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A CONVERSE FATOU THEOREM ON HOMOGENEOUS SPACES

BY

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0. Introduction

The classical Fatou theorem states that positive harmonic functions defined on $\mathbf{R}^n \times (0, \infty)$, $n \geq 1$, have limits at Lebesgue almost every boundary point x_0 of $\mathbf{R}^n$ provided approach is restricted to cones

$$\{(x, t) \in \mathbf{R}^n \times (0, \infty): |x - x_0| < \alpha t\}.$$

Let Ω be a subset of $\mathbf{R}^n \times (0, \infty)$ having the origin as its only limit point in $\mathbf{R}^n \times \{0\}$. It is shown in [12] that a similar almost everywhere convergence result holds if cones are replaced by sets of the form $(x_0, 0) + \Omega$, provided Ω satisfies the cone condition

$$(0.1)\quad \{(x, t) \in \mathbf{R}^n \times (0, \infty): |x - y| < \alpha(t - s) \text{ for some } (y, s) \in \Omega\} \subset \Omega,$$

and the cross sectional measure condition

$$(0.2)\qquad |\{x \in \mathbf{R}^n: (x, t) \in \Omega\}| \leq Ct^n,$$

where the absolute value bars denote Lebesgue measure and C depends only on n.

Similar Fatou theorems are obtained in [11] for functions on $\mathbf{R}^n \times (0, \infty)$ of the form

$$(0.3)\qquad Pf(x, t) = \int_{\mathbf{R}^n} P(x, t, z) f(z)\, dz, \quad f \in L^p(\mathbf{R}^n),\ p \geq 1,$$

where P is a kernel satisfying certain conditions. This includes harmonic functions, solutions of certain parabolic equations on the upper half space and solutions of the heat equation on the right half space. Associated with P is a pseudo-distance ρ. Then conditions on the approach region Ω analogous to

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(0.1) and (0.2) are:

(0.4) There exists $\Omega' \supset \Omega$ such that

$$\{(x, t) \in \mathbf{R}^n \times (0, \infty): \rho(x, y) < \alpha(t - s) \text{ for some } (y, s) \in \Omega\} \subset \Omega'$$

and

$$(0.5) \qquad |\{x \in \mathbf{R}^n: (x, t) \in \Omega'\}| \leq C(|B(0, t)|), \quad \text{for all } t > 0,$$

where $B(0, t) = \{x \in \mathbf{R}^n: \rho(x, 0) < \alpha t\}$. Examples of such Ω are constructed in [11] to contain sequences converging to the origin with arbitrary degree of tangency.

In [14] a Fatou theorem for Poisson-Szegö integrals of L^p functions on the boundary of the generalized half-plane

$$D = \left\{(z_1, \ldots, z_{n+1}) \in \mathbf{C}^{n+1}: \operatorname{Im} z_{n+1} > \sum_{i=1}^{n} |z_j|^2\right\}$$

is obtained. Examples are given of approach regions Ω inside which these functions have limits not implied by Koranyi's theorem [5]. If we identify D with $H_n \times (0, \infty)$, where H_n is the Heisenberg group in $\mathbf{C}^n$, the main conditions on Ω are analogous to (0.4) and (0.5).

The above results are improved in [9] in the following way. Functions are considered as in (0.3) with $\mathbf{R}^n$ replaced by X, a space of homogeneous type having a group structure, and Lebesgue measure replaced by the measure μ associated with X. A Fatou theorem is obtained guaranteeing μ almost everywhere limits of such functions at points of the boundary, $X \times \{0\}$, where approach is restricted to translates of an open set $\Omega \subset X \times (0, \infty)$ having the identity, e, of X as its only limit point in X. The set Ω is assumed to satisfy (0.4) (with $\mathbf{R}^n$ again replaced by X) and

$$(0.6) \quad \mu(\{x \in X: (x, t) \in \Omega'\}) = \mu(|\{x \in X: \rho(x, e) < t\}|) \quad \text{as } t \to 0^+.$$

Such a set Ω is said to be locally α-admissible. Notice that (0.6) is a weaker condition than (0.5) since it places no restriction on the size of the sections through Ω at height t for t bounded away from 0.

