ILLINOIS JOURNAL OF MATHEMATICS
Volume 33, Number 3, Fall 1989

THE SCHRÖDINGER-HILL EQUATION $-y''(x) + q(x)y(x) = \mu \cdot y(x)$ ON ODD POTENTIALS q

BY

M. PAPADIMITRAKIS

Introduction

We consider Hill's equation with Dirichlet boundary conditions:

$$(*)\qquad \begin{gathered} -y''(x) + q(x)y(x) = \mu \cdot y(x), \quad x \in [0,1], \\ y(0) = y(1) = 0 \end{gathered}$$

where $\mu \in \mathbf{C}$, the set of complex numbers, and $q \in L^2_{\mathbf{R}}[0,1]$, the set of real valued square integrable functions on $[0,1]$. It is known (see [1] or [3]) that the eigenvalues of $(*)$ form a strictly increasing sequence of real numbers $\{\mu_k(q)\}_{k=1}^{\infty}$ such that: $\mu_1 < \mu_2 < \cdots < \mu_k \to +\infty$. Also

$$\mu_k = k^2\pi^2 + \tilde{\mu}_k \quad \text{with } \sum \tilde{\mu}_k^2 < +\infty.$$

Conversely, for any choice of $\{\tilde{\mu}_k\}$, such that $\sum\tilde{\mu}_k^2 < +\infty$ and $\mu_k = k^2\pi^2 + \tilde{\mu}_k$ is increasing, we can find a potential $q \in L^2_{\mathbf{R}}[0,1]$ such that: $\mu_k(q) = \mu_k$. We can also choose q to be even, i.e., $q(x) = q(1-x)$, $x \in [0,1]$. When q is odd, i.e., $q(x) = -q(1-x)$, then one can prove that

$$\mu_k(q) = k^2\pi^2 + \tilde{\mu}_k(q) \quad \text{with } \sum (k\tilde{\mu}_k)^2 < +\infty.$$

The question is: given any sequence $\{\tilde{\mu}_k\}$ such that $\sum(k\tilde{\mu}_k)^2 < +\infty$ and $\mu_k = k^2\pi^2 + \tilde{\mu}_k$ is strictly increasing, does there exist an odd $q \in L^2_{\mathbf{R}}[0,1]$ such that $\mu_k(q) = \mu_k$? Let

$$l^2_\alpha = \left\{\{c_k\} : \sum (k^\alpha c_k)^2 < +\infty\right\}.$$

Received May 18, 1987.

We know [2, Theorem 4.1] that if q is odd and continuously differentiable, then

$$\tilde{\mu}_k = A/k^2 + c_k,$$

with $\{c_k\} \in l_2^2$ and A some constant depending on q. This implies that

$$\frac{\tilde{\mu}_1 + 2\tilde{\mu}_2 + \cdots + k\tilde{\mu}_k}{\log k} \to A \quad \text{as } k \to +\infty.$$

In the following we will prove:

THEOREM. *Given any odd $q \in L^2_{\mathbf{R}}[0,1]$, the eigenvalues μ_k of $(*)$ are such that*

$$\mu_k = k^2\pi^2 + \tilde{\mu}_k$$

where

$$\frac{\tilde{\mu}_1 + 2\tilde{\mu}_2 + \cdots + k\tilde{\mu}_k}{\log k} \to \frac{1}{4\pi^2}\int_0^1 q^2(t)\, dt, \quad k \to +\infty.$$

Consequently, $\{\tilde{\mu}_k\} \notin l^2_{3/2}$ and so the answer to the question posed is negative.

Section 1

In this part we prove that, for an odd q,

$$\mu_k(q) = k^2\pi^2 + \tilde{\mu}_k, \quad \sum (k\tilde{\mu}_k)^2 < +\infty.$$

This is already proven in [3], but we will find a form of $\tilde{\mu}_k$, which we will need below.

