\cdot}24$ distinct pairs (x, y) with $x \in 2B$, $y \in 3C$ and xy a fixed representative of $7A$. (We have used the fact that xy lies in precisely 24 conjugates from each class.) Thus $\Delta_{Suz}^*(2B, 3C, 7A) = 0$ and Suz is not $(2B, 3C, 7A)$-generated. The only remaining non-zero $(2, 3, 7)$-structure constant is $\Delta_{Suz}(2A, 3C, 7A)$, which equals 77. As the centralizer of a $7A$-element has order 84, non-generation follows from (1.1).

LEMMA 3.5. *The group O'N is non-Hurwitz.*

Proof. We calculate $\Delta_{O'N}(2A, 3A, 7A) = 343$ and $\Delta_{O'N}(2A, 3A, 7B) = 931$. In the former case, non-generation follows from (1.1) as the centralizer of a $7A$-element has order 1372. In the latter case, R.A. Wilson [11] has accounted for all relevant pairs within conjugates of $F \cong 4^{2\cdot}L_3(4)$, $J \cong J_1$, and $L \cong L_3(7)$ in $O'N$. The lemma follows.

LEMMA 3.6. *The group Co_2 is non-Hurwitz.*

Proof. We first calculate

$$\Delta_{Co_2}(2C, 3A, 7A) = 28, \quad \Delta_{Co_2}(2B, 3B, 7A) = 91, \quad \Delta_{Co_2}(2C, 3B, 7A) = 238,$$

with all remaining $(2,3,7)$-structure constants being zero. As $\Sigma_M(2B, 3B, 7A) = 49$ for $M \leq Co_2$ isomorphic to McL, we have $\Delta^*_{Co_2}(2X, 3Y, 7A) < 84 = |C_{Co_2}(s)|$ for $s \in 7A$ and $(X, Y) \in \{(C, A), (B, B)\}$. Thus, by (1.1), it remains only to establish non-generation of type $(2C, 3B, 7A)$.

Consider the action of Co_2 on its rank-3 graph of valence 891 on 2300 vertices [2]. Under this action the stabilizer of a vertex is isomorphic to $U_6(2).2$, while $U_4(3).2^2$ and $2^9{:}L_3(4){:}2$ represent the double point stabilizers. Thus we may choose U, U^x with $U \cong U^x \cong U_6(2).2$ and $s \in U \cap U^x \cong U_4(3).2^2$. Restriction of the character $\psi \in \mathrm{Irr}(Co_2)$ of degree 23 identifies the involutions in the derived group of U as $2B$-elements. Thus $\Sigma_{U \cap U^x}(2C, 3B, 7A) = 0$, as a Hurwitz group cannot have solvable quotient. We also calculate $\Sigma_U(2C, 3B, 7A) = 84$.

Let now $G \leq Co_1$ be isomorphic to Co_2. The restriction $\varphi \downarrow G (\varphi \in \mathrm{Irr}(Co_1)$ of degree 276) reveals that the G-classes $2C, 3B, 7A$ are of Co_1-type $2C, 3B, 7B$. Under the assumption that G is Hurwitz, we thereby obtain

$$\Delta_{Co_1}(2C, 3B, 7B) \geq |C_{Co_1}(s)| + 2\,\Sigma_U(2C, 3B, 7A) = 1176 + 2(84) = 1344,$$

where s is a representative of $7B$. (Indeed as $C_{Co_1}(G) = 1$, $|C_{Co_1}(s)|$ is a lower bound on the number of distinct pairs (x, y) with x of Co_1-type $2C$, y of Co_1-type $3B$, $xy = s$ and $\langle x, y\rangle$ conjugate to G.) This contradicts the fact that $\Delta_{Co_1}(2C, 3B, 7B) = 1274$. Thus Co_2 is non-Hurwitz as claimed.

LEMMA 3.7. *The group Co_1 is non-Hurwitz.*

Proof. We list below all non-zero $(2,3,7)$-structure constants for Co_1.

