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A REMARK ON THE DEGREES OF COMMUTATIVE ALGEBRAIC GROUPS

BY

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Let G denote a connected commutative algebraic group over an algebraically closed field k and G' an algebraic subgroup on G. Then there is a canonical commutative diagram with exact rows

$$(*)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & L' & \longrightarrow & G' & \longrightarrow & A' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} & & \\ 0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

where L denotes the maximal connected linear subgroup of G, A an abelian variety, $L' = G' \cap L$ and A' the image of G' in A. G is a locally trivial principal L-bundle over A. Hence if X is any L-variety over k, then the space of orbits of $G \times X$ under the action of L,

$$G(X) := G \times^L X,$$

is a locally trivial fibre bundle with fibre X over A. Moreover if $j\colon L \to P$ is an open L-equivariant immersion of L into a projective L-variety P, $G(P)$ is a (projective) compactification of G (cf. [2]). Let $\pi\colon G(P) \to A$ denote the natural projection map. Let M be an L-linearized line bundle on P and N be a line bundle on A. $G(M)$ is a line bundle on $G(P)$ (cf. [2, Lemma 1.2]). It is the aim of the present note to prove the following theorem which answers a question of [1] (cf. Remark 2 below).

THEOREM 1. *Assume that M, N and $G(M) \otimes \pi^*N$ are very ample on P, A and $G(P)$ respectively. Then for the degrees of L', G', and A' for the corresponding projective embeddings we have*

$$\deg_{G(M)\otimes\pi^*N} G' = \binom{\dim G'}{\dim L'} \deg_M L' \cdot \deg_N A'.$$

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Here, if X is any projective variety over k with embedding $X \hookrightarrow \mathbf{P}_n$ associated to a very ample line bundle H on X and Y is any quasiprojective subvariety of X (not necessarily irreducible), we denote by $\deg_H Y$ the degree of the Zariski closure $\overline{Y}$ of Y in $\mathbf{P}_n$.

Remark 2. In the terminology of [1], Theorem 1 can be written as follows:

$$\deg_\varphi(G') = \binom{\dim G'}{\dim L'} \deg_\varphi(L') \cdot \deg_\psi(A')$$

where the hypotheses are as above and $\varphi\colon G \hookrightarrow \mathbf{P}_n$ and $\psi\colon A \hookrightarrow \mathbf{P}_m$ respectively denote the projective embeddings and their restrictions to subvarieties associated to the line bundles $G(M) \otimes \pi^*N$ and N. (Note that $\deg_{G(M)\otimes\pi^*N} L' = \deg_M L'$). In Proposition 7 of [1] the inequality

$$\deg_\varphi(G') \geqq \deg_\varphi(L') \cdot \deg_\psi(A')$$

was given. The above equality over the field of complex numbers implies, as conjectured in Remark 4 of [1], that the volume of the maximal compact subgroup of G' and the degree of G' are equivalent quantities. In the special cases of abelian varieties and linear groups this was proved already in [1] (cf. [1, Propositions 3 and 5]).

Remark 3. In all important cases the first two assumptions that M and N are very ample imply that $G(M) \otimes \pi^*N$ is very ample. For example in the case of Serre's compactification (i.e., using a splitting of L into a product of multiplicative and additive groups and natural embeddings of these groups into $\mathbf{P}_1$) this was proved by Serre in [3] (cf. [3, Corollaire and Remarque 1 of § 1.3]). Moreover if M and N are normally generated, $G(M) \otimes \pi^*N$ is normally generated as well (cf. [2, Theorem 3.5]) which implies in particular the very ampleness.

The proof is divided into 3 parts: First we prove the theorem for the special case $G' = G$. Then we reduce the case in which G' and L' are connected to the first case and finally the general case to the connected one.

1. Proof of the theorem in the case $G' = G$

If for a very ample line bundle H on a projective variety X,

$$P_H(n) = \chi(X, H^n)$$

denotes its Hilbert polynomial, the degree $\deg_H X$ of X with respect to H is given by the highest coefficient of the polynomial $P_H(n)$ multiplied by (dim X)! It follows that it suffices to prove the polynomial identity

$$P_{G(M)\otimes\pi^*N}(n) = P_M(n) \cdot P_N(n).$$

To see this we use the following lemma, for the proof of which we refer to Theorem 3.6 in [2]. (Note that M^n is normally generated for $n \gg 0$.)