Consider the solution $y(x, \mu, q)$ of $(*)$, such that $y(0, \mu, q) = 0$, $y'(0, \mu, q) = 1$. It is known [3] that

$$\begin{aligned}
y(x,\mu,q) = {} & \frac{\sin\sqrt{\mu}\,x}{\sqrt{\mu}} + \int_0^x \frac{\sin\sqrt{\mu}\,(x-t)}{\sqrt{\mu}} \cdot \frac{\sin\sqrt{\mu}\,t}{\sqrt{\mu}} q(t)\, dt \\
& + \sum_{m=2}^{\infty} \int_0^x \int_0^{x_m} \cdots \int_0^{x_2} \frac{\sin\sqrt{\mu}\,(x-x_m)}{\sqrt{\mu}} \\
& \cdot \frac{\sin\sqrt{\mu}\,(x_m - x_{m-1})}{\sqrt{\mu}} \cdots \frac{\sin\sqrt{\mu}\,(x_2 - x_1)}{\sqrt{\mu}} \\
& \cdot \frac{\sin\sqrt{\mu}\,x_1}{\sqrt{\mu}} \cdot q(x_1) \cdots q(x_m)\, dx_1 \dots dx_m,
\end{aligned} \tag{1}$$

that for any fixed x, $y(x, \mu, q)$ is an analytic function of $\mu \in \mathbf{C}$, and that the $\mu_k(q)$ are the roots of the equation $y(1, \mu, q) = 0$.

From now on we consider (1) with $x = 1$.

We observe that if q is odd, then the second and fourth term of the right side of (1) vanish (in fact every term of even order vanishes). By writing

$$\sqrt{\mu} = k\pi + r, \quad \mu, r \in \mathbf{C}, \ |r| < 1, \ k \in \mathbf{Z}_+,$$

we get the expansion:

$$\frac{\sin\sqrt{\mu}\, t}{\sqrt{\mu}} = \frac{\sin k\pi t}{k\pi} + \left(\frac{t\cos k\pi t}{k\pi} - \frac{\sin k\pi t}{k^2\pi^2}\right) r + \frac{1}{k\pi} O(|r|^2)$$

where "big-O" comprises a constant independent of k, r, t ($|r| < 1$, $t \in [0,1]$). Using this expression in every integral of the right side of (1) we arrive at

$$y(1, \mu, q) = P_k(q) + Q_k(q) \cdot r + R_k(q, r) \cdot r^2$$

where

$$P_k(q) = \sum_{m=2}^{\infty} \int_0^1 \int_0^{x_m} \cdots \int_0^{x_2} \frac{\sin k\pi(1 - x_m)}{k\pi} \cdots \frac{\sin k\pi(x_2 - x_1)}{k\pi}$$
$$\cdot \frac{\sin k\pi x_1}{k\pi} \cdot q(x_1) \cdots q(x_m)\, dx_1 \cdots dx_m,$$

$$Q_k(q) = \frac{(-1)^k}{k\pi} + \sum_{m=2}^{\infty} \int_0^1 \cdots \int_0^{x_2} Q_{k,m}(x_1, \ldots, x_m)$$
$$\times q(x_1) \cdots q(x_m)\, dx_1 \cdots dx_m$$

where $Q_{k,m}$ is a sum of $m + 1$ expressions all, in absolute values, less than $2/(k\pi)^{m+1}$ and

$$R_k(q, r) = \frac{1}{k\pi} \cdot O(1) + \sum_{m=2}^{\infty} \int_0^1 \cdots \int_0^{x_2} R_{k,m}(x_1, \ldots, x_m, r)$$
$$\times q(x_1) \cdots q(x_m)\, dx_1 \cdots dx_m$$

where $|R_{k,m}| \leq c/(k\pi)^{m+1}$ and c is independent of k, r, $x_1, \ldots, x_m$ ($|r| < 1$). Hence

$$\text{(2)} \quad |P_k(q)| \leq \sum_{m=2}^{\infty} \frac{1}{(k\pi)^{m+1}} \int_0^1 \cdots \int_0^{x_2} |q(x_1)| \cdots |q(x_m)|\, dx_1 \cdots dx_m$$
$$= \sum_{m=2}^{\infty} \frac{1}{(k\pi)^{m+1}} \cdot \frac{1}{m!} \left(\int_0^1 |q(x)|\, dx\right)^m$$
$$\leq \frac{c(q)}{(k\pi)^3}.$$