In this paper we consider the converse question. We obtain our results in a framework more general than above. We assume that $X = G/K$ where G is a locally compact Hausdorff topological group, K is a compact subgroup of G and (G, K) is equipped with a gauge, thus making X into a space of homogeneous type. (The X in [9] corresponds to the case $K = \{e\}$.) This is the framework considered in [6]. We show in our main result that almost everywhere Ω-limits of Pf for every $f \in L^p$, for some $p \geq 1$, implies that Ω is contained in a locally α-admissible set for every $\alpha > 0$, thus showing that the limit results of [9] are best possible. In the special case described in paragraph 2 above, this converse is obtained in [10]. However, the technique employed in that paper does not work in the generality considered in the present setting. Here we use a constructive approach, based on the method of A. Zygmund in [15], though complicated by the fact that the sections

$$\Omega(t) = \{x \in \Omega: (x, t) \in \Omega\}$$

are not necessarily connected. Lemmas 3.1 and 3.2 allow us to get around this difficulty.

We wish to thank A. del Junco for helping us to prove Lemma 3.1.

1. Assumptions and statement of principal result

We begin by recalling some definitions in [6]. Let G be a locally compact Hausdorff topological group and let K be a compact subgroup of G. Denote the identity element of G by e. Let μ denote a left invariant Haar measure on G, normalized in case G is compact. Let π: $G \to G/K$ be the canonical map with G/K topologized so that π is continuous and open.

A gauge for (G, K) is a map $G \to [0, \infty)$, denoted by $g \to |g|$, such that for all $g, k \in G$, $k \in K$:

(i) $|g \cdot k| = |g|$.
(ii) $|g^{-1}| = |g|$.
(iii) The gauge balls $B(r) = \{g \in G: |g| < r\}$, $r > 0$, are relatively compact and measurable; the sets $\pi(B(r))$, $r > 0$, form a neighbourhood base at $\pi(e)$ in G/K.
(iv) $|g \cdot h| \leq \gamma(|g| + |h|)$, where $\gamma \geq 1$ is independent of g and h.
(v) $\mu(B(2r)) \leq A\mu(B(r))$, where A is independent of r.

It is shown in [6] that μ is necessarily right invariant, hence for each measurable subset E of G,

$$\mu(E) = \mu(E^{-1}) \tag{1.1}$$

[8, Section 30]. It is also shown that each compact subset of G is contained in a gauge ball $B(r)$ for some $r > 0$.

Let X be the homogeneous space G/K. Let m be the measure on X that is the image of μ under π. The action of G on X defined by $\pi(g \cdot \pi^{-1}(x))$ is written as $g \cdot x$. Using the gauge we can define the pseudo-distance

$$\rho(\pi(g), \pi(h)) = |g^{-1} \cdot h|.$$

Properties (i) and (ii) of the gauge ensure that ρ is well defined. This makes X into a left invariant space of homogeneous type [1, Chapter 3]. That is, for all $x, y, z \in X$, $g \in G$,

(1.2) (i) $\rho(g \cdot x, g \cdot y) = \rho(x, y)$,
(ii) $\rho(x, y) = \rho(y, x)$,
(iii) $\rho(x, y) = 0$ if and only if $x = y$,
(iv) $\rho(x, z) \leq \gamma(\rho(x, y) + \rho(y, z))$,
(v) the sets $B(x, r) = \{y \in X: \rho(x, y) < r\}$, $r > 0$, form a neighbourhood base at x,
(vi) $m(B(x, 2r)) \leq Am(B(x, r))$, for all $r > 0$.

Note that (iii) is a consequence of the fact that $|g| = 0$ precisely when $g \in K$ [6, page 578].

Let Ω be an open subset of $X \times (0, \infty)$ such that $(\pi(e), 0)$ is its only limit point in $X \times \{0\}$. The section through Ω at height $t > 0$ is

$$\Omega(t) = \{x \in X: (x, t) \in \Omega\}$$

Assume that for all t sufficiently close to 0

$$\Omega(t) = K \cdot \Omega(t) = \bigcup\{k \cdot x: k \in K, x \in \Omega(t)\}. \tag{1.3}$$

We say that $(x, t) \in X \times (0, \infty)$ Ω-converges to $x_0 \in X$ if t converges to 0 and x converges to x_0 in such a way that $x \in \pi^{-1}\{x_0\} \cdot \Omega(t)$. Assumption (1.3) implies that if (x, t) Ω-converges to $\pi(e)$ then $x \in \Omega(t)$, so we can think of Ω as an approach region at $\pi(e)$. We say that a function $u: X \times (0, \infty) \to \mathbf{R}$ has Ω-limit L at $x_0 \in X$ if $u(x, t)$ converges to L as (x, t) Ω-converges to x_0. Of course this definition makes sense whether or not Ω is open.

