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# $L^p$ ESTIMATES FOR CERTAIN GENERALIZATIONS OF $k$-PLANE TRANSFORMS.

BY

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## Introduction

Estimates for the $k$-plane transform have been obtained by a variety of authors [1], [3], [5], [7], the definitive results being obtained by M. Christ [2]. We denote by $M_{n,k}$ the manifold of affine $k$-planes in $\mathbf{R}^n$. For $\Pi$ an element of $M_{n,k}$, we denote by $\lambda_\Pi$ the Lebesgue measure carried by $\Pi$. The $k$-plane transform $T_k f$ of a suitable function $f$ defined on $\mathbf{R}^n$ is given by

$$T_k f(\Pi) = \int f(x)\, d\lambda_\Pi(x).$$

There is a natural measure $\mu$ carried by $M_{n,k}$ and invariant under rigid motions of $\mathbf{R}^n$. The typical estimate alluded to above takes the form

$$\|T_k f\|_{L^q(M_{n,k},\mu)} \leq C\|f\|_{L^p(\mathbf{R}^n)}.$$

The $l$-plane to $k$-plane transform $T_{l,k}$ is a generalization of the $k$-plane transform first mentioned (to our knowledge) by Strichartz [7, p. 701]. There is an analogous transform $S_{l,k}$ for vector planes. Here we adapt Christ's methods to obtain general estimates for these transforms. The novel element in the present article is the use of stereographic projection as a link between the two transforms $T_{l,k}$ and $S_{l,k}$.

The results presented here have the bizarre property of being invariant under all affine motions (cf. [4]). The affinely invariant results in this area are generally the sharpest and were not obtained by the earlier workers—compare for example [8, Theorem 4.2] with [2], Theorem 2.1(B). On the other hand, even in the earliest work, smoothness estimates are obtained and it is an open question how to formulate such estimates in an affinely invariant way.

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The reader may consult [9] for a comprehensive survey of the area. We should like to express our thanks to the referee for helpful comments and suggestions.

## 1. Statement of results

Let $G_{n,k}$ denote the space of linear $k$-planes (i.e., $k$-planes passing through the origin) in $\mathbf{R}^n$. This is a compact manifold and indeed a homogenous space of the orthogonal group $O(n)$. It possesses an invariant probability measure $\gamma$ invariant under the action of $O(n)$. The space $M_{n,k}$ of all $k$-planes can be viewed as a bundle over $G_{n,k}$. For $\Pi$ an element of $M_{n,k}$ we can find a unique translate $\pi$ of $\Pi$ passing through the origin. We can then specify $\Pi$ by $\Pi = \pi + x$ for a unique $x$ in $\pi^{\perp}$. Thus for $\pi \in G_{n,k}$, the fibre of $M_{n,k}$ over $\pi$ can be realized as the Euclidean space $\pi^{\perp}$. We use the notation

$$\Pi = \pi + x = (\pi, x).$$

It is easy to see that $d\mu(\Pi) = d\lambda_{\pi^{\perp}}(x)\, d(\pi)$ so that there are natural mixed norm spaces of functions on $M_{n,k}$ corresponding to the bundle structure. Specifically

$$\|f\|_{L^q(L^r)(M_{n,k})} = \int\left\{\int |f(\pi, x)|^r \, d\lambda_{\pi^{\perp}}(x)\right\}^{q/r} d\gamma(\pi))\}^{1/q}.$$