In the same way,

$$(3)\qquad Q_k(q) = \frac{(-1)^k}{k\pi} + \tilde{Q}_k(q), \quad |\tilde{Q}_k(q)| \leq \frac{c'(q)}{(k\pi)^3},$$

$$(4)\qquad |R_k(q, r)| \leq \frac{c''(q)}{k\pi}.$$

Let $r_1 = P_k/Q_k + r$. Then

$$y(r_1) = y(1, \mu, q) = Q_k \cdot r_1 + R_k(r_1) \cdot \left(r_1 - \frac{P_k}{Q_k}\right)^2.$$

Consider also $g(r_1) = Q_k \cdot r_1$. Then, because of the asymptotic relations (2), (3), (4), if k is larger than some $k_0 = k_0(q)$,

$$|y(r_1) - g(r_1)| = |R_k(r_1)| \left|r_1 - \frac{P_k}{Q_k}\right|^2 < |Q_k|\,|r_1| = |g(r_1)|$$

for all r_1 such that

$$|r_1| = \delta = 2\frac{|P_k|^2}{|Q_k|^3} \cdot \frac{c''(q)}{k\pi}.$$

By Rouche's theorem $y(1, \mu, q) = 0$ has exactly one solution μ_k such that

$$\sqrt{\mu_k} = k\pi - (P_k(q)/Q_k(q)) + r_1$$

where $|r_1| < \delta = O(1/k^4)$, if $k \geq k_0(q)$. Remembering that the fourth term in (1) vanishes,

$$\frac{P_k(q)}{Q_k(q)} = \frac{\dfrac{1}{k^3\pi^3}\displaystyle\int_0^1\int_0^{x_2} \sin k\pi(1 - x_2)\sin k\pi(x_2 - x_1) \sin k\pi x_1 q(x_1)q(x_2)\, dx_1\, dx_2 + O\left(\frac{1}{k^5}\right)}{\dfrac{(-1)^k}{k\pi} + O\left(\dfrac{1}{k^3}\right)}$$

$$= \frac{(1)^k}{k^2\pi^2}\int_0^1\int_0^{x_2} K_k(x_2, x_1)q(x_1)q(x_2)\, dx_1\, dx_2 + O\left(\frac{1}{k^4}\right)$$

where

$$K_k(x_2, x_1) = 4\sin k\pi(1 - x_2) \cdot \sin k\pi(x_2 - x_1) \cdot \sin k\pi x_1. \tag{5}$$

So $\mu_k = k^2\pi^2 + \tilde{\mu}_k$ where

$$\begin{aligned} \tilde{\mu}_k &= \frac{1}{2k\pi}\int_0^1\int_0^t K_k(t, x)q(t)q(x)\,dx\,dt + O\left(\frac{1}{k^3}\right) \\ &= \frac{1}{2k\pi}\tilde{\tilde{\mu}}_k + O\left(\frac{1}{k^3}\right). \end{aligned} \tag{6}$$

Now we observe that

$$\int_0^1\int_0^t K_l(t, x)K_k(t, x)\,dx\,dt = \begin{cases} \frac{3}{4}, & k = l, \\ 0, & k \neq l. \end{cases}$$

So $\{K_k\}$ is an orthogonal system for $L^2[0 \leq x \leq t \leq 1]$ and hence

$$\sum(k\tilde{\mu}_k)^2 < +\infty, \text{ i.e. } \{\tilde{\mu}_k\} \in l_1^2.$$

Section 2

By (5),

$$K_k(t, x) = \sin 2k\pi x - \sin 2k\pi t + \sin 2k\pi(t - x)$$

from which we have

$$\begin{aligned} \tilde{\tilde{\mu}}_k &= \int_0^1\int_0^t K_k(t, x)q(x)q(t)\,dx\,dt \\ &= \int_0^1 \sin 2k\pi x\left\{-2q(x)\int_0^x q(t)\,dt + \int_x^1 q(t)q(t - x)\,dt\right\}dx \\ &= \int_0^1 \sin 2k\pi x \cdot G(x)\,dx. \end{aligned}$$

Hence

$$G(x) \sim 2\sum_{k=1}^{\infty}\tilde{\tilde{\mu}}_k \sin 2k\pi x + \text{even function}.$$

