ILLINOIS JOURNAL OF MATHEMATICS
Volume 33, Number 3, Fall 1989

ON THE THEORY OF THE REAL EXPONENTIAL FIELD

BY

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1. Introduction and notation

Let L be the usual first order language of ordered rings together with a new unary function symbol e. We are interested in the L-structure $\mathbf{R}_e = \langle \mathbf{R}, 0, 1, +, \cdot, -, <, e \rangle$ consisting of the ordered field of real numbers with $e(x)$ interpreted as the exponential function e^x (and we shall henceforth write e^x for $e(x)$ in any L-structure). We denote by T_e the L-theory of $\mathbf{R}_e$. This theory and its subtheories have been investigated by many authors and we refer the reader to Macintyre [4] for a comprehensive survey. We are concerned here with the problem of determining whether T_e is model complete, that is whether $k, K \models T_e$ and $k \subseteq K$ imply $k \preccurlyeq K$, or equivalently $k \preccurlyeq_1 K$ (i.e., existential formulas with parameters in k are preserved down from K to k). We shall prove the following:

THEOREM 1. *Suppose $k, K \models T_e$, $k \subseteq K$ and k is cofinal in K (i.e., if $a \in K$ then $b < a < c$ for some $b, c \in k$). Then $k \preccurlyeq_1 K$.*

(Unfortunately there seems to be no general model theoretic argument that allows us to deduce that $k \preccurlyeq K$ here.)

We shall actually prove a result slightly stronger than Theorem 1 which allows us to isolate a plausible conjecture that would imply the model completeness of T_e. To state this result we require some notation.

Let us fix a model K of T_e and a substructure k of K. We also assume that k is a field. For $n \in \mathbf{N}$ we denote by $k[\vec{x}]^e$ the set of all terms of $L(k)$ (defined as L together with a constant symbol for each element of k) in the variables $\vec{x} = x_1, \ldots, x_n$ factored by the equivalence relation

$$f \sim g \quad \text{iff} \quad T_e \vdash \forall \vec{x} f = g.$$

Since it is known (see [4]) that $f \sim g$ iff $k \models \forall \vec{x} f = g$ it will be harmless to

Received May 18, 1987.

identify the elements of $k[\vec{x}]^e$ with the corresponding functions on k (or on K) or with the terms themselves.

Apart from being naturally an L-structure, $k[\vec{x}]^e$ also admits a differential structure: for $i = 1, \dots, n$ and $f \in k[\vec{x}]^e$ we define

$$\frac{\partial f}{\partial x_i} \in k[\vec{x}]^e$$

by induction on f by

$$\frac{\partial a}{\partial x_i} = 0 \text{ for } a \in k;$$

$$\frac{\partial x_j}{\partial x_i} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise;} \end{cases}$$

$$\frac{\partial (f \pm g)}{\partial x_i} = \frac{\partial f}{\partial x_i} \pm \frac{\partial g}{\partial x_i};$$

$$\frac{\partial (f \cdot g)}{\partial x_i} = f \cdot \frac{\partial g}{\partial x_i} + g \cdot \frac{\partial f}{\partial x_i}; \; \frac{\partial (e^f)}{\partial x_i} = \frac{\partial f}{\partial x_i} \cdot e^f.$$

It can be shown (see [4]) that $\partial / \partial x_i$ respects the equivalence relation $\sim$ and that the ring of absolute constants in the differential ring

$$\left\langle k[\vec{x}]^e, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle$$

is the field k. It is also known that $k[\vec{x}]^e$ is an integral domain and we denote by $k(\vec{x})^e$ the field of fractions of $k[\vec{x}]^e$, but note that $k(\vec{x})^e$ is not closed under exponentiation although the partial derivatives extend naturally to $k(\vec{x})^e$. If $h \in k(\vec{x})^e$ and $\vec{\alpha} \in K^n$ we say that *h is defined at $\vec{\alpha}$* if h can be written as $f \cdot g^{-1}$ with $f, g \in k[\vec{x}]^e$ and $g(\vec{\alpha}) \neq 0$. Note that if h is defined at $\vec{\alpha}$ then so are all its partial derivatives.

We now need to introduce Jacobians and a convenient way to do this here is via the notation of differential forms.

For $p \in \mathbf{N}$ and $M = k[\vec{x}]^e$ or $k(\vec{x})^e$, the set $F_p(M)$ of *differential p-forms* (over M) is defined to be M for $p = 0$, $\{0\}$ for $p > n$, and, for $1 \leq p \leq n$, the collection of objects of the form

$$\sigma = \sum_{\vec{i}} f_{\vec{i}} \left(dx_{i_1} \wedge \cdots \wedge dx_{i_p} \right)$$

where the summation is over all increasing p-tuples $\vec{i} = i_1 \cdots i_p$ taken from the set $\{1, \dots, n\}$ and each $f_{\vec{i}}$ is an element of M.

Thus (in all cases) $F_p(M)$ is a free M-module on $\binom{n}{p}$ generators.

The *exterior product* $\wedge : F_p(M) \times F_q(M) \to F_{p+q}(M)$ is defined as follows: if σ is the p-form given above and

$$\tau = \sum_{\vec{j}} g_{\vec{j}}\left(dx_{j_1} \wedge \cdots \wedge dx_{j_q}\right)$$

is a q-form, then

$$(\sigma \wedge \tau) = \sum_{\vec{i}, \vec{j}} \left(f_{\vec{i}} \cdot g_{\vec{j}}\right) \cdot \left(dx_{i_1} \wedge \cdots \wedge dx_{ip} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}\right)$$

where the summation is taken over all increasing p-tubles $\vec{i} = i_1 \cdots i_p$ and increasing q-tuples $\vec{j} = j_1 \cdots j_q$ from $\{1, \dots, n\}$ and is put into the correct shape for a $(p+q)$-form by invoking the rule

$$\left(dx_i \wedge dx_j\right) = -\left(dx_j \wedge dx_i\right) \quad \text{for } 1 \leq i,\ j \leq n$$

(so $dx_i \wedge dx_i = 0$ for $1 \leq i \leq n$) and specifying that $\wedge$ is associative and distributive with respect to addition.

The *exterior derivative* $d: M \to F_1(M)$ is defined by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot dx_i \quad \text{for } f \in M.$$

The reader may easily verify that if $f_1, \dots, f_p \in M$ $(1 \leq p \leq n)$ and $1 \leq i_1 < \cdots < i_p \leq n$, then the coefficient of $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ in the p-form $df_1 \wedge \cdots \wedge df_p$ is the determinant of the *Jacobian matrix*

$$\frac{\partial(f_1, \dots, f_p)}{\partial(x_{i_1}, \dots, x_{i_p})} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_{i_1}} & \cdots & \dfrac{\partial f_1}{\partial x_{i_p}} \\ \vdots & & \vdots \\ \dfrac{\partial f_p}{\partial x_{i_1}} & \cdots & \dfrac{\partial f_p}{\partial x_{i_p}} \end{pmatrix}.$$

If σ is the p-form given above and $\vec{\alpha} \in K^n$ then we write $\sigma(\vec{\alpha}) = 0$ if each $f_{\vec{i}}$ is defined at $\vec{\alpha}$ and $f_{\vec{i}}(\vec{\alpha}) = 0$. We call a point $\vec{\alpha} \in K^n$ *exponential-algebraic* (e.a.) *over* k if for some $f_1, \dots, f_n \in k[\vec{x}]^e$ we have

$$f_1(\vec{\alpha}) = \cdots = f_n(\vec{\alpha}) = 0 \quad \text{and} \quad (df_1 \wedge \cdots \wedge df_n)(\vec{\alpha}) \neq 0.$$

Our main theorem can now be stated.

THEOREM 2. *Suppose* $k, K \models T_e$, $k \subseteq K$, *and for all* $n \in \mathbf{N}$ *and all e.a. points over* k, $\langle \alpha_1, \dots, \alpha_n \rangle \in K^n$, *there exist* $a, b \in k$ *such that* $a < \alpha_i < b$ *for* $i = 1, \dots, n$. *Then* $k \preccurlyeq_1 K$.

