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L^p WEIGHTED NORM INEQUALITIES FOR THE SQUARE FUNCTION, $0 < p < 2$

BY

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1. Introduction

In a recent paper [4], the author has settled the following question of C. Fefferman [2]: what is the "smallest" homogeneous, positive operator $\tilde{M}$ such that

$$\int |f^*|^2 V\,dx \leq C(\tilde{M}) \int S^2(f) \tilde{M} V\,dx \tag{1}$$

for all weights V and all $f \in C_0^\infty$? It was conjectured [2] that (1) might hold for $\tilde{M} = M$, the Hardy-Littlewood operator, but this turned out to be false [1].

However, the "minimal" $\tilde{M}$'s discovered in [4] are only slightly larger than M. Therefore, it is either very surprising or very natural that the L^p version of (1) *does* hold, if $0 < p < 2$; and this is the result which we shall prove.

We shall now define our terms. For $Q \subset \mathbf{R}^d$ a dyadic cube, we let $l(Q)$ denote its sidelength and $|Q|$ its Lebesgue measure; Q will always denote a cube and all cubes are assumed dyadic. For $f \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ and Q a cube we define

$$f_Q = \frac{1}{|Q|} \int_Q f.$$

If k is an integer we let

$$f_k = \sum_{l(Q) = 2^{-k}} f_Q \chi_Q$$

where χ_Q is the characteristic function of Q. We set

$$f^*(x) = \sup_k |f_k(x)|.$$

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For $l(Q) = 2^{-k}$ we define

$$a_Q(f) = (f_{k+1} - f_k)\chi_Q.$$

We define the dyadic square function

$$S(f) = \left(\sum_{Q \ni x} \frac{\|a_Q(f)\|_2^2}{|Q|} \right)^{1/2}.$$

Lastly, the dyadic Hardy-Littlewood maximal operator, Mf, is defined by

$$Mf(x) = \sup_{Q \ni x} |f|_Q.$$

All of these are standard definitions. This next one is not quite so standard. For every cube Q and non-negative weight V we set

$$Y(Q, V) = \begin{cases} \dfrac{\int_Q M(\chi_Q V)}{\int_Q V} & \text{if } \int_Q V > 0 \\ 1 & \text{if } \int_Q V = 0. \end{cases}$$

The functional $Y(Q, V)$ measures how "peaky" V is on Q: $Y(Q, V)$ is large if V has most of its mass, relative to Q, concentrated on a small set. It is a natural object to look at when studying weighted inequalities for the square function, because for *any* weight V [5],

$$\sup_{f \in \mathscr{C}_0^\infty} \frac{\int |f^*|^2 V\,dx}{\int S^2(f) V\,dx} \sim \sup_Q Y(Q, V); \tag{2}$$

i.e., the left-hand side of (2) is bounded above and below by constant multiples of the right-hand side, where the (positive) constants depend only on d.

Let ψ: $[0, \infty) \mapsto [1, \infty)$ be increasing and satisfy $\psi(2x) \leq A\psi(x)$ for some A. Define

$$M_\psi V(x) = \sup_{Q \ni x} \psi(\log Y(Q, V)) V_Q.$$

In [4] it is proved that, if $\Sigma 1/\psi(k) \leq 1$ then

$$\int |f^*|^2 V\,dx \leq C(A, d) \int S^2(f) M_\psi V\,dx \tag{3}$$

for all f and V as above, and that (3) fails, for any finite constant, if $\Sigma 1/\psi(k) = \infty$.

An immediate consequence of this theorem is that (1) holds when $\tilde{M} = M(M)$. (Indeed, any M_ψ is going to be much smaller than $M(M)$.) These operators M_ψ are just a little larger than M. It turns out that when $0 < p < 2$, we get an extra factor out in front, of the form $2^{\varepsilon(p-2)k}$—here k is a positive integer which depends on Q—that completely washes out the $\psi(\log Y(Q, V))$ (the meaning of this will become clear in the next section). This is what makes the theorem true.