X	Y	Z	$\Delta_{Co_1}(2X, 3Y, 7Z)$
A	B	B	147
A	D	A	497
B	B	B	14
B	D	A	17640
B	D	B	2352
C	C	B	392
C	D	B	16464
C	B	B	1274

As the respective centralizers of a $7A$ and $7B$ element have orders 17640 and 1176, we conclude from (1.1) that Co_1 is not $(2X, 3Y, 7Z)$-generated for (X, Y, Z) equal to any of $(A, B, B), (A, D, A), (B, B, B), (C, C, B)$. We now treat the remaining cases.

(a) $(X, Y, Z) = (B, D, A)$. Here $\Sigma_G(2B, 3D, 7A) = 336$ for $G \leq Co_1$ isomorphic to $G_2(4)$. Thus

$$\Delta^*_{Co_1}(2B, 3D, 7A) \leq \Delta_{Co_1}(2B, 3D, 7A) - 336 = 17304$$

and non-generation follows from (1.1).

(b) $(X, Y, Z) = (B, D, B)$. Wilson [12] has accounted for all relevant pairs in conjugates of $N \cong 2^{11}{:}M_{24}$ in Co_1. Thus every Hurwitz subgroup of type $(2B, 3D, 7B)$ is proper.

(c) $(X, Y, Z) = (C, D, B)$. Let K, M denote subgroups of Co_1 isomorphic to Co_3, M_{24}, respectively. Once again we use character restriction to identify relevant conjugate classes:

Co_1	$2C$	$3D$	$7B$
K	$2B$	$3C$	$7A$
M	$2B$	$3B$	$7A$.

Let us further assume M is a complement in $X = 2^{11}{:}M_{24}$. Letting L denote a copy of $L_2(7)$ in M, we see from [12] that $N_{Co_1}(L)$ is contained in X, which is the stabilizer of a vector of type (4) in the 24-dimensional 2-modular representation of the Leech lattice. Furthermore, involutions in L are of M-type $2B$, while elements of order 3 in L are of M-type $3B$. As $O_2(X)$ consists only of elements of type $2A$ and $2C$, we now see that L acts fixed point freely on $O_2(X)$ (as $3D$ fails to commute with $2A$ and $7B$ fails to commute with $2C$) whence L is self-normalizing in Co_1. Thus a fixed element s of order 7 is in precisely 336 conjugates of L. A similar but much easier count yields that s lies in precisely 28 conjugates of K. Recalling from Lemma 2.12 that $\Delta^*_K(2B, 3C, 7A) = 504$, we now obtain

$$\Delta^*_{Co_1}(2C, 3D, 7B) = \Delta_{Co_1}(2C, 3D, 7B) - 28(504) - 336(7) = 0.$$

Thus Co_1 is not $(2C, 3D, 7B)$-generated.

(d) $(X, Y, Z) = (C, B, B)$. As $\Sigma_K(2C, 3B, 7A) = 238$ for $K \leq Co_1$ isomorphic to Co_2 (the K-class $7A$ lying within the Co_1-class $7B$), we have

$$\Delta^*_{Co_1}(2C, 3B, 7B) \leq \Delta_{Co_1}(2C, 3B, 7B) - 238 = 1036.$$

Non-generation now follows from (1.1), and the proof of the lemma is complete.

THEOREM 3.8. *The groups* J_1, J_2, He, Ru, Co_3, HN, Ly *are Hurwitz; the groups* M_{11}, M_{12}, M_{22}, M_{23}, HS, J_3, M_{24}, McL, Suz, $O'N$, Co_2, Co_1 *are non-Hurwitz; the question is unresolved for* Fi_{22}, Th, Fi_{23}, J_4, Fi'_{24}, B, M.

Proof. This is merely a restatement of Lemmas 2.3, 2.4, 2.7, 2.8, 2.9, 2.12, 2.15, 2.16, and Lemmas 3.1 through 3.7.