Clearly Theorem 1 follows from Theorem 2. Of course to prove the model completeness of T_e it would be sufficient to show that the hypothesis of Theorem 2 on the models k, K is always satisfied. This has been shown for $n = 1$ by Dahn [1] but a proof even for the case $n = 2$ seems to be beyond present methods. Dahn's result actually establishes something stronger, namely the case $n = 1$ of the following:

Conjecture. Let $n, r \in \mathbf{N}$, $n \geq 1$, and suppose

$$f_i(y_1, \ldots, y_r, x_1, \ldots, x_n)$$

is a term of L for $i = 1, \ldots, n$. Then there are terms

$$g_1(y_1, \ldots, y_r), \ldots, g_s(y_1, \ldots, y_r)$$

of $L \cup \{^{-1}\}$ (where $^{-1}$ is interpreted as multiplicative inverse, and is undefined at 0) such that for all

$$\vec{\alpha} = \langle \alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_{r+n} \rangle \in \mathbf{R}^{r+n},$$

if (working in the structure $\mathbf{R}_e$ throughout) $f_i(\vec{\alpha}) = 0$ for $i = 1, \ldots, n$ and

$$(df_1 \wedge \cdots \wedge df_n)(\vec{\alpha}) \neq 0$$

(where the exterior derivatives are taken with respect to $x_1, \ldots, x_n$) then, for some j, $1 \leq j \leq s$, we have $g_j(\alpha_1, \ldots, \alpha_r)$ defined and

$$|\alpha_t| < g_j(\alpha_1, \ldots, \alpha_r) \quad \text{for } t = r+1, \ldots, r+n.$$

The truth of this conjecture would clearly allow us to remove the hypothesis of theorem 2, and hence would imply the model completeness of T_e. However, under present knowledge it is possible that T_e is model complete yet the conjecture false.

2. Transfer

Since $K \models T_e$ we may use results from calculus (say) when working in K provided such results are first-order expressible in L uniformly in any parameters that occur. When doing this we shall simply use the phrase "by transfer".

For example, suppose

$$f_1, \ldots, f_p \in k(\vec{x})^e \quad (\vec{x} = x_1, \ldots, x_n,\ 1 \leq p < n)$$

and

$$\vec{\alpha} = \langle \alpha_1, \ldots, \alpha_n \rangle \in K^n$$

satisfies $f_i(\vec{\alpha}) = 0$ (and is defined) for $i = 1, \ldots, p$ and

$$(df_1 \wedge \cdots \wedge df_p)(\vec{\alpha}) \neq 0.$$

For convenience suppose the coefficient of

$$dx_{n-p+1} \wedge \cdots \wedge dx_n \quad \text{in} \quad df_1 \wedge \cdots \wedge df_p,$$

i.e.,

$$\det \frac{\partial(f_1, \ldots, f_p)}{\partial(x_{n-p+1}, \ldots, x_n)},$$

does not vanish at $\vec{\alpha}$. Then by the implicit function theorem and transfer, there are neighbourhoods U of $\langle \alpha_1, \ldots, \alpha_{n-p} \rangle$ in K^{n-p} and U' of $\langle \alpha_{n-p+1}, \ldots, \alpha_n \rangle$ in K^p (i.e.,

$$U = \left\{ \langle q_1, \ldots, q_{n-p} \rangle \in K^{n-p} \colon \sum_{i=1}^{n-p} (\alpha_i - q_i)^2 < \beta \right\}$$

for some $\beta \in K$, $\beta > 0$, and similarly for U') such that for any

$$\langle q_1, \ldots, q_{n-p} \rangle \in U$$

there is a unique

$$\langle q_{n-p+1}, \ldots, q_n \rangle \in U'$$

such that

$$f_1(q_1, \ldots, q_n) = \cdots = f_p(q_1, \ldots, q_n) = 0$$

(and, of course, these are all defined). Further, the uniqueness here guarantees that there are K-definable functions $\phi_1, \ldots, \phi_p$: $U \to K$ such that for all $\vec{q} \in U$, $f_i(\vec{q}, \phi_i(\vec{q}), \ldots, \phi_p(\vec{q})) = 0$ $(i = 1, \ldots, p)$ and these functions will be r-times differentiable in U (for any $r \in \mathbf{N}$) according to the usual ε-δ definition interpreted in K, and their derivatives will be given by the usual formula associated with the implicit function theorem (see [2] for example). More generally, suppose $g \in k(\vec{x})^e$ and let us consider the K-definable function

$$\bar{g} \colon U \to K, \quad \vec{q} \mapsto g(\vec{q}, \phi_1(\vec{q}), \ldots, \phi_p(\vec{q})),$$

which we assume defined throughout U. For $i = 1, \ldots, n - p$ let

$$\frac{\partial \bar{g}}{\partial x_i} \colon U \to K$$

denote the ith partial derivative of $\bar{g}$. Then by the chain rule, for $\vec{q} \in U$ we have

$$\frac{\partial \bar{g}}{\partial x_i}(\vec{q}) = \left(\frac{\partial g}{\partial x_i} + \sum_{j=1}^{p} \frac{\partial g}{\partial x_{n-p+j}} \cdot \frac{\partial \phi_j}{\partial x_i} \right) \left(\vec{q}, \phi_1(\vec{q}), \ldots, \phi_p(\vec{q}) \right).$$

In particular, since $\bar{f}_s$ is identically zero for $s = 1, \ldots, p$, the right-hand side is too for $g = f_s$. These equations can be expressed in matrix form as follows. Let

$$A = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_p}{\partial x_1} & \cdots & \dfrac{\partial f_p}{\partial x_n} \\ \dfrac{\partial g}{\partial x_1} & \cdots & \dfrac{\partial g}{\partial x_n} \end{pmatrix},$$

$$v^{(i)} = \begin{pmatrix} \delta_{i,0} \\ \vdots \\ \delta_{i,n-p} \\ \dfrac{\partial \phi_1}{\partial x_i} \\ \vdots \\ \dfrac{\partial \phi_p}{\partial x_i} \end{pmatrix}$$

where

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and

$$u^{(i)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \dfrac{\partial \bar{g}}{\partial x_i} \end{pmatrix} \quad (p \text{ zeroes}).$$

Then we have $Av^{(i)} = u^{(i)}$ for $i = 1, \ldots, n-p$, where we evaluate at the point $\vec{\alpha}$. (Note that $\phi_i(\alpha_1, \ldots, \alpha_{n-p}) = \alpha_{n-p+i}$ for $i = 1, \ldots, p$.)

Now if the rows of A are linearly independent then

$$\dim(\operatorname{Ker} A) = n - (p+1),$$

so for some $i = 1, \dots, n-p$, $0 \neq Av^{(i)}$ (since the $v^{(i)}$'s are clearly linearly independent), hence

$$\frac{\partial \bar{g}}{\partial x_i}(\alpha_1, \dots, \alpha_{n-p}) \neq 0.$$

Also, the converse of this is clear from the original equations. Now by elementary linear algebra, the rows of A are linearly dependent if and only if all its $(p+1) \times (p+1)$ submatrices have vanishing determinants. But these determinants are exactly the coefficients of the $p+1$-form

$$df_1 \wedge \cdots \wedge df_p \wedge dg.$$

To sum up, we have (working in K)

$$\left(df_1 \wedge \cdots \wedge df_p \wedge dg\right)(\vec{\alpha}) = 0$$

if and only if all the partial derivatives of $\bar{g}$ vanish at $\langle \alpha_1, \dots, \alpha_{n-p} \rangle$. In particular if the $p+1$-form

$$df_1 \wedge \cdots \wedge df_p \wedge dg$$

vanishes on

$$(U \times U') \cap \left\{\vec{\beta} \in K^n \colon f_i\left(\vec{\beta}\right) = 0 \text{ for } i = 1, \dots, p\right\},$$

then g is constant on this set, and conversely.

3. Exponential varieties

Suppose $f_1, \dots, f_p \in k[\vec{x}]^e$. We define

$$V\left(f_1, \dots, f_p\right) = \left\{\vec{\alpha} \in K^n \colon f_i(\vec{\alpha}) = 0 \text{ for } i = 1, \dots, p\right\},$$

and

$$V^{\mathrm{ns}}\left(f_1, \dots, f_p\right) = \left\{\vec{\alpha} \in V\left(f_1, \dots, f_p\right) \colon \left(df_1 \wedge \cdots \wedge df_p\right)(\vec{\alpha}) \neq 0\right\}.$$

Thus $V(f_1, \dots, f_p)$ is the "variety" determined by $f_1, \dots, f_p$, and

$$V^{\mathrm{ns}}\left(f_1, \dots, f_p\right)$$

consists of its "non-singular" points.