Let P: $X \times (0, \infty) \times X \to [0, \infty)$ be m-measurable in the last component. We assume there is a positive constant B such that for all $x \in X$ and $s > 0$,

$$\int_{B(x, 2s)} P(x, s, z)\, dm(z) > B. \tag{1.4}$$

We also assume there is a subset C of $X \times (0, \infty)$ such that

$$\int_D P(x, t, z)\, dm(z) \text{ has } C\text{-limit } \chi_D \text{ at } m\text{-a.e. point of } X, \tag{1.5}$$

for every m-measurable subset D of X. Here χ_D is the function which is 1 on D and 0 on $X \setminus D$.

Remark 1.1. In case $K = \{e\}$ we identify X with G. If in addition there is a one parameter group $\{a_s\}_{s>0}$ of automorphisms of G, a nonnegative function p defined on G and $q > 0$ such that

(i) $a_{1/s}$ maps $B(2s)$ to $B(2)$,

(ii) $s^q\, dm(z) = dm(a_s(z))$ and

(iii) $P(x, s, z)$ is bounded below by $s^{-q}p(a_{1/s}(x^{-1} \cdot z))$,

then (1.4) holds by invariance of m and the variable change $w = a_{1/s}(z)$ as the integral is just $\int_{B(2)} p(z)\, dm(z)$.

Let $\alpha > 0$. We say Ω is locally α-admissible if there exists an open set Ω' containing Ω such that

$$(1.6)\quad \{(x, t) \in X \times (0, \infty): \rho(x, y) < \alpha(t - s) \text{ for some } (y, s) \in \Omega\} \subset \Omega',$$

and

$$(1.7)\qquad m(\Omega'(t)) = O(m(B(\pi(e), t)) \quad \text{as } t \to 0^+.$$

We can now state our main result.

THEOREM 1.2. *Let Ω be an open subset of $X \times (0, \infty)$ with $(\pi(e), 0)$ as its only limit point in $X \times \{0\}$. Assume that $K \cdot \Omega(t) = \Omega(t)$ for all t sufficiently close to 0. Then if Ω is not contained in a locally α-admissible set, there is a subset F of X such that the function $\int_F P(x, t, z)\, dm(z)$ fails to have Ω-limits on a set of positive m-measure.*

Remark 1.3. For any $\alpha > 0$ define

$$(1.8)\quad \Omega_\alpha = \{(x, t) \in X \times (0, \infty): \rho(x, x_0) < \alpha(t - t_0) \text{ for some } (x_0, t_0) \in \Omega\}.$$

The following properties are easy to show:

(i) $\Omega \subset \Omega_\alpha$.

(ii) $\pi(e) \in \Omega_\alpha(t)$ for all $t > 0$.

(iii) $0 < s < t$ implies that $\Omega_\alpha(s) \subset \Omega_\alpha(t)$.

(iv) If $(y, s) \in \Omega_\alpha$ and $\rho(x, y) < \alpha(t - s)$, then $(x, t) \in \Omega_{\gamma\alpha}$.

Property (i) holds since if $(x, t) \in \Omega$, then $(x, t - \varepsilon) \in \Omega$ for some $\varepsilon > 0$ (recall Ω is open); hence $(x, t) \in \Omega_\alpha$ as $\rho(x, x) = 0 < \alpha(t - (t - \varepsilon))$. Property (ii) holds for the following reason. Let $t > 0$. Since $(\pi(e), 0)$ is a limit point of Ω, there exists $(x, s) \in \Omega$ such that $s < t/2$ and $\rho(\pi(e), x) < \alpha t/2$. Thus $\rho(\pi(e), x) < \alpha(t - t/2) < \alpha(t - s)$ and so $(\pi(e), t) \in \Omega_\alpha$. Property (iii) is obvious and property (iv) comes from (1.2). Observe that, by (1.2)(i), Ω_α satisfies (1.3) if Ω does. It follows that (1.6) is satisfied if we replace Ω by Ω_α and Ω' by $\Omega_{\gamma\alpha}$. Thus, if, as in the statement of Theorem 1.2, Ω fails to be contained in a locally α-admissible set, it must be true that $m(\Omega_{\gamma\alpha}(t)) \neq$