We prove our theorem in Section 2. We give as a corollary (of the proof) a sufficient condition for the two-weight inequality

$$\int |f^*|^p V\,dx \le \int S^p(f) W\,dx$$

to hold.

At the end we make some remarks about the analogues of these results for the continuous square function, and when $p > 2$.

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2. The theorem

We shall prove:

THEOREM. *For every $0 < p < 2$ there is a $C(p, d) < \infty$, such that*

$$\int |f^*|^p V\,dx \le C(p, d) \int S^p(f) MV\,dx$$

for all $f \in \mathscr{C}_0^\infty$ and non-negative $V \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$.

Our first and only lemma is an analogue of Lemma 1 in [4].

LEMMA. *Let $0 < p < \infty$ and let A be a positive number. Let $\mathscr{F}$ be a family of cubes such that $Y(Q, V) \le A$ for all $Q \in \mathscr{F}$. If*

$$f = \sum_{Q \in \mathscr{F}} a_Q(f),$$

and if $f^* \in L^p(V\,dx)$, then

$$\int |f^*|^p V\,dx \le C(p, d) A^{p/2} \int S^p(f) V\,dx.$$

Proof. Let $V\{\cdots\}$ denote $V\,dx$ measure. By standard arguments, it is enough to show that, for all $\lambda > 0$,

$$V\{f^* > 2\lambda,\ S(f) \leq \gamma\lambda\} \leq \varepsilon(p)V\{f^* > \lambda\} \tag{4}$$

for some $\gamma > C(p, d)A^{-1/2}$. Let $\{Q_\lambda^i\}$ be the maximal cubes such that $|f_{Q_\lambda^i}| > \lambda$. It is enough to show that

$$V\left\{x \in Q_\lambda^i\colon \left(f - f_{Q_\lambda^i}\right)^* > (.9)\lambda,\ S(f) \leq \gamma\lambda\right\} \leq \varepsilon(p)V\left(Q_\lambda^i\right) \tag{5}$$

for all Q_λ^i such that $|f_{Q_\lambda^i}| \leq (1.1)\lambda$. So fix Q_λ^i as above, and let $\{Q_k\}$ be the maximal subcubes of Q_λ^i which are elements of $\mathscr{F}$. A little thought shows that we must have $f_{Q_k} = f_{Q_\lambda^i}$, and therefore

$$\begin{aligned}\text{left-hand side of (5)} &\leq \sum_k V\left\{x \in Q_k\colon \left(f - f_{Q_k}\right)^* > (.9)\lambda,\ S(f) \leq \gamma\lambda\right\}\\ &= \sum_k V(E_k).\end{aligned}$$

By Theorem 3.1 of [1], each E_k satisfies

$$\frac{|E_k|}{|Q_k|} \leq B\exp\left(-C\gamma^{-2}\right)$$

where B and C are positive constants that depend on d. We have $Y(Q_k, V) \leq A$ for each k, and therefore [3, p. 23]

$$V(E_k) \leq C(d)A\left(\log\left(1 + \frac{|Q_k|}{|E_k|}\right)\right)^{-1} V(Q_k),$$

and thus we can get (4) by taking $\gamma \sim A^{-1/2}$. QED.

Henceforth we shall assume that p is fixed, $0 < p < 2$.

We shall need one more definition. Let f be as in the lemma, i.e.,

$$f = \sum_{Q \in \mathscr{F}} a_Q(f)$$

for some $\mathscr{F}$. For any cube Q^* we define

$$c_{Q^*}(f) = \left(\sum_{Q^* \subseteq Q} \frac{\|a_Q(f)\|_2^2}{|Q|}\right)^{p/2} - \left(\sum_{\substack{Q^* \subset Q\\ Q^* \neq Q}} \frac{\|a_Q(f)\|_2^2}{|Q|}\right)^{p/2}.$$

Clearly, $S^p(f) = \Sigma_{Q \ni x} c_Q(f)$, and each $c_Q \geq 0$. More importantly, $c_Q(f) = 0$ if $Q \notin \mathscr{F}$.