We wish to show now that if

$$\vec{\alpha} \in V^{\mathrm{ns}}(f_1, \ldots, f_p) \quad (p < n)$$

and

$$\vec{\alpha} \notin V^{\mathrm{ns}}(f_1, \ldots, f_{p+1}) \quad \text{for any } f_{p+1} \in k[\vec{x}]^e,$$

then the f_i's can be chosen with the additional property that whenever $g \in k[\vec{x}]^e$, if $g(\vec{\alpha}) = 0$ then g vanishes on $U \cap V(f_1, \ldots, f_p)$ for some neighbourhood U of $\vec{\alpha}$ in K^n, a property we shall usually refer to by saying "g vanishes on $V(f_1, \ldots, f_p)$ close to $\vec{\alpha}$". The proof of this goes by induction on terms and the ranking of terms is defined (at least for the present purpose) as follows. Let us suppose that k is countable. We define

$$M_i = k[x_1, \ldots, x_1] \quad \text{for } 0 \leq i \leq n$$

and

$$M_{i+1} = M_i[e^{g_i}] \quad \text{for } i \geq n,$$

where each $g_i \in M_i$ is chosen in some way so that $k[\vec{x}]^e = \bigcup_{i \in \mathbf{N}} M_i$. Clearly this is possible and note that each M_i (and its field of fractions) is closed under partial differentiation. We now define *rank*(f) (for $f \in k[\vec{x}]^e$) as the least i such that $f \in M_i$.

LEMMA 1. *Suppose $n \in \mathbf{N}$, $n \geq 1$, and let S be any non-empty subset of K^n. Then for some $p \in \mathbf{N}$, $0 \leq p \leq n$, there are $h_1, \ldots, h_p \in k[\vec{x}]^e$ such that:*

(1) $\operatorname{rank}(h_1) < \cdots < \operatorname{rank}(h_p)$.

(2) *For some $\vec{\alpha} \in S$,*

$$h_1(\vec{\alpha}) = \cdots = h_p(\vec{\alpha}) = 0 \quad \text{and} \quad (dh_1 \wedge \cdots \wedge dh_p)(\vec{\alpha}) \neq 0.$$

(3) *Either $p = n$ or for any $\vec{\beta} \in S$ and $h \in k[\vec{x}]^e$, if*

$$h_1(\vec{\beta}) = \cdots = h_p(\vec{\beta}) = h(\vec{\beta}) = 0 \quad \text{and} \quad (dh_1 \wedge \cdots \wedge dh_p)(\vec{\beta}) \neq 0$$

then h vanishes on $V(h_1, \ldots, h_p)$ close to $\vec{\beta}$.

Proof. Suppose we have proved the lemma with M_j in place of $k[\vec{x}]^e$ for some $j \geq 0$ (it being trivial for $j = 0$), Denote the corresponding three conditions by $(1)_j$, $(2)_j$ and $(3)_j$. We wish to extend the set $\{h_1, \ldots, h_p\}$ so that $(1)_{j+1}$, $(2)_{j+1}$ and $(3)_{j+1}$ are satisfied for the extended set.

Case 1. $(3)_{j+1}$ is satisfied with the same $h_1, \ldots, h_p$.

Clearly there is nothing to do here since $(1)_j$ and $(2)_j$ are certainly still satisfied.

Case 2. Not Case 1.
Then there is some $h \in M_{j+1}$ and:
(*) There exists $\vec{\beta} \in S$ such that

$$h_1(\vec{\beta}) = \cdots = h_p(\vec{\beta}) = h(\vec{\beta}) = 0,$$
$$(dh_1 \wedge \cdots \wedge dh_p)(\vec{\beta}) \neq 0$$

and h does not vanish on $V(h_1, \ldots, h_p)$ close to $\vec{\beta}$.

Subcase 2(a). $j < n$.
Then $h = \sum_{i=0}^{s} a_i x_{j+1}^i$, where $a_0, \ldots, a_s \in M_j$ and we may suppose that s is minimal such that (*) holds, witnessed by $\vec{\beta} = \langle \beta_1, \ldots, \beta_n \rangle \in S$ say. By $(3)_j$, $h \notin M_j$ and hence $(1)_{j+1}$ holds for $\{h_1, \ldots, h_p, h\}$. Also

$$h_1(\vec{\beta}) = \cdots = h_p(\vec{\beta}) = h(\vec{\beta}) = 0$$

by (*). Now suppose, for contradiction that

$$(dh_1 \wedge \cdots \wedge dh_p \wedge dh)(\vec{\beta}) = 0.$$

Since

$$(dh_1 \wedge \cdots \wedge dh_p)(\vec{\beta}) \neq 0$$

we have

$$F(\vec{\beta}) \neq 0 \quad \text{for some } 1 \leq i_1 < \cdots < i_p \leq j$$

where F is the coefficient of $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ in $dh_1 \wedge \cdots \wedge dh_p$ (note that $\partial f / \partial x_t = 0$ for all $f \in M_j$ and $t > j$). Since the coefficient of

$$dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j+1} \quad \text{in} \quad dh_1 \wedge \cdots dh_p \wedge dh$$

is clearly

$$F \cdot \frac{\partial h}{\partial x_{j+1}},$$

we have

$$\frac{\partial h}{\partial x_{j+1}}(\vec{\beta}) = 0.$$

By the minimality of s this implies that $\partial h / \partial x_{j+1}$ vanishes on

$$V(h_1, \ldots, h_p) \cap U = Y,$$

say for some neighbourhood U of $\vec{\beta}$ in K^n. But clearly

$$\{\langle \beta_1, \dots, \beta_j \rangle\} \times U' \subseteq Y$$

for some neighbourhood U' of $\langle \beta_{j+1}, \dots, \beta_n \rangle$ in K^{n-j}, so the polynomial

$$\sum_{i=1}^{s} i a_i(\vec{\beta}) x_{j+1}^{i-1} \quad (\in K[x_{j+1}])$$

is identically zero. So

$$a_s(\vec{\beta}) = \cdots = a_1(\vec{\beta}) = 0,$$

and since $h(\vec{\beta}) = 0$, we have $a_0(\vec{\beta}) = 0$ also. But by $(3)_j$ this implies a_i vanishes close to $\vec{\beta}$ on $V(h_1, \dots, h_p)$ for $i = 0, \dots, s$, and hence so does h, contradicting $(*)$. This establishes $(2)_{j+1}$.

To show that $(3)_{j+1}$ is satisfied for $\{h_1, \dots, h_p, h\}$ consider any $H \in M_{j+1}$ and $\vec{\gamma} \in S$ such that

$$h_1(\vec{\gamma}) = \cdots = h_p(\vec{\gamma}) = h(\vec{\gamma}) = H(\vec{\gamma}) = 0$$

and

$$(dh_1 \wedge \cdots \wedge dh_p \wedge dh)(\vec{\gamma}) \neq 0.$$

Now (by the Euclidean algorithm) there exist $F_1, F_2 \in M_{j+1}$, $m \in \mathbf{N}$ such that

$$a_s^m \cdot H = F_1 \cdot h + F_2$$

(this being an identity in the ring M_{j+1}) where F_2 has degree $< s$ (as a polynomial in x_{j+1} over M_j). Clearly $F_2(\vec{\gamma}) = 0$ so by the minimality of s, F_2 vanishes on $V(h_1, \dots, h_p)$ close to $\vec{\gamma}$. Since h obviously vanishes everywhere on $V(h_1, \dots, h_p, h)$ it follows that $a_s^m \cdot H$ vanishes on $V(h_1, \dots, h_p, h)$ close to $\vec{\gamma}$. However, using the minimality of s again and $(3)_j$ it is easy to show that $a_s(\vec{\gamma}) \neq 0$ and hence (by transfer) a_s is non-zero throughout some sufficiently small neighbourhood of $\vec{\gamma}$ in K^n. It follows that H vanishes on $V(h_1, \dots, h_p, h)$ close to $\vec{\gamma}$ as required.

Subcase 2(*b*). $j \geq n$.

Write g for g_j. Then $h = \sum_{i=0}^{s} a_i e^{ig}$, where $a_0, \dots, a_s \in M_j$ and we may suppose again that s is minimal such that $(*)$ holds, witnessed by $\vec{\beta} \in S$, say. As in subcase 2(a) we have $(1)_{j+1}$ holding for $\{h_1, \dots, h_p, h\}$ and

$$h_1(\vec{\beta}) = \cdots = h_p(\vec{\beta}) = h(\vec{\beta}) = 0.$$

To show $(dh_1 \wedge \cdots \wedge dh_p \wedge dh)(\vec{\beta}) \neq 0$, we let

$$\sigma = dh_1 \wedge \cdots \wedge dh_p$$

and consider the $(p+1)$-form

$$\tau = a_0 \cdot (\sigma \wedge dh) - h \cdot (\sigma \wedge da_0).$$

Now

$$\tau = \sigma \wedge e^g \cdot \sum_{l=1}^{n} F_l \, ds_l,$$

where, for $l = 1, \dots, n$,

$$\begin{aligned} F_l &= e^{-g}\left(a_0 \frac{\partial h}{\partial x_l} - h \frac{\partial a_0}{\partial x_l}\right) \\ &= e^{-g} \cdot \sum_{i=0}^{s} \left[a_0\left(\frac{\partial a_i}{\partial x_l} + i a_i \frac{\partial g}{\partial x_l}\right) - a_i \frac{\partial a_0}{\partial x_l}\right] \cdot e^{ig} \\ &= \sum_{i=0}^{s-1} \left(a_0 \frac{\partial a_{i+1}}{\partial x_l} + a_0 (i+1) a_{i+1} \frac{\partial g}{\partial x_l} - a_{i+1} \frac{\partial a_0}{\partial x_l}\right) \cdot e^{ig} \\ &= p_l(e^g) \text{ say,} \end{aligned}$$

where p_l is a polynomial over M_j of degree $< s$.