Now let $0 \le l \le k \le n$. For $\Pi \in M_{n,k}$ there is a natural measure $\mu_{\Pi}$ on the space $M_{k,l}(\Pi)$ of $l$-planes contained in $\Pi$. This measure is invariant under the rigid transformations of $\Pi$ and the normalizations for different $\Pi$ "agree". Similarly for $\pi \in G_{n,k}$, there is a natural probability measure $\gamma_{\pi}$ on the space $G_{k,l}(\pi)$ of linear $l$-planes contained in $\pi$. For $f$ a suitable function defined on $M_{n,l}$ we define

$$(T_{l,k}f)(\Pi) = \int f(\Theta)\, d\mu_{\Pi}(\Theta),$$

a function on $M_{n,k}$. Similarly if $g$ is given on $G_{n,l}$ we define the function $S_{l,k}g$ on $G_{n,k}$ by

$$S_{l,k}g(\pi) = \int g(\theta)\, d\gamma_{\pi}(\theta).$$

THEOREM 1. *Let $1 \le l \le k < n$. Then*

$$\|S_{l,k}g\|_{n/l} \le C_{n,k,l}\|g\|_{n/k}. \tag{1}$$

This result was previously known [2] if $l = 1$, or if $k = n - 1$, or if $l = k$. The result for $k = n - 1$ can be deduced from the case $l = 1$, because $S_{n-l,n-1}$ is essentially the adjoint of $S_{1,l}$. If $l = k$ then $S_{l,k}$ is of course the identity map.

THEOREM 2. *Let* $1 \leq l \leq k < n$. *Then*

$$\|T_{l,k}f\|_{L^B(L^R)} \leq C\|f\|_{L^A(L^P)} \tag{2}$$

*where*

$$(n-l)R' = (n-k)P' = Q, \quad (n-l)B' = (n-k)A' = nQ'$$

*and where* $1 \leq R \leq (n+1)/(l+1)$ *in general and* $1 \leq R < \infty$ *if* $2k > n + l$. *Here the constant* $C$ *depends on* $n$, $k$, $l$ *and* $R$. *The exponent* $Q$ *has been used only to link the other exponents.*

Simple examples show that the result fails if $(n-l)R' \neq (n-k)P'$ or if $(n-l)B' < (n-k)A'$. The result was previously known if $k = n - 1$ [2]. The reader should consult [4] for this formulation of the estimate. It is used to obtain control of the standard $k$-plane transform.

## 2. Proof of the theorems

Theorems 1 and 2 will be proved together by an induction procedure. We will show the following:

PROPOSITION 1. *If* (1) *holds for fixed* $l$, $k$ *and* $n$ *then so does* (2) *for the same* $l$, $k$ *and* $n$.

PROPOSITION 2. *If* (2) *holds for fixed* $l$, $k$ *and* $n$ *then so does the statement* (1) *in which* $l$, $k$ *and* $n$ *are replaced by* $l + 1$, $k + 1$ *and* $n + 1$ *respectively.*

Since Theorem 1 is known for $l = 1$ and general $k$ and $n$ $(1 \leq k < n)$, an obvious induction argument establishes Theorem 1 and Theorem 2. It remains then to prove the propositions.

Proposition 1 is essentially a reworking of the ideas of Christ [2] in a marginally different context. An easy calculation yields the following way of writing $T_{l,k}$:

$$T_{l,k}f(\pi, x) = \iint f(\theta, y)\, d\lambda_{\theta^\perp \cap (\pi + x)}(y)\, d\gamma_\pi(\theta). \tag{3}$$

Here $\pi \in G_{n,k}$, $x \in \pi^\perp$, $\theta \in G_{n,l}$ and $y \in \theta^\perp$. The function $f$ is defined on $M_{n,l}$ and $T_{l,k}f$ on $M_{n,k}$. The affine $l$-plane $\Theta = \theta + y$ is contained in the affine $k$-plane $\Pi = \pi + x$ if and only if $\theta \subseteq \pi$ (i.e., $\theta \in G_{k,l}(\pi)$) and $y \in \theta^{\perp} \cap \Pi$. It is easy to check that

$$d\mu_{\pi}(\Theta) = d\lambda_{\theta^{\perp} \cap (\pi + x)}(y)\, d\gamma_{\pi}(\theta).$$