LEMMA 4. *Under the above assumptions we have, for $n \gg 0$,*

$$h^\circ(G(P), G(M^n) \otimes \pi^*N^n) = h^\circ(P, M^n) \cdot h^\circ(A, N^n).$$

Here h° denotes the dimension of the corresponding cohomology group. Note that Lemma 4 is valid for every n $\geqq$ 1. In fact the proof of Theorem 3.6 in [2] works in all these cases but we do not need this. Hence

$$\begin{aligned} P_{G(M)\otimes\pi^*N}(n) &= h^\circ(G(P)),(G(M)\otimes\pi^*N)^n) \text{ for } n \gg 0 \\ &\qquad\qquad\qquad\qquad \text{(Kodaira's vanishing theorem)} \\ &= h^\circ(G(P), G(M^n)\otimes\pi^*N^n) \quad [2, \text{Corollary } 1.5] \\ &= h^\circ(P, M^n)\cdot h^\circ(A, N^n) \text{ for } n \gg 0 \quad \text{(Lemma 4)} \\ &= P_M(n)\cdot P_N(n) \text{ for } n \gg 0 \quad \text{(Kodaira's vanishing theorem)}. \end{aligned}$$

Since two polynomials are equal if and only if they have the same values for all integers $n \gg 0$, this implies the assertion.

2. Proof in case G' and L' are connected

In this case L' is the unique maximal connected linear subgroup of G' and the diagram $(*)$ factors as follows:

$$\begin{tikzcd} 0 \arrow[r] & L' \arrow[r] \arrow[d, "i_1"] & G' \arrow[r] \arrow[d, "i_{21}"] & A' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r] \arrow[d, equal] & G'' \arrow[r] \arrow[d, "i_{22}"] & A' \arrow[r] \arrow[d, "i_3"] & 0 \\ 0 \arrow[r] & L \arrow[r] & G \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

where the middle row is the pullback of the lower row, i.e., $G'' = G \times_A A'$ and the pushout of the upper row, i.e., $G'' = G' \times^{L'} L$. Let P' denote the Zariski closure of L' in P. The natural map $j'\colon L' \to P'$ is L'-equivariant and we have the commutative diagram

$$(**)\qquad \begin{array}{ccc} L' & \xrightarrow{j'} & P' \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle \bar{i}_1} \\ L & \xrightarrow[j]{} & P \end{array}.$$

The L-equivariant map i_{22}: $G'' \to G$ induces a map

$$\alpha \colon G''(P) \to G(P)$$

which is a closed immersion, since i_{22}: $G'' \to G$ is so. The L'-equivariant map $\bar{i}_1$: $P' \to P$ induces a map

$$G'(\bar{i}_1) \colon G'(P') \to G'(P)$$

which combined with the canonical isomorphism

$$G'(P) = G' \times^{L'} P = G' \times^{L'} L \times^{L} P = G'' \times^{L} P = G''(P)$$

gives a natural map

$$\beta \colon G'(P') \to G''(P).$$

Since $\bar{i}_1$: $P' \to P$ is a closed immersion, so is β. In particular we get:

LEMMA 5. *$G'(P')$ is the Zariski-closure of G' in $G(P)$.*

In order to prove the theorem in this case it suffices to prove:

LEMMA 6. *There is an isomorphism of line bundles*

$$G'(\bar{i}_1{}^*M) \simeq (\alpha\beta)^* G(M).$$

Since then

$$\begin{aligned} \deg_{G(M)\otimes\pi^*N} G' &= \deg_{(\alpha\beta)^*G(M)\otimes\pi'^*N} G' \quad \text{(definition of deg)} \\ &= \deg_{G'(\bar{i}_1{}^*M)\otimes\pi'^*N} G' \quad \text{(Lemma 6)} \\ &= \binom{\dim G'}{\dim L'} \cdot \deg_{\bar{i}_1{}^*M}(L') \cdot \deg_{\pi'^*N} A' \\ &\qquad\qquad\qquad\qquad \text{(Part 1 of the proof)} \\ &= \binom{\dim G'}{\dim L'} \cdot \deg_M L' \cdot \deg_N A' \quad \text{(definition of deg)} \end{aligned}$$

Here π': $G'(P') \to A'$ denotes the natural projection map.