Now since the coefficients of σ are all elements of M_j it follows that the coefficients of τ are all of the form $e^g \cdot \bar{p}(e^g)$ where $\bar{p}$ is a polynomial over M_j of degree $< s$. Hence if $(\sigma \wedge dh)(\vec{\beta}) = 0$, then $\tau(\vec{\beta}) = 0$ (since $h(\vec{\beta}) = 0$) and so by the minimality of s, τ vanishes on $V(h_1, \dots, h_p)$ close to $\vec{\beta}$, that is $\sigma \wedge (a_0 \, dh - h \, da_0)$ vanishes on $V(h_1, \dots, h_p)$ close to $\vec{\beta}$. Now if $a_0(\vec{\beta}) = 0$ then

$$\left(e^{-g} \cdot (h - a_0)\right)\left(\vec{\beta}\right) = 0,$$

so by $(3)_j$ and the minimality of s (and the fact that $e^{-g}(\vec{\beta}) \neq 0$) both a_0 and $e^{-g} \cdot (h - a_0)$ would vanish on $V(h_1, \dots, h_p)$ close to $\vec{\beta}$—hence so would h, contradicting $(*)$. Thus $a_0(\vec{\beta}) \neq 0$ and we may write

$$a_0 \, dh - h \, da_0 = a_0^2 \cdot d\left(h \cdot a_0^{-1}\right)$$

and conclude that $\sigma \wedge d(h \cdot a_0^{-1})$ is defined and vanishes on $V(h_1, \dots, h_p)$ close to $\vec{\beta}$. However, by the comments in Section 2, this implies that $h \cdot a_0^{-1}$ is constant, hence 0, on $V(h_1, \dots, h_p)$ close to $\vec{\beta}$ which contradicts $(*)$ and establishes $(2)_{j+1}$ for $\{h_1, \dots, h_p, h\}$. The proof of $(3)_{j+1}$ for $\{h_1, \dots, h_p, h\}$ is similar to the proof in subcase 2(a) and is left to the reader.

This completes our inductive construction, which clearly implies the lemma, since for some j_0, case 1 must hold for all $j \geq j_0$ and every $g \in k[\vec{x}]^e$ lies in some M_j. □

We now slightly modify our notion of rank.

We call a subring M of $k[\vec{x}]^e$ *of height* 0 if $M = k[\vec{x}]$, and *of height* $\leq j+1$ if $M = \overline{M}[e^g]$ for some $g \in \overline{M}$, where $\overline{M}$ has height $\leq j$. An element, h, of M has degree $\leq s$ (in M) if $h = \sum_{i=0}^{s} a_i e^{ig}$ for some $a_0, \dots, a_s \in \overline{M}$.

LEMMA 2. *Let $j, n \in \mathbf{N}$, $n \geq 1$ and suppose $\vec{\alpha} \in K^n$. Let M have height j and suppose p is maximal such that for some $g_1, \dots, g_p \in M$,*

$$\vec{\alpha} \in V^{\mathrm{ns}}(g_1, \dots, g_p).$$

If $j \geq 1$, suppose that $g_1, \dots, g_p$ all have degree $\leq s$. Then there are $h_1, \dots, h_p \in M$ such that:

(1) *If $j \geq 1$ then $h_1, \dots, h_{p-1}$ have degree ≤ 0 in M and h_p has degree $\leq s$ (in M).*

(2) $\vec{\alpha} \in V^{\mathrm{ns}}(h_1, \dots, h_p)$.

(3) *If $h \in M$, $p < n$ and $h(\vec{\alpha}) = 0$, then h vanishes on $V(h_1, \dots, h_p)$ close to $\vec{\alpha}$.*

Proof. By the proof of Lemma 1 (i.e., using the result of Lemma 1 with M in place of $k[\vec{x}]^e$) we can find $h_1, \dots, h_{p'}$ satisfying the first clause of (1), (2) and (3) (where we are applying Lemma 1 with $S = \{\vec{\alpha}\}$). Now $p' \leq p$, by the maximality of p, and since $g_1, \dots, g_p$ all vanish, in particular are constant, on $V(h_1, \dots, h_{p'})$ close to $\vec{\alpha}$, we have (by Section 2) that

$$(dh_1 \wedge \cdots \wedge dh_{p'} \wedge dg_i)(\vec{\alpha}) = 0 \quad \text{for } i = 1, \dots, p.$$

The fact that this implies $p = p'$ now follows from the following result, the simple proof of which is left to the reader:

(∗∗) Suppose σ is a q-form and $\sigma_1, \dots, \sigma_{q+1}$ are 1-forms such that

$$(\sigma \wedge \sigma_i)(\vec{\alpha}) = 0 \quad \text{for } i = 1, \dots, q+1.$$

Then either $\sigma(\vec{\alpha}) = 0$ or $(\sigma_1 \wedge \cdots \wedge \sigma_{q+1})(\vec{\alpha}) = 0$.

Now recall that h_p was chosen (in the proof of Lemma 1) of minimal degree such that $h_p(\vec{\alpha}) = 0$ but such that h_p did not vanish on $V(h_1, \dots, h_{p-1})$ close to $\vec{\alpha}$. Thus, if this degree is $> s$, a similar argument to the above shows that $(dh_1 \wedge \cdots \wedge dh_{p-1} \wedge dg_i)(\vec{\alpha}) = 0$ for $i = 1, \dots, p$, which again contradicts (∗∗). □

4. Constructing e.a. points

Let $A(\vec{y})$ be an existential formula of L. Since the sentences

$$\forall x, y\big((x = 0 \vee y = 0) \leftrightarrow x \cdot y = 0\big),$$
$$\forall x, y\big((x = 0 \wedge y = 0) \leftrightarrow x^2 + y^2 = 0\big),$$

and

$$\forall x, y\big(x < y \leftrightarrow \exists z\big((y - x)z^2 - 1 = 0\big)\big)$$

are all in T_e, we may assume (modulo T_e) that $A(\vec{y})$ has the form

$$\exists \vec{x} F(\vec{y}, \vec{x}) = 0,$$

where $F(\vec{y}, \vec{x})$ is a term of L. Now if $k, K \models T_e$ and $k \subseteq K$, it follows to show that $k \preccurlyeq_1 K$ it is sufficient to show that for any $F(\vec{x}) \in k[\vec{x}]^e$, if F has a zero in K, then it has one in k. The next lemma reduces this problem to one of studying e.a. points.

Lemma 3. *Suppose $F(\vec{x}) \in k[\vec{x}]^e$ and $V(F) \neq \varnothing$. Then $V(F)$ contains an e.a. point of K^n over k. (We only assume here that $k \subseteq K$, k a field.)*

Proof. We may clearly suppose k countable, so let $h_1, \ldots, h_p$ be as given by Lemma 1 for $S = V(F)$. Since (2) asserts that

$$S \cap V^{\mathrm{ns}}(h_1, \ldots, h_p) \neq \varnothing,$$

it is sufficient to show $p = n$, so suppose for contradiction that $p < n$.
Choose

$$\vec{\alpha} \in S \cap V^{\mathrm{ns}}(h_1, \ldots, h_p)$$

and let $f(\vec{x})$ be a coefficient of $dh_1 \wedge \cdots \wedge dh_p$ such that $f(\vec{\alpha}) \neq 0$. Let $\alpha_{n+1} = f(\vec{\alpha})^{-1}$. Set

$$h_{p+1}(\vec{x}, x_{n+1}) = x_{n+1} \cdot f(\vec{x}) - 1.$$

Then

$$\langle \vec{\alpha}, \alpha_{n+1} \rangle \in (S \times K) \cap V(h_1, \ldots, h_{p+1}).$$

Further, we have

$$(*) \qquad V(h_1, \ldots, h_{p+1}) = V^{\mathrm{ns}}(h_1, \ldots, h_{p+1}).$$

To see this suppose

$$\langle \vec{\beta}, \beta_{n+1} \rangle \in V(h_1, \dots, h_{p+1}).$$

Now if $f(\vec{x})$ is the coefficient of $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ (where $1 \leq i_1 < \cdots < i_p \leq n$) in $dh_1 \wedge \cdots \wedge dh_p$, then $f(\vec{x})^2$ is clearly the coefficient of $dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{n+1}$ in $dh_1 \wedge \cdots \wedge dh_{p+1}$, and since $h_{p+1}(\vec{\beta}, \beta_{n+1}) = 0$, $f(\vec{\beta}) \neq 0$, which proves $(*)$.