If we consider

$$\begin{aligned} g(x) &= \frac{G(x) - G(1-x)}{2} \\ &= -2q(x)\int_0^x q(t)\,dt \\ &\quad + \frac{1}{2}\left[\int_x^1 q(t)q(t-x)\,dt - \int_{1-x}^1 q(t)q(t-1+x)\,dt\right] \\ &= A(x) + B(x), \end{aligned}$$

then

$$g(x) \sim 2\sum_{k=1}^{\infty} \tilde{\tilde{\mu}}_k \sin 2k\pi x.$$

$B(x)$ is a continuous function of $x \in [0,1]$ and

(7) $$B(x) \to \frac{1}{2}\int_0^1 q^2(t)\,dt \quad \text{as } x \to 0^+.$$

Also

$$\begin{aligned} \frac{1}{\delta}\int_0^\delta |A(x)|\,dx &\le \frac{2}{\delta}\left(\int_0^\delta |q(x)|^2\,dx\right)^{1/2}\left(\int_0^\delta\left(\int_0^x |q(t)|\,dt\right)^2 dx\right)^{1/2} \\ &\le \frac{2}{\delta}\left(\int_0^\delta |q(x)|^2\,dx\right)^{1/2}\left(\int_0^\delta x\int_0^x |q(t)|^2\,dt\,dx\right)^{1/2} \\ &\le \frac{2}{\delta}\int_0^\delta |q(x)|^2\,dx \cdot \sqrt{\frac{\delta^2}{2}} \\ &= \sqrt{2}\int_0^\delta |q(x)|^2\,dx, \end{aligned}$$

i.e.,

(8) $$\frac{1}{\delta}\int_0^\delta |A(x)|\,dx \to 0 \quad \text{as } \delta \to 0^+.$$

From (7) and (8),

(9) $$\frac{1}{\delta}\int_0^\delta \left| g(x) - \frac{1}{2}\int_0^1 q^2(t)\,dt \right| dx \to 0 \quad \text{as } \delta \to 0^+.$$

Let $\Psi(x) = (\frac{1}{2} - x)\int_0^1 q^2(t)\,dt$.

Then (9) becomes

$$\frac{1}{\delta}\int_0^\delta |g(x) - \Psi(x)|\,dx \to 0 \quad \text{as } \delta \to 0^+. \tag{10}$$

Now

$$\int_0^{1/2} g(x)\sin 2k\pi x\,dx = \frac{\tilde{\tilde{\mu}}_k}{2},$$

$$\int_0^{1/2} \Psi(x)\sin 2k\pi x\,dx = \frac{1}{4k\pi}\int_0^1 q^2(t)\,dt.$$

Hence

$$\begin{aligned} A_n &:= \frac{1}{2}\sum_{k=1}^n \tilde{\tilde{\mu}}_k - \frac{1}{4\pi}\int_0^1 q^2(t)\,dt \sum_{k=1}^n \frac{1}{k} \\ &= \int_0^{1/2} (g(x) - \Psi(x)) \sum_{k=1}^n \sin 2k\pi x\,dx \\ &= \int_0^{1/2} (g(x) - \Psi(x)) D_n(x)\,dx. \end{aligned}$$

$D_n(x)$ has the following two properties, see [4]:

(i) $|D_n(x)| \le n,$

(ii) $|D_n(x)| = \left|\dfrac{\cos \pi x - \cos(2n+1)\pi x}{2\sin \pi x}\right| \le \dfrac{1}{2x}, \quad 0 < x \le \frac{1}{2}.$

Therefore

$$|A_n| \le n\int_0^{1/n} |g(x) - \Psi(x)|\,dx + \frac{1}{2}\int_{1/n}^{1/2} \frac{|g(x) - \Psi(x)|}{x}\,dx.$$

The first term $\to 0$ as $n \to +\infty$ by (10). To handle the last term, we consider

$$F(x) = \frac{1}{x}\int_0^x |g(t) - \Psi(t)|\,dt.$$

Then $|g(x) - \Psi(x)| = (xF(x))' = F(x) + xF'(x)$ and

$$\text{“last term”} = \frac{1}{2}\int_{1/n}^{1/2} \frac{F(x)}{x}\,dx + \frac{1}{2}F\left(\frac{1}{2}\right) - \frac{1}{2}F\left(\frac{1}{n}\right).$$