The key observation is that the inner integral in (3) is really an $(k - l)$-plane transform in the $(n - l)$-dimensional space $\theta^{\perp}$. Let us denote this inner integral by

$$F(\theta, \tilde{\pi}, x) = \int f(\theta, y)\, d\lambda_{\theta^{\perp} \cap (\pi + x)}(y)$$

where $\tilde{\pi} = \theta^{\perp} \cap \pi$ is an element of $G_{n-l, k-l}(\theta^{\perp})$ and $x$ is an element of $\theta^{\perp}$ orthogonal to $\tilde{\pi}$ (which is a complicated way of saying that $x \in \pi^{\perp}$). Thus $\tilde{\pi} + x$ is the generic affine $(k - l)$-plane contained in the $(n - l)$-plane $\theta^{\perp}$. Using now the standard estimate for $k$-plane transforms, Christ [2], we have

$$\| F(\theta, \cdot, \cdot) \|_{L^Q(L^R)} \leq C \| f(\theta, \cdot) \|_{L^P}$$

where $(n - l)R' = (n - k)P' = Q$ and $1 \leq P \leq (n + 1 - l)(k + 1 - l)^{-1}$. If in addition $2k > n + l$ then $(k - l) > 1/2(n - l)$ and Christ's $L^2$ methods extend the range to

$$1 \leq P < (n - l)(k - l)^{-1}.$$

Now, taking the $L^A$ norm in $\theta$ we get

$$\|F\|_{L^A(L^Q(L^R))} \leq C \|f\|_{L^A(L^P)}.$$

Rewriting (3), we have

$$T_{l,k} f(\pi, x) = \int F(\theta, \tilde{\pi}, x)\, d\gamma_{\pi}(\theta)$$

and we need to show that $T_{l,k} f \in L^B(L^R)$. Thus taking $L^R$ norms in $x$ over $\pi^{\perp}$, it remains to show that if

$$g(\pi) = \int G(\theta, \tilde{\pi})\, d\gamma_{\pi}(\theta)$$

then

$$\|g\|_{L^B(G_{n,k})} \leq C \|G\|_{L^A(L^Q)}.$$

Towards this, we let $h$ be a function in the unit ball of $L^{B'}(G_{n,k})$. Then we have

$$\begin{aligned}
&\int g(\pi)h(\pi)\,d\gamma(\pi)\\
&\quad = \int G(\theta,\tilde{\pi})h(\pi)\,d\gamma_\pi(\theta)\,d\gamma(\pi)\\
&\quad = \int G(\theta,\tilde{\pi})h(\pi)\,d\gamma_\theta(\pi)\,d\gamma(\theta)\\
&\quad \le \int\left\{\int|h(\pi)|^{Q'}\,d\gamma_\theta(\pi)\right\}^{1/Q'}\left\{\int|G(\theta,\tilde{\pi})|^{Q}\,d\gamma_\theta(\pi)\right\}^{1/Q}\,d\gamma(\theta)
\end{aligned}$$

where $d\gamma_\theta$ is the invariant measure on the space of $k$-planes containing $\theta$. We have

$$d\gamma_\theta(\pi) = d\gamma_{\theta^\perp}(\tilde{\pi}).$$

Now

$$H(\theta) = \int|h(\pi)|^{Q'}\,d\gamma_\theta(\pi) = \left(S_{l,k}^*|h|^{Q'}\right)(\theta)$$

so by the estimate adjoint to (1), we see that $H^{1/Q'}$ is controlled in $L^{nQ'(n-k)^{-1}}$ provided that

$$nQ'(n-l)^{-1} = B'.$$

The desired estimate now follows if

$$A' = nQ'(n-k)^{-1}.$$

This completes the proof of Proposition 1.

