ILLINOIS JOURNAL OF MATHEMATICS
Volume 33, Number 4, Winter 1989

PERTURBATION THEORY IN DIFFERENTIAL HOMOLOGICAL ALGEBRA I

BY

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1. Introduction and notation

1.1. Introduction. Differential homological algebra extends the classical machinery of homological algebra to differential algebras and modules. As first introduced by Eilenberg and Moore [5], the functor *Tor* (differential Tor) can be constructed in terms of resolutions relative to the category of differential modules, that is, in terms of *bicomplexes*. The category **DGA** of differential graded augmented algebras and differential graded augmented algebra maps was enlarged to the category **DASH** in [7]. **DASH** and **DGA** have the same objects; but **DASH**$(A, A') =$ **DC**$(\bar{B}A, \bar{B}A')$ where **DC** is the category of differential graded augmented coalgebras and $\bar{B}$ denotes the bar construction functor. The functoriality of *Tor* and *Cotor* was extended to this larger category and used to explain certain "collapse theorems" for the Eilenberg-Moore spectral sequence. These collapse theorems are subsumed by the theory in [6] which uses *multicomplexes*, a useful generalization of bicomplexes [22]. Perturbation theory has come to refer to a systematic way of constructing multicomplexes and the purpose of this paper is to begin a study of such algorithms.

DASH itself can be expanded to include more objects and this can be done in a way that generalizes results from [7]. More specifically, it is shown in [10] that if $A \in$ **DGA** is chain homotopy equivalent to a differential graded *module* M in a certain way (see 1.2 below) then M inherits an A_∞ structure and there is a differential graded coalgebra map which is a homology isomorphism from the "tilde construction" (which is what is meant by an "A_∞ structure" [20], [21]) of M to the bar construction of A. With strong conditions on a chain homotopy equivalence between an algebra A and another algebra M, an algorithm was given in [7, 4.1] for the construction of a chain homotopy equivalence which is a differential graded coalgebra map $\bar{B}M \to \bar{B}A$. In §3 we will show that the algorithm given in [10] reduces to this one under these special hypotheses. In other words, a special case of the proof in [10] is the proof of the special case in [7]. Next, we consider the algorithm presented in [8] and called "the basic perturbation lemma" in [14]. We show that the

Received July 31, 1987.

coalgebra map $\bar{B}M \to \bar{B}A$ above can also be obtained by this method. In the final section, we examine some multicomplexes constructed using the twisting cochain studied in §3.

As already mentioned, when the special hypothesis (see 2.5) is not assumed, the algorithm in [10] generally produces a non-associative multiplication on M and a "classifying space", $\tilde{B}M$ for it. Connections between this general procedure and other constructions, along with some applications, will appear in part II. Also see [14].

1.2. Notation and side conditions. We adopt the conventions for signs involving tensor product of maps, suspension, twisting cochains, etc., from [7]. All modules are considered to be differential graded modules over a commutative ring R with 1. The identity map on an object X is denoted by the same symbol X. Strong deformation retraction data is a collection of modules and maps,

$$\left(X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y, \nu\right),$$

which satisfy

$$\text{(i)} \qquad \beta\alpha = X,$$

$$\text{(ii)} \qquad \alpha\beta = X + D(\nu),$$

where we have used the notation $D(\nu) = d\nu + \nu d$ for the chain homotopy ν. Consider the additional conditions

$$\text{(iii)} \qquad \nu\alpha = 0$$

$$\text{(iv)} \qquad \beta\nu = 0$$

$$\text{(v)} \qquad \nu^2 = 0.$$

SDR-data which satisfies (iii) and (iv) is said to provide a contraction of Y onto X [3]. Initial SDR-data can always be altered to satisfy these conditions. If (iii) and (iv) are not satisfied we can replace ν by $\nu' = D(\nu)\nu D(\nu)$ to obtain a contraction. If (v) is not satisfied, we can now replace ν' by $\nu'' = \nu' d\nu'$ and all 3 conditions will hold. This was pointed out in [14]. We will assume that our SDR-data satisfies (iii)–(v). If X and Y are coalgebras and α is a coalgebra map then such data is called "Eilenberg-Zilber data" [7]. The objects called "Koszul algebras" in [16, §3] give rise to Eilenberg-Zilber data. The dual case in which X and Y are algebras and β is multiplicative will concern us here.