Finally as $g(G) = 1 + |G|/84$ for G a Hurwitz group, we trivially obtain:

COROLLARY 3.9. *The genera of* J_1, J_2, He, Ru, Co_3, HN, Ly *are given by*:

G	$g(G)$
J_1	2091
J_2	7201
He	47980801
Ru	1737216001
Co_3	5901984001
HN	3250368000001
Ly	616252131000001

4. Automorphism groups of surfaces of least genus

LEMMA 4.1. *Let G be a sporadic simple group and S_G a surface of least genus on which G is effectively and conformally represented. Then the full automorphism group* $\mathrm{Aut}(S_G)$ *of S_G embeds faithfully in* $\mathrm{Aut}\, G$.

Proof. In [14] it is shown that M_{11} and M_{22} admit (2,4,10)- and (2,5,7)-generation respectively. We show directly that McL is (2,4,11)-generated. Indeed $\Delta_{McL}(2A,4A,11A) = 143$, while $\Sigma_M(2A,4A,11A) = 22$ for $M \cong M_{22}$ and $\Sigma_A(2A,4A,11A) = 11$ for $A \cong M_{11}$. As a fixed element of order 11 is easily seen to lie in two $\mathrm{Aut}(McL)$-conjugates M, M^α of M (one from each McL-class) and in a unique conjugate of A, and as no other maximal subgroup of McL has order divisible by 11, we conclude that

$$\Delta^*_{McL}(2A,4A,11A) \geq \Delta_{McL}(2A,4A,11A) - 2(22) - 11 = 88$$

and McL is (2,4,11)-generated as claimed.

By [6], the above generations, together with those established in Theorem 2.24, yield the existence of a surface H/Δ for G (G non-isomorphic to M_{23}) where Δ is the torsion-free kernel of the canonical epimorphism $T(2,s,t) \to G$

and H is the classical hyperbolic plane. (Here $T(r, s, t)$ denotes the Fuchsian triangle group with presentation $\langle X, Y, Z | X^r = Y^s = Z^t = XYZ = 1\rangle$.) From the Riemann-Hurwitz equation, we compute

$$\text{genus}(H/\Delta) = 1 + \frac{|G|}{2}\left[-2 + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{s}\right) + \left(1 - \frac{1}{t}\right)\right],$$

whence it follows that

$$g(G) \leq \text{genus}(H/\Delta) < 1 + \frac{|G|}{12}.$$

To complete the lemma, we shall need the following result from [13], which we state below without proof.

THEOREM B. *Let G be a finite simple $(2, s, t)$-group with genus action on the Riemann surface S arising from the short exact sequence*

$$1 \to \Delta \to \Gamma \to G \to 1$$

Then G is normal in $\operatorname{Aut} S$. *Moreover, if Γ is a triangle group, then* $\operatorname{Aut} S$ *embeds faithfully in* $\operatorname{Aut} G$.

Clearly, from Theorem B and Lemma 6.2, the lemma is proved for G a sporadic group non-isomorphic to M_{23}. But a genus action for M_{23} arises from either $(3,4,4)$- or $(2,6,7)$-generation [14]; indeed non-generation of type $(2,2,2,3)$ has been established by S.P. Norton [7]. Thus Γ is a triangle group, and a final application of Theorem B yields the desired conclusion for M_{23} as well.

We are now in a position to prove our main result.

THEOREM 4.2. *Let G be a sporadic simple group other than McL, Fi_{24}'. Then* $\operatorname{Aut}(S_G) \cong G$ *where S_G is a surface of least genus for G. Moreover if G is isomorphic to one of McL, Fi_{24}', then we have either* $\operatorname{Aut}(S_G) \cong G$ *or* $\operatorname{Aut}(S_G) \cong \operatorname{Aut} G$.