For Proposition 2 we will need only the special case

$$P = A = (n+1)(k+1)^{-1},\quad Q = n+1,\quad R = B = (n+1)(l+1)^{-1}$$

of (2). This is the situation in which the norms are unmixed. We aim to show that $T_{l,k}$ in $\mathbf{R}^n$ and $S_{l+1,k+1}$ in $\mathbf{R}^{n+1}$ are intertwined by stereographic projection. Towards this we need to calculate some Jacobian determinants. We start out with the formula

$$d\lambda_\Pi(x_0)\ldots d\lambda_\Pi(x_k)\,d\mu(\Pi) = c\Delta^{-(n-k)}\,d\lambda(x_0)\ldots d\lambda(x_k) \tag{4}$$

for the affine $k$-planes $\Pi$ in $\mathbf{R}^n$. We have denoted by $\Delta$ the volume of the $k$-simplex with vertices $x_0, x_1, \ldots, x_k$. This formula is proved in [3], but (we have since learnt) was originally due to Blaschke; see [6, p. 200]. We also need the Blaschke formula in the essentially equivalent form

$$d\lambda_\pi(z_0)\dots d\lambda_\pi(z_k)\, d\gamma(\pi) = cD^{-(n-k)}\, d\lambda(z_0)\dots d\lambda(z_k) \tag{5}$$

for $k+1$ planes $\pi$ through the origin in $\mathbf{R}^{n+1}$. Here $D$ denotes the volume of the $(k+1)$-simplex with vertices $0, z_0, z_1, \dots, z_k$.

Now let $E$ be a hyperplane in $\mathbf{R}^{n+1}$ a distance $a > 0$ from the origin. For every $k$-plane $\Pi$ in $E$ let $\pi$ be the linear span of $\Pi$ in $\mathbf{R}^{n+1}$. Then $\pi$ is a $k+1$ plane through the origin in $\mathbf{R}^{n+1}$. Furthermore, almost every $k+1$ plane $\pi$ through the origin in $\mathbf{R}^{n+1}$ arises in this way. For $\Pi$ a $k$-plane in $E$, we denote by $b(\Pi)$ the distance from 0 to $\Pi$. Let $x_0, \dots, x_k$ be points of $\Pi$. Let $t_0, \dots, t_k$ be real numbers and set $z_j = t_j x_j$ $(0 \le j \le k)$ points of $\pi$. Then one easily sees that

$$d\lambda(z_j) = a t_j^n\, d\lambda_E(x_j)\, dt_j, \tag{6}$$

$$d\lambda_\pi(z_j) = b(\Pi) t_j^k\, d\lambda_\Pi(x_j)\, dt_j \tag{7}$$

for $0 \le j \le k$. Taking (4), "multiplying" each side by $dt_0 \dots dt_k$ and rewriting with (6) and (7) we get

$$\begin{aligned}(b(\Pi))^{-(k+1)} \prod_{j=0}^{k} t_j^{-k}\, d\lambda_\pi(z_0)\dots d\lambda_\pi(z_k)\, d\mu(\Pi) \\ = c\Delta^{-(n-k)} a^{-(k+1)} \prod_{j=0}^{k} t_j^{-n}\, d\lambda(z_0)\dots d\lambda(z_k).\end{aligned}$$

Now $D = cb(\Pi)\Delta\prod_{j=0}^{k} t_j$, whence

$$\begin{aligned}(b(\Pi))^{-(n+1)}\, d\lambda_\pi(z_0)\dots d\lambda_\pi(z_k)\, d\mu(\Pi) \\ = cD^{-(n-k)} a^{-(k+1)}\, d\lambda(z_0)\dots d\lambda(z_k).\end{aligned}$$

Finally, comparison with (5) yields

$$d\gamma(\pi) = ca^{k+1}(b(\Pi))^{-(n+1)}\, d\mu(\Pi). \tag{8}$$

Now repeat the argument with $\mathbf{R}^{n+1}$ replaced by $\pi$, $n$ by $k$, $E$ by $\Pi$, $\pi$ by $\theta$, $k$ by $l$ and $\Pi$ by $\Theta$. We obtain

$$d\gamma_\pi(\theta) = c(b(\Pi))^{l+1}(b(\Theta))^{-(k+1)}\, d\mu_\Pi(\Theta). \tag{9}$$

Finally in (8) we replace $k$ by $l$, $\pi$ by $\theta$ and $\Pi$ by $\Theta$ to find

$$d\gamma(\theta) = ca^{l+1}(b(\Theta))^{-(n+1)}\, d\mu(\Theta) \tag{10}$$

where now $\gamma$ and $\mu$ are the standard measures on $G_{n+1,l+1}$ and $M_{n,l}$ respectively.