Now suppose $\vec{\delta}$ is any point of k^n and (by transfer) let $\langle \vec{\gamma}, \gamma_{n+1} \rangle$ be a point of (the "closed" set) $V(h_1, \dots, h_{p+1}, F)$ at minimal distance from $\langle \vec{\delta}, 0 \rangle$; that is, $\langle \vec{\gamma}, \gamma_{n+1} \rangle$ is a minimum of the function

$$D_{\vec{\delta}}(\vec{x}, x_{n+1}) = \sum_{i=1}^{n} (x_i - \delta_i)^2 + x_{n+1}^2 \quad \left(\in k[\vec{x}, x_{n+1}]^e\right)$$

on $V(h_1, \dots, h_{p+1}, F)$.

Since $\vec{\gamma} \in V^{\mathrm{ns}}(h_1, \dots, h_p)$ (this follows from $(*)$) and $F(\vec{\gamma}) = 0$ (i.e., $\vec{\gamma} \in S$) we have, from (3) of Lemma 1 (note $p < n$), that F vanishes on

$$V(h_1, \dots, h_p)$$

close to $\vec{\gamma}$, and hence on

$$V(h_1, \dots, h_{p+1})$$

close to $\langle \vec{\gamma}, \gamma_{n+1} \rangle$. Thus $\langle \vec{\gamma}, \gamma_{n+1} \rangle$ is actually a local minimum of the function $D_{\vec{\delta}}$ on $V(h_1, \dots, h_{p+1})$. But then clearly the function

$$D'_{\vec{\delta}}(\vec{x}) = \sum_{i=1}^{n} (x_i - \delta_i)^2 + f(\vec{x})^{-2} \quad \left(\in k(\vec{x})^e\right)$$

is defined on $V(h_1, \dots, h_p)$ close to $\vec{\gamma}$ and has a local minimum there. Thus (by Section 2) $(\sigma \wedge dD'_{\vec{\delta}})(\vec{\gamma}) = 0$, where $\sigma = dh_1 \wedge \cdots \wedge dh_p$.

Let

$$G(\vec{x}) = F(\vec{x})^2 + f(\vec{x})^6 \cdot \left(\text{sum of the squares of the coefficients of } \sigma \wedge dD'_{\vec{\delta}}\right).$$

Then $G(\vec{x}) \in k[\vec{x}]^e$, $\vec{\gamma} \in V(G) \subseteq V(F)$, and $\vec{\gamma} \in V^{\mathrm{ns}}(h_1, \dots, h_p)$ so we may clearly repeat the above argument with G in place of F, $\vec{\gamma}$ in place of $\vec{\alpha}$ (note $f(\vec{\gamma}) \neq 0$) and any point $\vec{\delta}'$ of k^n in place of $\vec{\delta}$, to produce a point $\vec{\gamma}'$ of $V^{\mathrm{ns}}(h_1, \dots, h_p)$ such that

$$\left(\sigma \wedge D'_{\vec{\delta}'}\right)(\vec{\gamma}') = 0, \quad f(\vec{\gamma}') \neq 0, \quad G(\vec{\gamma}') = 0.$$

The latter two imply that we also have

$$\left(\sigma \wedge dD'_{\vec{\delta}}\right)\left(\vec{\gamma}'\right) = 0 \quad \text{and} \quad F\left(\vec{\gamma}'\right) = 0.$$

Continuing, we see that for any $r \in \mathbf{N}$ and $\vec{\delta}^{(0)}, \dots, \vec{\delta}^{(r)} \in k^n$, there is

$$\vec{\eta} \in V^{\text{ns}}\left(h_1, \dots, h_p\right) \cap V(F)$$

such that $(\sigma \wedge dD'_{\vec{\delta}^{(i)}})(\vec{\eta}) = 0$ for $i = 0, \dots, r$. We now apply this with

$$\delta_j^{(0)} = 0 \quad \text{for } j = 1, \dots, n,$$

and

$$\delta_j^{(i)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i, j = 1, \dots, n,$$

so that $dD'_{\vec{\delta}^{(i)}} = dD'_{\vec{\delta}^{(0)}} - 2\, dx_i$ for $i = 1, \dots, n$, and obtain a point

$$\vec{\eta} \in V^{\text{ns}}\left(h_1, \dots, h_p\right) \cap V(F)$$

such that

$$\left(\sigma \wedge dD'_{\vec{\delta}^{(0)}}\right)\left(\vec{\eta}\right) = 0$$

and

$$\left(\sigma \wedge \left(dD'_{\vec{\delta}^{(0)}} - 2\, dx_i\right)\right)\left(\vec{\eta}\right) = 0 \quad \text{for } i = 1, \dots, n.$$

These equations imply

$$\left(\sigma \wedge dx_i\right)\left(\vec{\eta}\right) = 0 \quad \text{for } i = 1, \dots, n.$$

However, since σ is a p-form and $p < n$ this contradicts the results $(**)$ mentioned in the proof of Lemma 2 (since $\sigma(\vec{\eta}) \neq 0$ and $(dx_1 \wedge \cdots \wedge dx_{p+1})(\vec{\eta}) \neq 0$), and Lemma 3 is proved. □

By the remarks at the beginning of this section we have the following immediate consequence of Lemma 3.

COROLLARY 1. *Suppose that $k, K \models T_e$, $k \subseteq K$, and for all $n \in \mathbf{N}$ and all e.a. points $\vec{\alpha} \in K^n$ over k we have $\vec{\alpha} \in k^n$. Then $k \preccurlyeq_1 K$.*

5. More results for transfer

To prove Theorem 2 it only remains to show (by Corollary 1) that if k, K satisfy the hypotheses of that theorem then every e.a. point of K^n (for all $n \in \mathbf{N}$) lies in k^n. To do this we require generalizations of the intermediate value theorem and some results on functions defined on space curves.

LEMMA 4. *Suppose $B \in \mathbf{R}$, $B > 0$, $n \in \mathbf{N}$, $n \geq 2$, and let*

$$g_1, \ldots, g_{n-1}: \mathbf{R}^n \to \mathbf{R}$$

be continuously differentiable. Let

$$V = \left\{ \vec{\alpha} \in \mathbf{R}^n : g_i(\vec{\alpha}) = 0 \text{ for } i = 1, \ldots, n-1 \right\}$$

and suppose that for each $\vec{\alpha} \in V$,

$$\left(\det \frac{\partial(g_1, \ldots, g_{n-1})}{\partial(x_2, \ldots, x_n)} \right)(\vec{\alpha}) \neq 0.$$

Suppose further that V is the union of finitely many connected components.

For $\alpha \in \mathbf{R}$, define

$$U_\alpha = \left\{ \langle \alpha_2, \ldots, \alpha_n \rangle \in \mathbf{R}^{n-1} : |\alpha_i| < \alpha \text{ for } i = 2, \ldots, n \right\},$$

and let $\overline{U}_\alpha$ be the closure of U_α. Let $\beta_1 \in \mathbf{R}$, $|\beta_1| < B$, $r \in \mathbf{N}$, $r \geq 1$, and assume that

$$V \cap (\{\beta_1\} \times U_B) = V \cap (\{\beta_1\} \times U_{B+2})$$

and that these sets contain exactly r points.

Then there exists $c, d \in \mathbf{R}$, $-B \leq c < \beta_1 < d \leq B$ such that for each $\alpha \in [c, d]$, $V \cap (\{\alpha\} \times \overline{U}_B)$ contains exactly r points and, further, if $\alpha_1 \in \{c, d\}$ then for some $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in V$ we have

$$\max\{|\alpha_i| : 1 \leq i \leq n\} \in \{B, B+1\}.$$

Also, for $a_1 \in (c, d)$ and any $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in V$, setting

$$\max\{|\alpha_i| : 1 \leq i \leq n\} = \theta$$

we have either $\theta < B$ or $\theta > B + 1$.