$O(mB(\pi(e), t))$ as $t \to 0^+$, hence there exists a sequence $t_j \to 0^+$ such that

$$\frac{m\big(B\big(\pi(e), t_j\big)\big)}{m\big(\Omega_{\gamma\alpha}\big(t_j\big)\big)} \to 0 \quad \text{as } j \to \infty.$$

2. Examples

For the reader's convenience we describe in more detail the examples given in [11] and [14] that were mentioned in the introduction and indicate why they satisfy the conditions of Section 1. For a proof of the corresponding Fatou theorem where approach is restricted to a locally admissible region, see [9].

In each of the examples $K = \{e\}$ so we identify X with G. In the first three examples m is Lebesgue measure on $X = \mathbf{R}^n$. In all of the examples (1.4) follows as P is bounded below by a dilated convolution kernel as in Remark 1.1. Also in all cases the set C of (1.5) can be taken to be $\{(0, t)\colon t > 0\}$. In what follows, c_n denotes a constant depending only on n, not necessarily the same at each occurrence.

Harmonic functions on $\mathbf{R}^n \times (0, \infty)$.

$p(x) = (1 + |x|^2)^{-(n+1)/2}$, $a_s(x) = s \cdot x$, $\rho(x, y) = |x - y|$, $\gamma = 1$ and $q = n$. The functions

$$Pf(x, t) = \int_{\mathbf{R}^n} P(x, t, z) f(z)\, dm(z), \quad f \in L^p(\mathbf{R}^n),\ p \geq 1,$$

are solutions of Laplace's equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial t^2} = 0.$$

Parabolic functions on $\mathbf{R}^n \times (0, \infty)$.

$P(x, t, z)$ is bounded above and below by

$$c_n t^{-n/2} \exp\big(-|x - y|^2/2r_i t\big), \quad i = 1, 2,$$

where r_1, r_2 are positive constants, $\rho(x, z) = |x - z|^2$, $\gamma = 2$ and $q = n/2$. To verify (1.4) we may apply Remark 1.1 since $P(x, s, z)$ is bounded below by

$$c_n s^{-n/2} p\big(a_{1/s}(x - z)\big),$$

where $p(x) = \exp(-|x|^2/4r_2)$ and $a_s(x) = s^{1/2} \cdot x$. The functions Pf satisfy the second order linear parabolic equation in divergence form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_1}\left(A_{ij}(x,t)\frac{\partial u}{\partial x_i} + A_j(x,t)u\right) + \sum_{j=1}^{n} B_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u - \frac{\partial u}{\partial t} = 0,$$

where the coefficients satisfy certain general conditions (cf. [11]).

Solutions of the heat equation on the right half-space.
We represent a point of $\mathbf{R}^n$ as (x', x_n) where $x' = (x_1, \dots, x_{n-1})$. Then

$$p(x) = x_n^{-(n+2)/2}\exp\left(-(1 + |x'|^2)/4x_n\right)$$

if $x_n > 0$ and 0 otherwise, $a_s(x', x_n) = (s \cdot x', s^2 \cdot x_n)$. We define

$$\rho(x, y) = \left(|x' - y'|^2 + |x_n - y_n|\right)^{1/2}.$$

Then $\gamma = 1$ and $q = n + 1$. The functions Pf satisfy

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x_n}.$$

The remaining conditions of Section 1 are shown in [11] to hold for these examples.

Poisson-Szegö integrals on the generalized half-plane. (See [5] and [14].)
$X = \mathbf{C}^n \times \mathbf{R}$, with the Heisenberg group law

$$(z, t) \cdot (w, s) = (z + w, t + s + 2\operatorname{Im}\langle z, w\rangle),$$

where $\langle z, w\rangle = \sum_{i=1}^{n} z_i \bar{w}_i$. Then

$$p(z,t) = \left(t^2 + \left(|z|^2 + 1\right)^2\right)^{-n-1}, \quad a_h(z,t) = (h^{1/2} \cdot z, h \cdot t),$$

m is Lebesgue measure on $\mathbf{R}^{2n+1}$,

$$\rho((z,t),(w,s)) = \max\left\{|z - w|^2, |t - s - 2 \cdot \operatorname{Im}\langle w, z\rangle|\right\},$$

$\gamma = 2$ and $q = n + 1$.