We assume that our algebras are augmented over R and are complete with respect to the augmentation filtration and that the corresponding dual notions hold for our coalgebras.

2. Review of some constructions

2.1. Coproducts in tensor modules. Let M be a connected module, i.e., $M_0 = R$ and $M = R \oplus \overline{M}$ where $\overline{M}$ consists of the elements of M of positive degree. We set

$$T^0 = R$$

and

$$T^n = s\overline{M} \otimes \cdots \otimes s\overline{M} \; n\text{-times} \quad \text{if } n \geq 1.$$

Let $T = \bigotimes_{n \geq 0} T^n$ and $\pi_n \colon T \to T^n$ be the nth projection. Generators of T^n are written $[m_1, \ldots, m_n]$ for $n > 0$, and $[\ \ ]$ for $n = 0$. The coproduct $\psi \colon T \to T \otimes T$ is defined by the commutativity of the diagram:

$$\begin{array}{ccc} T & \xrightarrow{\psi} & T \otimes T \\ {\scriptstyle \pi_{a+b}} \downarrow & & \downarrow {\scriptstyle \pi_a \otimes \pi_b} \\ T^{a+b} & \xrightarrow{E(a,b)} & T^a \otimes T^b \end{array}$$

where

$$T^{a+b} \underset{F(a,b)}{\overset{E(a,b)}{\rightleftarrows}} T^a \otimes T^b$$

are the natural isomorphisms. Note that $\psi[\ \ ] = [\ \ ] \otimes [\ \ ]$. T is the *tensor coalgebra* on $s\overline{M}$. If M is an augmented algebra then T is the underlying coalgebra structure of the bar construction $\overline{B}(M)$. In general, if we want to emphasize the role of the module M, we will write $T(M)$.

2.2. Coderivations of tensor coalgebras. An R-linear map $\partial \colon T \to T$ is a *coderivation* if

$$\pi_0 \partial = 0,$$
$$(\partial \otimes 1 + 1 \otimes \partial)\psi = \psi \partial.$$

If ∂ is a coderivation we have

$$(\pi_a \partial \otimes \pi_b + \pi_a \otimes \pi_b \partial)\psi = F(a,b)\pi_{a+b}\partial$$

or equivalently

$$E(a,b)(\pi_a \partial \otimes \pi_b + \pi_a \otimes \pi_b \partial)\psi = \pi_{a+b}\partial$$

and so by induction

$$\partial_n = \sum E(1,\dots,1)(\pi_1 \otimes \cdots \otimes \pi_1\partial \otimes \cdots \otimes \pi_1)\Psi$$

where $E(1,\dots,1)\colon T^1 \otimes \cdots \otimes T^1 \to T^n$ is the natural isomorphism and Ψ is the iterated coproduct (n-fold). Thus ∂ is determined by $\pi_1\partial\colon T \to T^1$ and conversely, given any map $\delta\colon T \to T^1$ we have a unique coderivation ∂ with $\pi_1\partial = \delta$. We write $\omega(\delta) = \partial$. Note that we *do not* claim that $\omega(\delta)^2 = 0$.

2.2.1. Remark. The following special cases will arise later. Let δ be defined by

$$\begin{aligned} \delta([\ \ ]) &= 0, \\ \delta([m]) &= -[dm], \\ \delta|_{T^a} &= 0, \quad a > 1. \end{aligned}$$

Then

$$\begin{aligned} \omega(\delta)([\ \ ]) &= 0, \\ \omega(\delta)[m_1,\dots,m_n] &= \sum \pm [m_1,\dots,dm_k,\dots,m_n]. \end{aligned}$$