Proof. By the previous lemma, the result follows at once for those sporadics with trivial outer automorphism group, viz. M_{11}, J_1, M_{23}, M_{24}, Ru, Co_3, Co_2, Ly, Th, Fi_{23}, Co_1, J_4, B, M. For the remaining sporadics we have

$$|\operatorname{Aut} G : \operatorname{Aut}(S_G)| \leq |\operatorname{Aut} G : G| = 2.$$

Thus, by Theorem 1 of [8], G is extendible in a genus action only if S_G arises from a short exact sequence of the form

$$1 \to \Delta \to T(r, r, t) \to G \to 1.$$

The minimal genus attainable from such an action is given by $g^* = 1 + |G|/24$, which occurs precisely when $r = 3$ and $t = 4$. Using the earlier established $(2, s, t)$-generations, one easily verifies that $g(G) < g^*$ for G isomorphic to any of M_{12}, M_{22}, J_2, HS, J_3, He, Suz, $O'N$, Fi_{22}, HN. We conclude that only McL and Fi'_{24} can be extendible in their genus actions, and the theorem is proved.

Remark. We comment briefly on the status of Theorem 4.2 for the "exceptional" groups McL and Fi'_{24}.

$M = McL$. As $(2,4,11)$-generation has been established there are three possibilities which give rise to an extendible genus action for M [8].

(1) M fails to admit $(2,4,5)$-generation and there exists a commutative row exact diagram of the form

$$\begin{tikzcd} 1 \arrow[r] & \Delta \arrow[r] & T(2,3,8) \arrow[r] & \operatorname{Aut} M \arrow[r] & 1 \\ 1 \arrow[r] & \Delta \arrow[r] \arrow[u] & T(3,3,4) \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & 1. \end{tikzcd}$$

(In this, and all successive diagrams, the rows indicate canonical epimorphisms while the vertical arrows are (left to right) the identity, inclusion, and inclusion maps, respectively.)

(2) M fails to admit $(2,4,5)$-, $(2,4,6)$- and $(3,3,4)$-generation and there exists a commutative row exact diagram of the form

$$\begin{tikzcd} 1 \arrow[r] & \Delta \arrow[r] & T(2,4,5) \arrow[r] & \operatorname{Aut} M \arrow[r] & 1 \\ 1 \arrow[r] & \Delta \arrow[r] \arrow[u] & T(2,5,5) \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & 1. \end{tikzcd}$$

(3) M fails to admit $(2,4,t)$-, $(3,3,4)$-, and $(2,5,5)$-generation $(5 \leq t \leq 8)$ and there exists a commutative row exact diagram of the form

$$\begin{tikzcd} 1 \arrow[r] & \Delta \arrow[r] & T(2,5,6) \arrow[r] & \operatorname{Aut} M \arrow[r] & 1 \\ 1 \arrow[r] & \Delta \arrow[r] \arrow[u] & T(3,3,5) \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & 1. \end{tikzcd}$$

Each of these cases must be eliminated if one is to extend the result $\mathrm{Aut}(S_G) \cong G$ of Theorem 4.2 to include McL.

$F = Fi'_{24}$. As $(2,3,29)$-generation has been established there are two possibilities which give rise to an extendible genus action for F [8].

(1) F fails to admit $(2,3,t)$- and $(2,4,5)$-generation $(7 \leq t \leq 11)$ and there exists a commutative row exact diagram of the form

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & T(2,3,8) & \longrightarrow & \mathrm{Aut}\, F & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & T(3,3,4) & \longrightarrow & F & \longrightarrow & 1. \end{array}$$

(2) F fails to admit $(2,3,t)$-, $(2,4,5)$-, $(2,4,6)$-, and $(3,3,4)$-generation $(7 \leq t \leq 14)$ and there exists a commutative row exact diagram of the form

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & T(2,4,5) & \longrightarrow & \mathrm{Aut}\, F & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & T(2,5,5) & \longrightarrow & F & \longrightarrow & 1. \end{array}$$

Once again, both cases must be eliminated if one is to extend the result $\mathrm{Aut}(S_G) \cong G$ to include Fi'_{24}.

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