Proof. Using the implicit function theorem, the hypotheses imply that there exist open intervals (possibly infinite), $I_1, \ldots, I_m$ (say, where $m \in \mathbf{N}$)

and continuously differentiable functions

$$\Phi_i\colon I_i \to \mathbf{R}^n,\ t \to \left\langle t, \phi_2^{(i)}(t), \dots, \phi_n^{(i)}(t) \right\rangle \quad (\text{for } i = 1, \dots, m)$$

such that

$$V = \bigcup_{i=1}^{m} \bigcup_{t \in I_i} \{\Phi_i(t)\}$$

and the outer union is a disjoint one. In particular, for $\alpha \in \mathbf{R}$, distinct points of $V \cap (\{\alpha\} \times \mathbf{R}^{n-1})$ lie in distinct connected components (and so this set has at most m points). Let us suppose, without loss of generality, that the suffices $1, \dots, r$ correspond to the components of the points in $V \cap (\{\beta_1\} \times U_B)$. Define

$$|\Phi_i(t)| = \max_{2 \le j \le n} \left|\phi_j^{(i)}(t)\right| \quad (\text{for } t \in I_i).$$

By the continuity of the Φ_i at β_1 (for those i such that $\beta_1 \in I_i$), we can find $\varepsilon > 0$ such that for $t \in (\beta_1 - \varepsilon, \beta_1 + \varepsilon)$ we have

(∗) $t \in I_i$ and $|\Phi_i(t)| < B$ for $i = 1, \dots, r$, and, for $j = r + 1, \dots, m$, if $t \in I_j$ then $|\Phi_j(t)| > B + 1$,

(since for such j, $|\Phi_j(t)| \ge B + 3/2$ for t close to β_1).

Let d' be the supremum of those $\alpha \ge \beta_1 + \varepsilon$ for which (∗) holds for all $t \in (\beta_1 - \varepsilon, \alpha)$. If $d' > B$ (or $d' = \infty$) we may clearly set $d = B$. Otherwise, note that $\lim_{t \to d^-} \Phi_i(t)$ certainly exists for $i = 1, \dots, r$, and this point must lie in V (since V is closed). It clearly follows that $d' \in I_i$ and $|\Phi_i(d')| \le B$ for $i = 1, \dots, r$, and, for $j = r + 1, \dots, m$, if $d' \in I_j$ then $|\Phi_j(d)| \ge B + 1$. Now we must have either $|\Phi_i(d')| = B$ for some $i = 1, \dots, r$, or else $|\Phi_j(d')| = B + 1$ for some $j = r + 1, \dots, m$ such that $d' \in I_j$, for otherwise, by the continuity of the Φ's (when defined) at d' we could find $\eta > 0$ such that (∗) held for all $t \in (\beta_1 - \varepsilon, d' + \eta)$ contradicting the definition of d'. Thus we may set $d = d'$. The construction of c is similar. □

Lemma 5. *Suppose that $r \in \mathbf{N}$, $r \ge 1$, and*

$$f_1, \dots, f_r\colon [a, b] \to \mathbf{R} \quad (a, b \in \mathbf{R},\ a < b)$$

have continuous non-vanishing derivatives throughout $[a, b]$. For $\sigma, \tau \in \{+, -\}$ and $x \in [a, b]$ define

$$S(\sigma, \tau, x) = \{i\colon 1 \le i \le r,\ f_i(x) \text{ is } \sigma\text{'ve and } f_i'(x) \text{ is } \tau\text{'ve}\}$$

(*where* $'$ *denotes differentiation*).

Let $Z = \{i\colon 1 \le i \le r,\ f_i \text{ has a zero in } [a, b]\}$. *Then*

$$|Z| = r - |S(+,+,a)| - |S(-,-,a)| - |S(+,-,b)| - |S(-,+,b)|.$$

Proof. If f_i' is $+$'ve on $[a, b]$, then by the intermediate value theorem

$$\begin{aligned} i \in Z \quad &\text{iff} \quad f_i(a) = 0 \text{ or } f_i(b) = 0 \text{ or } i \in S(-,+,a) \cap S(+,+,b) \\ &\text{iff} \quad i \notin S(+,+,a) \cup S(-,+,b). \end{aligned}$$

Similarly if f_i' is $-$'ve on $[a, b]$,

$$i \in Z \quad \text{iff} \quad i \notin S(-,-,a) \cup S(+,-,b).$$

Since the sets $S(+,+,a)$, $S(-,-,a)$, $S(+,-,b)$, $S(-,+,b)$ are pairwise disjoint, the result follows.

Lemma 6. *Suppose* $n \ge 2$, $g_1, \ldots, g_{n-1}\colon \mathbf{R}^n \to \mathbf{R}$ *are continuously differentiable, and let*

$$V = \left\{\vec{\alpha} \in \mathbf{R}^n\colon g_i(\vec{\alpha}) = 0 \text{ for } i = 1, \ldots, n-1\right\}.$$

Suppose that for each $\vec{\alpha} \in V$,

$$\left(\det \frac{\partial(g_1, \ldots, g_{n-1})}{\partial(x_2, \ldots, x_n)}\right)(\vec{\alpha}) \ne 0.$$

Let $B, c, d \in \mathbf{R}$, $B > 0$, $-B \le c < d \le B$, *and suppose that for each* $\alpha \in [c, d]$, $V \cap (\{\alpha\} \times \overline{U}_B)$ *contains exactly* r *points where* $r \in \mathbf{N}$, $r \ge 1$, *and we use the notation of Lemma* 4. *Suppose further that for* $\alpha \in (c, d)$, $V \cap (\{\alpha\} \times \overline{U}_B) = V \cap (\{\alpha\} \times U_B)$.

Let $g\colon \mathbf{R}^n \to \mathbf{R}$ *be continuously differentiable and define* $g^*\colon V \to \mathbf{R}$ *by*

$$g^* = \frac{\partial g}{\partial x_1} - \left(\frac{\partial g}{\partial x_2}, \ldots, \frac{\partial g}{\partial x_n}\right)\left(\frac{\partial(g_1, \ldots, g_{n-1})}{\partial(x_2, \ldots, x_n)}\right)^{-1}\begin{pmatrix} \dfrac{\partial g_1}{\partial x_1} \\ \vdots \\ \dfrac{\partial g_{n-1}}{\partial x_1} \end{pmatrix}$$

and suppose that for all $\vec{\alpha} \in V$, $g^*(\vec{\alpha}) \ne 0$.

Then the number of zeroes of g *on* $V \cap ([c, d] \times \overline{U}_B)$ *is given by*

$$r - |S^{(c)}(+,+)| - |S^{(c)}(-,-)| - |S^{(d)}(+,-)| - |S^{(d)}(-,+)|$$

where for example,

$$S^{(d)}(+,-) = \left\{\vec{\alpha} \in V \cap \left(\{d\} \times \overline{U}_B\right) \colon g(\vec{\alpha}) > 0 \text{ and } g^*(\vec{\alpha}) < 0\right\}.$$

Proof. By the implicit function theorem there are continuously differentiable functions

$$\Phi_i \colon (c', d') \to \mathbf{R}^n, \quad t \to \left\langle t, \phi_2^{(i)}(t), \ldots, \phi_n^{(i)}(t)\right\rangle,$$

where (c', d') is some open interval containing $[c, d]$ such that

$$V \cap \left([c, d] \times \overline{U}_B\right) = \bigcup_{i=1}^{r} \bigcup_{t \in [c,d]} \{\Phi_i(t)\},$$

where the outer union is a disjoint one.

For $t \in (c', d')$ and $i = 1, \ldots, r$ define

$$f_i(t) = g\left(t, \phi_2^{(i)}(t), \ldots, \phi_n^{(i)}(t)\right).$$

Then a calculation similar to those of Section 2 shows that

$$f_i'(t) = g^*\left(t, \phi_2^{(i)}(t), \ldots, \phi_n^{(i)}(t)\right)$$

for $t \in (c', d')$, $i = 1, \ldots, r$. The result now follows from Lemma 5. □

(The importance of the formula for the number of zeroes in Lemma 6 will be that it makes no reference to any parameterization of the variety V.)

Unfortunately Lemmas 4 and 6 are not immediately transferable to an arbitrary model of T_e (in the case that $g_1, \ldots, g_{n-1}, g$ are terms) because of the connectedness hypothesis. However, all is well because of the following result of Khovansky [3].

PROPOSITION. *Let $m, n \in \mathbf{N}$, $n \geq 2$,*

$$\vec{y} = (y_1, \ldots, y_m), \quad \vec{x} = (x_1, \ldots, x_n)$$

and suppose $f_1(\vec{y}, \vec{x}), \ldots, f_p(\vec{y}, \vec{x})$ are terms of L. Then there is $N \in \mathbf{N}$ such that for any $\vec{\alpha} \in \mathbf{R}^m$ the subset

$$V^{\text{ns}}\left(f_1(\vec{\alpha}, \vec{x}), \ldots, f_p(\vec{\alpha}, \vec{x})\right)$$

of $\mathbf{R}^n$ has at most N connected components (and hence at most N points if $p = n$).