By (10), given $\varepsilon > 0$ there exists x_0 such that

$$F(x) \leq \varepsilon \quad \text{if } 0 < x \leq x_0.$$

Then

$$\frac{1}{2}\int_{1/n}^{1/2}\frac{F(x)}{x}\,dx = \frac{1}{2}\int_{x_0}^{1/2}\frac{F(x)}{x}\,dx + \frac{1}{2}\int_{1/n}^{x_0}\frac{F(x)}{x}\,dx$$
$$\leq \frac{1}{2}\int_{x_0}^{1/2}\frac{F(x)}{x}\,dx + \frac{\varepsilon}{2}(\log n + \log x_0).$$

So

$$\frac{|A_n|}{\log n} \leq \frac{n\int_0^{1/n}|g(x) - \Psi(x)|\,dx}{\log n} + \frac{\varepsilon}{2} + \frac{\varepsilon \log x_0}{2\log n}$$
$$+ \frac{1}{2\log n}\int_{x_0}^{1/2}\frac{F(x)}{x}\,dx + \frac{\frac{1}{2}F(\frac{1}{2}) - \frac{1}{2}F\left(\frac{1}{n}\right)}{\log n}.$$

Hence

$$\varlimsup_{n\to\infty}\frac{|A_n|}{\log n} \leq \frac{\varepsilon}{2}$$

and

$$\frac{A_n}{\log n} \to 0, \quad n \to +\infty,$$

$$\frac{\tilde{\tilde{\mu}}_1 + \cdots + \tilde{\tilde{\mu}}_n}{2\log n} - \frac{1}{4\pi}\int_0^1 q^2(t)\,dt\,\frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{\log n} \to 0,$$
$$\frac{\tilde{\tilde{\mu}}_1 + \cdots + \tilde{\tilde{\mu}}_n}{\log n} \to \frac{1}{2\pi}\int_0^1 q^2(t)\,dt, \quad n \to +\infty.$$

So, by (6),

$$\frac{\tilde{\mu}_1 + 2\tilde{\mu}_2 + \cdots + n\tilde{\mu}_n}{\log n} \to \frac{1}{4\pi^2}\int_0^1 q^2(t)\,dt, \quad n \to +\infty.$$

The last assertion of the theorem comes from the above because if

$$\sum\left(n^{3/2}\tilde{\mu}_n\right)^2 < +\infty,$$

then

$$\frac{\tilde{\mu}_1 + \cdots + n\tilde{\mu}_n}{\log n} \leq \frac{\left(\tilde{\mu}_1^2 + \cdots + \left(n^{3/2}\tilde{\mu}_n\right)^2\right)^{1/2}\left(1 + \cdots + \frac{1}{n}\right)^{1/2}}{\log n} \to 0$$

and so

$$\int_0^1 q^2(t)\,dt = 0, \quad q \equiv 0.$$

This theorem is sharp, because there exists a q,

$$q(x) = \begin{cases} 1, & 0 < x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x < 1, \end{cases}$$

for which $\{\tilde{\mu}_k\} \in l^2_\alpha$ for every α, $1 \leq \alpha < \frac{3}{2}$ since

$$\tilde{\tilde{\mu}}_k = \begin{cases} \dfrac{3}{2k\pi}, & k \text{ even}, \\ -\dfrac{1}{2k\pi}, & k \text{ odd}. \end{cases}$$

References

1. G. Borg, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertanfgabe*, Acta Math., vol. 78 (1946), pp. 1–96.
2. V.A. Marčenko and I.V. Ostrovskii, *A characterization of the spectrum of Hill's operator*, Math. USSR-Sbornik, vol. 97 (1975), pp. 493–554.
3. J. Poschel and E. Trubowitz, *Lectures on inverse spectral theory*, to appear.
4. A. Zygmund, *Trigonometric series*, Vol. 1., Cambridge Univ. Press, Cambridge, England, 1959.

University of California at Los Angeles
Los Angeles, California

University of Wisconsin, Madison
Madison, Wisconsin