We remark that a result analogous to Theorem 1.2 holds for the harmonic functions on the unit ball in $\mathbf{R}^n$ and Poisson-Szegö integrals on the unit ball in

$\mathbf{C}^n$ by applying the Kelvin transform [2, page 26] and the Cayley transform [5, page 511].

3. Proof of the theorem

We first prove two lemmas.

LEMMA 3.1. *Let E be a subset of G contained in $U = B(r)$, $r > 0$. Let $U_1 = B(s_1)$, $s_1 > r$ and let $U_2 = B(s_2)$ contain $U \cdot U_1$. Denote by $\alpha - 1$ the greatest integer in $(\mu(E))^{-1}$. Then there exist $g_1, \dots, g_\alpha \in U_2$ such that*

$$\mu(E \cup E \cdot g_1 \cup \cdots \cup E \cdot g_\alpha) \geq \mu(U_1)\big(1 - \exp\big((-\mu U_2)^{-1}\big)\big). \tag{3.1}$$

Proof. Let F be any subset of U_1. Then,

$$\begin{aligned} \int_{U_2} \mu(E \cdot g \cap F)\, d\mu(g) &= \int_{U_2}\int_F \chi_{E\cdot g}(h)\, d\mu(h)\, d\mu(g) \\ &= \int_F\int_{U_2} \chi_{E^{-1}\cdot y}(g)\, d\mu(g)\, d\mu(h) \\ &= \int_F \mu(E^{-1} \cdot h \cap U_2)\, d\mu(h) \\ &= \mu(E)\mu(F). \end{aligned} \tag{3.2}$$

The last equality follows by the right invariance of μ and the facts that $E^{-1} \cdot F \subset U_2$ and $\mu(E^{-1}) = \mu(E)$.

Now apply (3.2) with $F = U_1 \setminus E$. It follows there exists $g_1 \in U_2$ such that

$$\begin{aligned} \mu(E \cdot g_1 \cap (U_1 \setminus E)) &\geq \frac{\mu(E)\mu(U_1 \setminus E)}{\mu(U_2)} \\ &\geq \frac{\mu(E)\{\mu(U_1) - \mu(E)\}}{\mu(U_2)}. \end{aligned}$$

Therefore, since $E \cup E \cdot g_1 \supset E \cup \{E \cdot g_1 \cap (U_1 \setminus E)\}$, and the latter is a disjoint union, we get

$$\begin{aligned} \mu(E \cup E \cdot g_1) &\geq \mu(E) + \mu\{E \cdot g_1 \cap (U_1 \setminus E)\} \\ &\geq \mu(E) + \frac{\mu(E)\{\mu(U_1) - \mu(E)\}}{\mu(U_2)} \\ &= \mu(E)(1 - \mu(E)/\mu(U_2)) + \mu(E)\mu(U_1)/\mu(U_2). \end{aligned}$$

Now suppose we have chosen the points $g_1, \ldots, g_m \in U_2$, $m \geq 1$. Apply (3.2) with

$$F = U_1 \setminus E \cup E \cdot g_1 \cup \cdots \cup E \cdot g_m. \tag{3.3}$$

Then there exists $g_{m+1} \in U_2$ such that

$$\begin{aligned}\mu(E \cdot g_{m+1} \cap F) &\geq \mu(E)\mu(F)/\mu(U_2)\\ &\geq \frac{\mu(E)}{\mu(U_2)}\{\mu(U_1) - \mu(E \cup E \cdot g_1 \cup \cdots \cup E \cdot g_m)\}.\end{aligned}$$