It thus follows that Lemmas 4 and 6 can be expressed as first-order sentences of L (in the case $g_1, \dots, g_{n-1}, g$ are terms) uniformly in the parameters occurring in the g's.

6. The proof of Theorem 2

Recall the hypotheses: $k, K \models T_e$, $k \subseteq K$ and for all $n \in \mathbf{N}$, $n \geq 1$, and all e.a. points $\langle \alpha_1, \dots, \alpha_n \rangle$ of K^n over k, there is $a \in k$ such that $|\alpha_i| < a$ for $i = 1, \dots, n$. By Corollary 1 it is sufficient to show that for all $n \in \mathbf{N}$, $n \geq 1$, every e.a. point of K^n lies in k^n.

We shall prove the following by induction on $\langle j, s \rangle \in \mathbf{N}^2$ (ordered lexicographically).

$P_{j,s}$: Suppose $n \in \mathbf{N}$, $n \geq 1$, $\vec{x} = x_1, \dots, x_n$ and $M \subseteq k[\vec{x}]^e$ has height $\leq j$ (cf. the definition before Lemma 2). Suppose $g_1, \dots, g_n \in M$ all have degree $\leq s$ (in M) (in the case $j \geq 1$). If $\vec{\alpha} \in V^{\mathrm{ns}}(g_1, \dots, g_n)$ then $\vec{\alpha} \in k^n$.

For all $s \in \mathbf{N}$, $P_{0,s}$ is clear since it is well known that the coordinates of an $\vec{\alpha} \in V^{\mathrm{ns}}(g_1, \dots, g_n)$, where $g_1, \dots, g_n \in k[\vec{x}]$, are algebraic over k, and k, K are real-closed fields (being models of T_e).

Since $P_{j+1,0}$ is immediately implied by $\forall s \in \mathbf{N} P_{j,s}$, the inductive step amounts to showing that for each $j, s \in \mathbf{N}$, $P_{j+1,s}$ implies $P_{j+1,s+1}$.

So suppose $j, s, n \in \mathbf{N}$, $n \geq 1$, $\vec{x} = (x_1, \dots, x_n)$, $M \subseteq k[\vec{x}]^e$ has height $\leq j$, $g \in M$ and $g_1, \dots, g_n \in M[e^g]$, where each g_i has degree $\leq s+1$ as a polynomial in e^g over M, and $\vec{\alpha} \in V^{\mathrm{ns}}(g_1, \dots, g_n)$. ....(*)

Of course we also suppose that $P_{j+1,s}$ holds, and we want to show that $\vec{\alpha} \in k^n$.

Our first aim is to modify the g_i's so that Lemmas 4 and 6 are applicable.

By Lemma 2 (with $p = n$) there are $h_1, \dots, h_n \in M[e^g]$ such that

(1) $h_1, \dots, h_{n-1} \in M$ and $h_n = \Sigma_{i=0}^{s+1} a_i e^{ig}$ for some $a_0, \dots, a_{s+1} \in M$,
(2) $\vec{\alpha} \in V^{\mathrm{ns}}(h_1, \dots, h_n)$.

Let $\sigma = dh_1 \wedge \cdots \wedge dh_{n-1}$. If

$$a_0(\vec{\alpha}) = 0 \quad \text{and} \quad (\sigma \wedge da_0)(\vec{\alpha}) \neq 0$$

we may immediately apply $P_{j+1,s}$ (in fact $P_{j+1,0}$) to conclude that $\vec{\alpha} \in k^n$. If

$$a_0(\vec{\alpha}) = 0 \quad \text{and} \quad (\sigma \wedge da_0)(\vec{\alpha}) = 0,$$

then let $\bar{h}_n = e^{-g}(h_n - a_0)$ so that $\bar{h}_n$ has degree $\leq s$ (in M),

$$\vec{\alpha} \in V(h_1, \dots, h_{n-1}, \bar{h}_n)$$

and

$$\begin{aligned}(\sigma \wedge d\bar{h}_n)(\vec{\alpha}) &= \left(e^{-g}[(\sigma \wedge dh_n) - (\sigma \wedge da_0)] - (h - a_0)\cdot(\sigma \wedge d(e^{-g}))\right)(\vec{\alpha})\\ &= \left(e^{-g}(\sigma \wedge dh_n)\right)(\vec{\alpha}) \neq 0,\end{aligned}$$

so $\vec{\alpha} \in V^{\mathrm{ns}}(h_1, \dots, h_{n-1}, \bar{h}_n)$ and we may again conclude that $\vec{\alpha} \in k^n$ by $P_{j+1,s}$. Thus we may suppose that $a_0(\vec{\alpha}) \neq 0$.

Now define $h, f \in k[\vec{x}, x_{n+1}]^e$ by

$$h = x_{n+1} \cdot a_0 - 1$$

and

$$f = 1 + x_{n+1} \cdot \sum_{i=1}^{s+1} a_i e^{ig} \quad \left(= 1 + x_{n+1}(h_n - a_0)\right).$$

Let $\alpha_{n+1} = a_0(\vec{\alpha})^{-1}$ and set $\overline{M} = M[x_{n+1}]$, so that $\overline{M}$ has height $\leq j$ (as a subring of $k[\vec{x}, x_{n+1}]^e$). Then $h_1, \dots, h_{n-1}, h \in \overline{M}$, $f \in \overline{M}[e^g]$, f has degree $\leq s + 1$ (in $\overline{M}[e^g]$) and

$$\langle \vec{\alpha}, \alpha_{n+1} \rangle \in V(h_1, \dots, h_{n-1}, h, f).$$

Further, since $f = x_{n+1}h_n - h$, we have

$$\begin{aligned}(\sigma \wedge dh \wedge df)(\vec{\alpha}, \alpha_{n+1}) &= (\sigma \wedge dh \wedge d(x_{n+1}h_n))(\vec{\alpha}, \alpha_{n+1})\\ &\quad -(\sigma \wedge dh \wedge dh)(\vec{\alpha}, \alpha_{n+1})\\ &= (\sigma \wedge dh \wedge d(x_{n+1}h_n))(\vec{\alpha}, \alpha_{n+1})\\ &\qquad \text{(since } \tau \wedge \tau = 0 \text{ for any 1-form } \tau)\\ &= \left(x_{n+1} \cdot (\sigma \wedge dh \wedge dh_n)\right)(\vec{\alpha}, \alpha_{n+1})\\ &\qquad \left(\text{since } h_n(\vec{\alpha}) = 0\right)\\ &= \left(x_{n+1} \cdot a_0 \cdot (\sigma \wedge dx_{n+1} \wedge dh_n)\right)(\vec{\alpha}, \alpha_{n+1}).\end{aligned}$$

(The last equality follows since $dh = x_{n+1}\,da_0 + a_0\,dx_{n+1}$ and $\sigma \wedge da_0 \wedge dh_n = 0$ since it is an $(n+1)$-form over $k[x_1, \dots, x_n]^e$.) Now $\alpha_{n+1} \neq 0$, $a_0(\vec{\alpha}) \neq 0$ and since $(\sigma \wedge h_n)(\vec{\alpha}) \neq 0$ (by (2)) it follows that

$$(\sigma \wedge dx_{n+1} \wedge dh_n)(\vec{\alpha}, \alpha_{n+1}) \neq 0.$$

Thus we have shown that $\langle \vec{\alpha}, \alpha_{n+1} \rangle \in V^{\mathrm{ns}}(h_1, \dots, h_{n-1}, h, f)$.

It now follows that in (*) we may as well assume that $g_1, \dots, g_{n-1} \in M$, and $g_n = 1 + \sum_{i=1}^{s+1} a_i e^{ig}$ for some $a_1, \dots, a_{s+1} \in M$.

Now since $(dg_1 \wedge \cdots \wedge dg_{n-1})(\vec{\alpha}) \neq 0$ we may suppose (by permuting variables if necessary) that

$$(\det J)(\vec{\alpha}) \neq 0 \quad \text{where } J = \frac{\partial(g_1, \ldots, g_{n-1})}{\partial(x_2, \ldots, x_n)}.$$

Note now that $\det J \in M$, so by the "de-singularizing trick" of considering

$$g = x_{n+1} \cdot \det J - 1$$

and showing that

$$\langle \vec{\alpha}, (\det J)(\vec{\alpha})^{-1} \rangle \in V^{\mathrm{ns}}(g_1, \ldots, g_{n-1}, g, g_n)$$

and

$$\det \frac{\partial(g_1, \ldots, g_{n-1}, g)}{\partial(x_2, \ldots, x_{n+1})}$$

is non-vanishing throughout $V(g_1, \ldots, g_{n-1}, g)$ we may as well suppose that $\det J$ is non-vanishing throughout $V(g_1, \ldots, g_{n-1})$ (and that we still have $g_1, \ldots, g_{n-1} \in M$).