By the monotone convergence theorem, it is enough to prove our theorem when V is bounded. For k a non-negative integer, let $\mathscr{F}_k$ be the collection of cubes Q such that $2^k \leq Y(Q,V) < 2^{k+1}$; every Q will be in some $\mathscr{F}_k$ since $V \in L^\infty$. Set

$$f_{(k)} = \sum_{Q \in \mathscr{F}_k} a_Q(f).$$

If $f \in \mathscr{C}_0^\infty$ and some $f_{(k)} \notin L^p(V\,dx)$, then $S(f) \notin L^p(V\,dx)$, and there is nothing to prove. Therefore we may assume that each $f_{(k)} \in L^p(V\,dx)$. We write

$$\begin{aligned}
\int |f^*|^p V\,dx &\leq C\sum_k (1+k)^2 \int |f^*_{(k)}|^p V\,dx \\
&\leq C(p,d)\sum_k (1+k)^2 2^{kp/2} \int S^p\big(f_{(k)}\big) V\,dx \qquad (6)\\
&= C(p,d)\sum_k (1+k)^2 2^{kp/2} \int \sum_{x \in Q \in \mathscr{F}_k} c_Q\big(f_{(k)}\big) V\,dx \\
&= C(p,d)\sum_k (1+k)^2 2^{kp/2} \sum_{Q \in \mathscr{F}_k} c_Q\big(f_{(k)}\big) V(Q) \\
&\leq C(p,d)\sum_k (1+k)^2 \sum_{Q \in \mathscr{F}_k} c_Q\big(f_{(k)}\big) Y(Q,V)^{p/2} V(Q) \\
&= C(p,d)\sum_k (1+k)^2 \sum_{Q \in \mathscr{F}_k} c_Q\big(f_{(k)}\big) Y(Q,V)^{p/2-1} \int_Q M(\chi_Q V)\,dx \qquad (7)\\
&\leq C(p,d)\sum_k (1+k)^2 2^{k(p/2-1)} \sum_{Q \in \mathscr{F}_k} c_Q\big(f_{(k)}\big) \int_Q MV\,dx \qquad (8)\\
&= C(p,d)\sum_k (1+k)^2 2^{k(p/2-1)} \int S^p\big(f_{(k)}\big) MV\,dx \\
&\leq C(p,d) \int S^p(f) MV\,dx
\end{aligned}$$

since $p/2 - 1 < 0$. (Inequality (6) follows from the lemma and (7) is from the definition of $Y(Q,V)$.) The theorem is proved. QED.

The astute reader will have observed that inequality (8) has the following consequence.

COROLLARY. *Let $\eta > p/2$ and let V and W be non-negative weights. If for all cubes Q,*

$$Y(Q,V)^{\eta}\int_Q V\,dx \le \int_Q W\,dx,$$

then

$$\int |f^*|^p V\,dx \le A(p,\eta,d)\int S^p(f)W\,dx$$

for all $f \in \mathscr{C}_0^\infty$.

Remark. The author has a "machine" which turns dyadic results like the preceding into corresponding inequalities for the continuous square function(s). This machine, along with applications to singular integrals and Sobolev inequalities, will appear elsewhere [6].

Remark. If $2 \le p < \infty$ then the right $\tilde{M}$ is

$$M_{\psi,p}V = \sup_{Q \ni x} \psi(\log Y(Q,V))Y(Q,V)^{p/2-1}V_Q$$

where ψ: $[0,\infty) \mapsto [1,\infty)$ is increasing, $\psi(2x) \le A\psi(x)$, and

$$\sum_k \psi(k)^{-1/(p-1)} \le 1. \tag{9}$$

The proof follows from arguments like those here and in [4], plus the additional fact that

$$\sum_k S^p(f_{(k)}) \le S^p(f)$$

when $p \ge 2$. If the sum in (9) is infinite, then essentially the same construction as in [4] shows that the corresponding weighted norm inequality fails. We leave the details to the interested reader.

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