Proof of Lemma 6. First we have

$$\begin{aligned}\alpha^*G(M) &= G(M)\times_{G(P)}G''(P)\\ &= G\times^L M\times_{G\times^L P}G''\times^L P\\ &= G''\times^L M = G''(M)\end{aligned}$$

and finally

$$\begin{aligned}\beta^*G''(M) &= G'(P')\times_{G''(P)}(G''\times^L M)\\ &= (G'\times^{L'}P')\times_{G'\times^{L'}L\times^L P}(G'\times^{L'}L\times^L M)\\ &= (G'\times^{L'}P')\times_{G'\times^{L'}P}(G'\times^{L'}M)\\ &= G'\times^{L'}(P'\times_P M)\\ &= G'(\bar{i}_1{}^*M)\end{aligned}$$

which completes the proof in case 2.

3. Proof in the general case

It is obvious that the theorem is valid in general if it is proved for any connected subgroup G'. Hence we may assume that G' is connected. Let L_0' denote the component of L' containing the identity element. Then we have the following diagram with exact rows and columns (cf. [1, Proposition 6])

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & L'/L_0' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_0' & \longrightarrow & G' & \longrightarrow & B & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle i_0} & & \| & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & L' & \longrightarrow & G' & \longrightarrow & A' & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & L'/L_0' & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Here B is an abelian variety and φ an isogeny with kernel L'/L_0'. According to Lemma 5 and 6 (note that they are valid in this slightly more general situation) we have to show that

$$\deg_{G'(M')\otimes\pi'^*N'} G' = \binom{\dim G'}{\dim L'} \deg_{M'} L' \cdot \deg_{N'} A'$$

with $M' = \bar{i}_1{}^*M$ and $N' = i_3{}^*N$, where we use the notations of diagrams $(*)$ and $(**)$, the latter applied in the slightly more general situation of a not necessarily connected L'. Let P_0' denote the Zariski closure of L_0' in P' and let $\bar{i}_0\colon P_0' \to P'$ denote the natural embedding.

LEMMA 7. *$\bar{i}_0$ induces an isomorphism $G'(P_0') \xrightarrow{\sim} G'(P')$ of k-varieties.*

Note that $G'(P_0')$ is a fibre bunde over B and $G'(P')$ a fibre bunde over A'. The isomorphism does not respect any fibre bundle structure.

Proof. The morphism

$$\alpha\colon \begin{cases} G' \times^{L_0'} P_0' \longrightarrow G' \times^{L'} P' \\ (g', p_0') \longmapsto (g', \bar{i}_0 p_0') \end{cases}$$

is obviously injective. To see that it is surjective consider an element

$$(g', p') \in G' \times^{L'} P'.$$

Since L_0' is the identity component of L' and P_0' is the component of P' containing L_0' there is an $l \in L'$ such that $lp' \in P_0'$. Hence

$$(g', p') = (g'l^{-1}, \bar{i}_0(lp') = \alpha(g'l^{-1}, lp')$$

which means that α is surjective.

Identifying both sides of Lemma 7 we get

$$G'(\bar{i}_0{}^*M') = G' \times^{L_0'} P_0' \times_{P'} M' = G' \times^{L'} P' \times_{P'} M' = G' \times^{L'} M' = G'(M')$$

and hence

$$\deg_{G'(M')\otimes\pi'^*N'} G' = \deg_{G'(\bar{i}_0{}^*M')\otimes\pi_B^*\varphi^*N'} G'$$

where $\pi_B\colon G'(P_0') \to B$ denotes the natural projection map. Since φ is finite, φ^*N' is very ample and we may apply Part 1 of the proof to the last expression to get

$$\deg_{G'(\bar{i}_0^*M')\otimes\pi_B^*\varphi^*N'} G' = \binom{\dim G'}{\dim L_0'} \deg_{\bar{i}_0^*M'} L_0' \cdot \deg_{\varphi^*N'} B.$$

Now on the one hand we have

$$\deg_{i_0^* M'} L_0' = \frac{1}{|L'/L_0'|} \cdot \deg_{M'} L'$$

since L' has $|L'/L_0'|$ components one of which is L_0' and on the other hand

$$\deg_{\varphi^* N'} B = |L'/L_0'| \cdot \deg_{N'} A'$$

since φ is an isogeny of degree $|L'/L_0'|$. This completes the proof of the assertion and thus of the theorem.

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