Now let us consider g_n^* (in the notation of Lemma 6). By Section 2, the points on $V(g_1, \ldots, g_{n-1})$ at which g_n^* vanishes are exactly those points where $dg_1 \wedge \cdots \wedge dg_n$ vanishes. In particular, $g_n^*(\vec{\alpha}) \neq 0$. Now note that $(\det J) \cdot g_n^*$ is of the form

$$\sum_{i=1}^{n} b_i \cdot \frac{\partial g_n}{\partial x_i} \quad \text{where } b_1, \ldots, b_n \in M.$$

By our supposition on g_n above, this is of the form

$$e^g \cdot \sum_{i=1}^{n} b_i p_i(e^g) = e^g \cdot p(e^g)$$

where $p_1, \ldots, p_n$, and hence p, are polynomials over M of degree $\leq s$, for $i = 1, \ldots, n$. Thus using de-singularization again (i.e., considering $x_{n+1} \cdot p(e^g) - 1$, etc.) we may as well suppose that g_n^* is non-vanishing throughout

$$V(g_1, \ldots, g_{n-1}) = V,$$

say, and that $g_1, \ldots, g_{n-1}$ all have degree $\leq s$ in $M[e^g]$. (We possibly have to sacrifice the condition $g_1, \ldots, g_{n-1} \in M$, of course.)

We are now in a position to apply Lemma 4 (using transfer—see the remarks at the end of Section 5) because, by the proposition of Section 5 and the hypothesis of Theorem 2, there certainly exists a $B \in k$ such that for some $r \in \mathbf{N}$, $r \geq 1$, (in the notation of Lemma 4 applied in K), there are exactly r points in $V \cap (\{\alpha_1\} \times U_B)$, $\vec{\alpha}$ is one of them, and $V \cap (\{\alpha_1\} \times U_B) = V \cap (\{\alpha_1\} \times U_{B+2})$. We thus obtain $c, d \in K$ satisfying the (transferred to K) conclusion of Lemma 4. We want to show that $c, d \in k$.

To see that $d \in k$ we first choose a point (provided by Lemma 4)

$$\langle \delta_1, \ldots, \delta_n \rangle = \vec{\delta} \in V^{\text{ns}}(g_1, \ldots, g_{n-1})$$

with $\delta_1 = d$ and $\max\{|\delta_i|: 1 \leq i \leq n\} = \gamma$, where $\gamma \in \{B, B+1\}$ (so $\gamma \in k$). We may suppose $|d| \neq \gamma$, for otherwise we are done. Now choose $\varepsilon \in K$, $\varepsilon > 0$, and a neighbourhood U of $\vec{\delta}$ in K^n so that $d - \varepsilon > c$ and:

(3) For each $\eta_1 \in (d - \varepsilon, d + \varepsilon)$ there is a unique $\vec{\eta} = \langle \eta_1, \ldots, \eta_n \rangle$ in
$$U \cap V^{\text{ns}}(g_1, \ldots, g_n);$$

(4) If $\eta_1 \in (d - \varepsilon, d]$ and i, i' are such that $1 \leq i, i' \leq n$ and $|\delta_i| < |\delta_{i'}|$ then $|\eta_i| < |\eta_{i'}|$ (in the notation of (3)).

(This is possible by the implicit function theorem and transfer, since

$$\det\left(\frac{\partial(g_1, \ldots, g_{n-1})}{\partial(x_2, \ldots, x_n)} \right)(\vec{\delta}) \neq 0$$

and the local parameterization functions are continuous.)

Now the final conclusion of Lemma 4 and (4) clearly imply (in the notation of (3)):

(5) For each $\eta_1 \in (d - \varepsilon, d)$, there is some i such that $1 \leq i \leq n$, $|\delta_i| = \gamma$ and $|\eta_i| \neq \gamma$.

We now let p be maximal such that for some $\bar{g}_1, \ldots, \bar{g}_p \in M[e^g]$,

$$\vec{\delta} \in V^{\text{ns}}(\bar{g}_1, \ldots, \bar{g}_p).$$

Clearly $p = n - 1$ or $p = n$.

Case 1. $p = n - 1$.

By Lemma 2 we can find $h_1, \ldots, h_{n-1}$ satisfying (2) and (3) of that lemma. In particular $g_1, \ldots, g_{n-1}$ all vanish on $V^{\text{ns}}(h_1, \ldots, h_{n-1})$ close to $\vec{\delta}$, as does any function of the form $x_i \pm \gamma$ which happens to vanish at $\vec{\delta}$. But it now follows from (5) that if

$$\vec{\eta} = \langle \eta_1, \ldots, \eta_n \rangle \in V^{\text{ns}}(h_1, \ldots, h_{n-1})$$

and $\vec{\eta}$ is sufficiently close to $\vec{\delta}$ then $\eta_1 \geq d$. However, since $\delta_1 = d$ this implies (by considering a (necessary continuous) parameterization of $V^{\mathrm{ns}}(h_1, \ldots, h_{n-1})$ close to $\vec{\delta}$—see Section 2) that for any $\nu > 0$, $\nu \in K$, there is $\eta_1 \in [d, d + \nu)$ and distinct points

$$\vec{\eta}, \vec{\eta}' \in V^{\mathrm{ns}}(h_1, \ldots, h_{n-1})$$

both having first coordinate η_1. Further, $\vec{\eta}$ and $\vec{\eta}'$ may be chosen arbitrarily close to $\vec{\delta}$ (for sufficiently small choice of ν). Since

$$V^{\mathrm{ns}}(h_1, \ldots, h_{n-1}) \subseteq V^{\mathrm{ns}}(g_1, \ldots, g_{n-1}) \quad \text{close to } \vec{\delta},$$

this contradicts (3).

Case 2. $p = n$.
By Lemma 2 we can find $h_1, \ldots, h_{n-1} \in M$ and $h_n \in M[e^g]$ such that

$$\vec{\delta} \in V^{\mathrm{ns}}(h_1, \ldots, h_n).$$

Now if we can choose h_n of degree $\leq s$ then we can apply the inductive hypothesis $P_{j+1, s}$ to deduce that $\vec{\delta} \in k^n$ and so, in particular, that $d \in k$ as required. However, in the proof of Lemma 2 (or, rather, Lemma 1 with $S = \{\vec{\delta}\}$) recall that h_n is chosen of minimal degree so that $h_n(\vec{\delta}) = 0$ and h_n does not vanish on $V^{\mathrm{ns}}(h_1, \ldots, h_{n-1})$ close to $\vec{\delta}$. It therefore follows that if h_n cannot be chosen of degree $\leq s$, then any $h \in M[e^g]$ of degree $\leq s$ with $h(\vec{\delta}) = 0$ vanishes on $V^{\mathrm{ns}}(h_1, \ldots, h_{n-1})$ close to $\vec{\delta}$. In particular this is so for $h = g_1, \ldots, g_{n-1}, x_i \pm \gamma$ and we may proceed to a contradiction as in case 1.
This completes the proof that $d \in k$.

The same argument shows that $c \in k$ and a similar one shows that for any $\beta_1 \in k$ with $c \leq \beta_1 \leq d$, the r points of $V \cap (\{\beta_1\} \times \overline{U_B})$ all lie in k^n (and these points are in $\{\beta_1\} \times U_B$ if $c < \beta_1 < d$). It now follows from this (and the assumptions above on $g_1, \ldots, g_n$) that the hypotheses of Lemma 6 are satisfied when interpreted in K *and* when interpreted in k. But clearly the formula given there for the number of zeroes of g_n on

$$V \cap ([c, d] \times \overline{U_B})$$

gives the same answer no matter whether it is computed in K or in k (since it only depends on the signs of g_n and g_n^*, which are elements of $k(\vec{x})^e$, at certain points of k^n and k is a substructure of K). Thus all points of

$$V^{\mathrm{ns}}(g_1, \ldots, g_n) \cap ([c, d] \times \overline{U_B})$$

lie in k^n. In particular $\vec{\alpha} \in k^n$, as required.

This completes the induction and establishes that $P_{j,s}$ holds for all $j, s \in \mathbf{N}$, which clearly implies Theorem 2.

Acknowledgement. I would like here to express my thanks to Kobi Peterzil and Ernest Schimmerling of UC Berkeley. Not only did they suggest several minor corrections to my original draft of this paper but also found a rather serious error towards the end of the proof of Theorem 2.

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