Thus, since

$$\begin{aligned}&E \cup E \cdot g_1 \cup \cdots \cup E \cdot g_{m+1}\\ &\quad \supset (E \cup E \cdot g_1 \cup \cdots \cup E \cdot g_m) \cup (E \cdot g_{m+1} \cap F)\end{aligned}$$

and the latter is a disjoint union, we have

$$\begin{aligned}&\mu(E \cup E \cdot g_1 \cup \cdots \cup E \cdot g_{m+1})\\ &\quad \geq \mu(E \cup E \cdot g_1 \cup \cdots \cup E \cdot g_m) + \mu(E \cdot g_{m+1} \cap F)\\ &\quad \geq \mu(E \cup E \cdot g_1 \cup \cdots \cup E \cdot g_m)\\ &\qquad + \frac{\mu(E)}{\mu(U_2)}\{\mu(U_1) - \mu(E \cup E \cdot g_1 \cup \cdots \cup E \cdot g_m)\}\\ &\quad = \mu(E \cup E \cdot g_1 \cup \cdots \cup E \cdot g_m)(1 - \mu(E)/\mu(U_2))\\ &\qquad + \mu(E)\mu(U_1)/\mu(U_2).\end{aligned}$$

Put

$$a_0 = \mu(E), \quad a_m = \mu(E \cup E \cdot g_1 \cup \cdots \cup E \cdot g_m),$$
$$b = 1 - \mu(E)/\mu(U_2)$$

and

$$c = \mu(E)\mu(U_1)/\mu(U_2).$$

Then we have shown that $a_{m+1} \geq b \cdot a_m + c$ for all $m \geq 0$, hence

$$a_m \geq b^m \cdot a_0 + c(1 - b^m)/(1 - b) > \mu(U_1)\{1 - (1 - \mu(E)/\mu(U_2))^m\},$$

for all $m \geq 1$. Taking $m = \alpha$, we get

$$\begin{aligned}\mu(E \cup E \cdot g_1 \cup \cdots \cup E \cdot g_\alpha) &> \mu(U_1)\left\{1 - \left(1 - \frac{\mu(E)}{\mu(U_2)}\right)^\alpha\right\} \\ &\geq \mu(U_1)\left\{1 - \left[1 - \frac{1}{\alpha \cdot \mu(U_2)}\right]^\alpha\right\} \\ &\geq \mu(U_1)\left\{1 - \exp\left(\frac{-1}{\mu(U_2)}\right)\right\},\end{aligned}$$

since $(1 - t/\alpha)^\alpha \leq \exp(-t)$ for $0 < t < \alpha$. This completes the proof. ∎

The following lemma is a modification of Lemma 1.24 in Chapter XIII of [16].

LEMMA 3.2. *Let notation be as in Lemma* 3.1. *Let V and V_1 be gauge balls such that $U \cdot U_2 \subset V \subset V \cdot V \subset V_1$. (Notice that the translates, $E \cdot g_j$, of Lemma* 3.1 *are all contained in V.) Let $\{g_j\}$ be a sequence of subsets of V such that $\sum \mu(G_j) = \infty$. Then there is a sequence $\{g_j\}$ in V_1 such that*

$$\mu \bigcup_{p=1}^{\infty} \bigcap_{j=p}^{\infty} (V \setminus G_j \cdot g_j) = 0.$$

Proof. Let h be a point of V. Then

$$\begin{aligned}\int_{V_1} \chi_{V \setminus G_j \cdot g}(h)\, d\mu(g) &= \int_{V_1} \chi_V(h)\big(1 - \chi_{G_j \cdot g}(h)\big)\, d\mu(g) \\ &= \chi_V(h)\big(\mu(V_1) - \mu\big(G_j^{-1} \cdot h \cap V_1\big)\big) \\ &= \chi_V(h)\big(\mu(V_1) - \mu(G_j)\big),\end{aligned}$$

since $G_j^{-1} \cdot h \subset V^{-1} \cdot V \subset V_1$.

Let p_1 be a positive integer to be chosen below. Then

$$\begin{aligned}&\frac{1}{(\mu(V_1))^{p_1}} \int_{V_1} \cdots \int_{V_1} \int_V \prod_{j=1}^{p_1} \chi_{V \setminus G_j \cdot g_j}(h)\, d\mu(h)\, d\mu(g_1) \cdots d\mu(g_{p_1}) \\ (3.4) \qquad &= \frac{\mu(V)}{(\mu(V_1))^{p_1}} \prod_{j=1}^{p_1} \big(\mu(V_1) - \mu(G_j)\big) \\ &= \mu(V) \prod_{j=1}^{p_1} \big(1 - \mu(G_j)/\mu(V_1)\big).\end{aligned}$$