With these Jacobian determinants computed we complete the proof of Proposition 2. We revert to the viewpoint that $E$ is a *fixed* hyperplane of $\mathbf{R}^{n+1}$, to be thought of as a copy of $\mathbf{R}^n$. We denote $\Pi$ a $k$-plane of $E$ and an $l$-plane of $\Pi$. Further, $\pi$ and $\theta$ denote the linear spans of $\Pi$ and $\Theta$ in $\mathbf{R}^{n+1}$ respectively. Let $f \in L^{(n+1)/(l+1)}(G_{n+1,l+1})$ and define

$$F(\Theta) = (b(\Theta))^{-(k+1)} f(\theta).$$

Then

$$\|F\|_{L^{(n+1)/(k+1)}(M_{n,l})} = ca^{-(k+1)(l+1)/(n+1)} \|f\|_{L^{(n+1)/(k+1)}(G_{n+1,l+1})}$$

by (10). Also

$$\begin{aligned} S_{l+1,k+1} f(\pi) &= \int f(\theta)\, d\gamma_\pi(\theta) \\ &= c \int F(\Theta)(b(\Pi))^{l+1}\, d\mu_\Pi(\Theta) \\ &= c(b(\Pi))^{l+1} T_{l,k} F(\Pi) \end{aligned}$$

by (9). Finally, by (8),

$$\|S_{l+1,k+1} f\|_{L^{(n+1)/(l+1)}(G_{n+1,k+1})} = a^{(k+1)(l+1)/(n+1)} \|T_{l,k} F\|_{(n+1)/(l+1)}.$$

But by (2),

$$\|T_{l,k} F\|_{(n+1)/(l+1)} \le C \|F\|_{(n+1)/(k+1)}$$

from which follows

$$\|S_{l+1,k+1} f\|_{(n+1)/(l+1)} \le C \|f\|_{(n+1)/(k+1)}.$$

This completes the proof of Proposition 2.

## References

1. A.P. Calderón, "On the Radon transform and some of its generalizations" in *Conference on Harmonic Analysis in Honor of Antoni Zygmund*, Beckner et al. Eds., vol. 1, Wadsworth, Belmont, Calif., 1983, pp. 673–689.
2. M. Christ, *Estimates for the k-plane transform*, Indiana Univ. Math. J., vol. 33 (1984), pp. 891–910.
3. S.W. Drury, *Generalizations of Riesz potentials and $L^p$ estimates for certain k-plane transforms*, Illinois J. Math., vol. 28 (1984), pp. 495–512.

4. ______, *An endpoint estimate for certain k-plane transforms*, Canad. Math. Bull., vol. 29 (1986), pp. 96–101.
5. D.M. Oberlin and E.M. Stein, *Mapping properties of the Radon transform*, Indiana Univ. Math. J., vol. 31 (1982), pp. 641–650.
6. L.A. Santaló, *Integral geometry and geometric probability*, Encyclopedia of Mathematics and its Applications, vol. I, Addison-Wesley, Reading, Mass., 1976.
7. R. Strichartz, *$L^p$ estimates for Radon transforms in Euclidean and non-Euclidean spaces*, Duke. J. Math., vol. 48 (1981), pp. 699–727.
8. ______, *Improved Sobolev inequalities*, Trans. Amer. Math. Soc., vol. 279 (1983), pp. 397–409.
9. S.W. Drury, *A survey of k-plane transform estimates*, Proceedings of the Conference in Harmonic Analysis, St. Lawrence University, July 27–29, 1987, Amer. Math. Soc., Providence, R.I., (to appear).

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