Since $\Sigma\mu(G_j)/\mu(V_1) = \infty$, the product $\prod_{j=1}^{\infty}(1 - \mu(G_j)/\mu(V_1))$ is 0. Thus we may choose p_1 so that the expression in (3.4) is less than 1. It follows there exist $g_1, g_2, \ldots, g_{p_1} \in V_1$ such that

$$\int_V \prod_{j=1}^{p_1} \chi_{v \setminus G_j \cdot g_j}(h)\, d\mu(h) = \mu \bigcap_{j=1}^{p_1} (V \setminus G_j \cdot g_j) < 1.$$

Continuing in this way we get $p_1 < p_2 < \cdots$ and a sequence $\{g_j\}$ in V_1 such that, for each $k \geq 1$,

$$\mu \bigcap_{j=p_k}^{p_{k+1}} (V \setminus G_j \cdot g_j) < 1/k.$$

Thus, for each integer $p > 1$, $\mu \bigcap_{j=p}^{\infty} (V \setminus G_j \cdot g_j) = 0$, and the result follows. ■

Proof of Theorem 1.2. Without loss of generality we may assume that Ω contains the subset C introduced in (1.5). Indeed if we construct a set F such $\int_F P(x, t, z)\, d\mu(z)$ fails to have $\Omega \cup C$ limits on a set of positive m measure, our assumption on C implies it must fail to have Ω limits on a set of positive m measure as well.

Let U be a gauge ball that contains K and let $\varepsilon > 0$ be less than $\mu(U)$. Our assumption on Ω and Remark 1.3 implies there is a sequence $\{t_j\}$ decreasing to 0 such that

$$\sum_{j=1}^{\infty} \frac{m\big(B(\pi(e), t_j)\big)}{m\big(\Omega_{\gamma\alpha}(t_j)\big)} < \varepsilon. \tag{3.5}$$

Choose b so that for each j,

$$m\big(\Omega_{\gamma\alpha}(t_j)\big) < b. \tag{3.6}$$

Let $E_j = \pi^{-1}(\Omega_{\gamma\alpha}(t_j))$. Since $(\pi(e)), 0)$ is the only limit point of $\Omega_{\gamma\alpha}$ in $X \times \{0\}$, we may assume that $E_j \subset U$ for all j. Let $\alpha_j - 1$ be the greatest integer in $(\mu(E_j))^{-1}$.

Let V be as in Lemma 3.2. According to Lemma 3.1, there is a $\delta > 0$ such that, for each j, there are α_j points $\{g_{(j,1)}, \ldots, g_{(j,\alpha_j)}\}$ with

$$\mu\big(E_j \cup E_j \cdot g_{(j,1)} \cup \cdots \cup E_j \cdot g_{(j,\alpha_j)}\big) > \delta, \tag{3.7}$$

and each of the translates in this union is contained in V.

Let $R_j = \{(g_{(j,k)})^{-1}: k = 1, \ldots, \alpha_j\} \cup \{e\}$. Put

$$G_j = \left\{g \in V: g^{-1} \cdot E_j \cap R_j \neq \emptyset\right\} = E_j \cup E_j \cdot g_{(j,1)} \cup \cdots \cup E_j \cdot g_{(j,\alpha_j)}.$$

Then, by (3.7) we have $\mu(G_j) > \delta$. It follows from Lemma 3.2 that there exists a sequence $\{h_j\}$ such that

$$\mu \bigcup_{p=1}^{\infty} \bigcap_{j=p}^{\infty} (V \setminus G_j \cdot h_j) = 0,$$

hence

$$\mu\left(V \cap \bigcup_{p=1}^{\infty} \bigcup_{j=p}^{\infty} G_j \cdot h_j\right) = \mu(V). \tag{3.8}$$

Let

$$S_j = \left\{\left(g_{(j,k)} \cdot h_j\right)^{-1}: k = 1, \ldots, \alpha_j\right\}.$$

Note that $G_j \cdot h_j = \{g: g^{-1} \cdot E_j \cap S_j \neq \emptyset\}$.

It follows from (3.8) that μ almost every g in V is in $G_j \cdot h_j$ for infinitely many j. Thus, for such g, $g^{-1} \cdot E_j \cap S_j \neq \emptyset$ for infinitely many j. By (1.1), we get that for μ almost every g in V, $g \cdot E_j \cap S_j \neq \emptyset$ for infinitely many j. Hence $\emptyset \neq \pi(g \cdot E_j) \cap \pi(S_j) = g \cdot \Omega_{\alpha\gamma}(t_j) \cap \pi(S_j)$ for infinitely many j. Noting that, by the right K-invariance of the gauge, $\pi^{-1}(\pi(V)) = V$, we get

$$\begin{aligned} m&\left\{x \in \pi(V): \pi^{-1}(x) \cdot \Omega_{\alpha\gamma}(t_j) \cap \pi(S_j) \neq \emptyset \text{ for infinitely many } j\right\} \\ &= \mu\left\{g \in V: g \cdot \Omega_{\alpha\gamma}(t_j) \cap \pi(S_j) \neq \emptyset \text{ for infinitely many } j\right\} \\ &= \mu(V) \\ &= m(\pi(V)). \end{aligned}$$

We thus deduce that for m-almost every $x \in \pi(V)$ there is a sequence in

$$T = \bigcup_{j=1}^{\infty} \left(\pi(S_j) \times \{t_j\}\right)$$

that $\Omega_{\alpha\gamma}$-converges to x.

Let F_j be the union of ρ-balls of radius $\gamma(\alpha\gamma + 2)t_j$ with centers the points of $\pi(S_j)$. Put $F = \bigcup F_j$. By (1.2)(vi), (3.5) and (3.6) we have

$$\begin{aligned}\mu(F) &\leq \sum_{j=1}^{\infty} \alpha_j \cdot \mu\big(B\big(\pi(e), \gamma(\alpha\gamma + 2)t_j\big)\big)\\ &\leq c_{\alpha\gamma} \sum_{j=1}^{\infty} \alpha_j \cdot \mu\big(B\big(\pi(e), t_j\big)\big)\\ &\leq c_{\alpha\gamma} \sum_{j=1}^{\infty} \left(\frac{1}{\mu\big(\Omega_{\alpha\gamma}(t_j)\big)} + 1\right) \cdot \mu\big(B\big(\pi(e), t_j\big)\big)\\ &< c_{\alpha\gamma}(1 + b)\varepsilon,\end{aligned}$$

where $c_{\alpha\gamma}$ depends on α and γ. For $(x, t) \in X \times (0, \infty)$ let

$$u(x, t) = \int_F P(x, t, z)\, dm(z).$$

We will show that u fails to have Ω limits at m-almost every point of $\pi(V) \setminus F$.

Since Ω contains C, it follows from (1.5) that if u had Ω limits on a subset of $\pi(V)$ of positive m measure, the limit would have to be 0 at m-almost every point of $\pi(V) \setminus F$. Thus we will be done if we can show that at m-almost every point of $\pi(V)$ there is a sequence that Ω-converges to it on which u is bounded away from 0.

Let $x = \pi(g)$ be a point of $\pi(V)$ and let (x_j, t_j) be a sequence in T that $\Omega_{\alpha\gamma}$ converges to x. We have seen that this is possible for m-almost every x in $\pi(V)$. Thus there are sequences $k_j \in K$ and $z_j \in \Omega_{\alpha\gamma}(t_j)$ such that $x_j = g \cdot k_j \cdot z_j$. By the definition of $\Omega_{\alpha\gamma}(t_j)$ there is a sequence $(w_j, s_j) \in \Omega$ such that $\rho(z_j, w_j) < \alpha\gamma \cdot (t_j - s_j)$. Put $y_j = g \cdot k_j \cdot w_j$. Then $y_j \in \pi^{-1}(x) \cdot \Omega(s_j)$, so by (1.2), (y_j, s_j) Ω converges to x. Since $(x_j, y_j) = \rho(z_j, w_j)$, we have $B(x_j, \gamma(\alpha\gamma + 2)t_j) \supset B(y_j, 2s_j)$. It follows that

$$\begin{aligned}u(y_j, s_j) &\geq \int_{B(x_j, \gamma(\alpha\gamma+2)t_j)} P(y_j, s_j, z)\, dm(z)\\ &\geq \int_{B(y_j, 2s_j)} P(y_j, s_j, z)\, dm(z)\\ &> B\end{aligned}$$

by (1.4). This completes the proof. ■

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