This coderivation will be denoted by $T(d)$. More generally, let $u\colon T \to T^1$ be any function such that $u|_{T^a} = 0$, for $a \neq n$; then

$$\omega(u)[m_1,\dots,m_a] = \begin{cases} 0 & \text{if } a < n \\ u[m_1,\dots,m_a] & \text{if } a = n \\ \Sigma[m_1,\dots,u[\cdots],\dots,m_a] & \text{if } a > n \end{cases}$$

where $[\cdots]$ has n entries. Also note that $\omega(u)(T^a) \subset T^{a-n+1}$. If M is an algebra and we take the cup product $\pi_1 \cup \pi_1$ of $\pi_1\colon T \to M$ with itself, i.e., form the composition

$$\overline{B}A \xrightarrow{\psi} \overline{B}A \otimes \overline{B}A \xrightarrow{\pi_1 \otimes \pi_1} A \otimes A \xrightarrow{\mu_A} A,$$

then $\omega(\pi_1 \cup \pi_1)$ is the algebraic bar construction differential since $\pi_1 \cup \pi_1$ restricted to T^a is zero for $a \neq 2$ and for $a = 2$ it is just given by the multiplication in M. Putting these together, if M is a differential graded augmented algebra and u is the sum of δ and $\pi_1 \cup \pi_1$ above, then $\omega(u)$ is just the full bar construction for differential Tor.

2.3. A_∞ structure for modules strongly homotopy equivalent to algebras. We will review the construction given in [10] so that we can conveniently refer to the details in the proofs that follow.

2.3.1 THEOREM [10]. *Given SDR-data*

$$\left(M \underset{\beta}{\overset{\alpha}{\rightleftarrows}} A, \nu\right)$$

with A a connected differential graded augmented algebra and a splitting [9, §1]

$$0 \to Z \rightleftarrows M \rightleftarrows \mathscr{B} \to 0$$

of M (where the boundaries and cycles of M are denoted by $\mathscr{B}$ and $\mathscr{Z}$) there exist two R-module maps of degree -1,

$$\tau\colon T(M) \to A$$

and

$$\partial\colon T(M)\colon \to T(M),$$

such that ∂ is a coderivation with $\partial^2 = 0$ and τ is a twisting cochain for which the induced differential graded coalgebra map [7, 2.4] *is a chain homotopy equivalence*

$$(T(M), \partial) \to \overline{B}(A).$$

The topological analog of this theorem was a motivation for the study of A_∞ spaces in [19], [20]. When the given differential d_M on M is zero it reduces to results in [9], [11] and [18]. We will often write $\tilde{B}M$ for the complex $(T(M), \partial)$ of the theorem and call it the *tilde construction* of M as in [21]. We emphasize that all of the constructions which follow depend on

(1) the given SDR-data,
(2) the given splitting of M.

With respect to the splitting, we have $s\overline{M} = s\overline{\mathscr{B}} \oplus s\overline{\mathscr{Z}}$ and

$$T = \sum \otimes^a s\overline{M} = \sum C_{i_1} \otimes \cdots \otimes C_{i_a}$$

where each C_{i_j} is either $s\overline{\mathscr{B}}$ or $s\overline{\mathscr{Z}}$. To construct the maps of the theorem, we use the following filtration, $F^{a,q}$. Let $X^{a,q}$ be the submodule of T^a equal to the sum

$$\sum C_{i_1} \otimes \cdots \otimes C_{i_a}$$

where exactly q factors are equal to $s\overline{\mathscr{B}}$. Let

$$F^{a,q} = \left(\bigoplus_{p \le q} X^{a,p}\right) \oplus \left(\bigoplus_{b<a} T^b\right).$$

Notice that the differential d_M of M takes the form

$$\mathscr{Z} \oplus \mathscr{B} \to \mathscr{B} \oplus 0,$$
$$(z, b) \mapsto (b, 0).$$

This shows that the differential decreases filtration. For more details about the splitting see [8, pp. 198–199].

If we denote the ordinary Eilenberg-Moore filtration by F^a, i.e.,

$$F^a = \bigoplus_{b \leq a} T^b$$

then $F^a = F^{a,a}$. Let $G^n = F^{a,q}$ where $n = \gamma(a, q)$ and the function γ is given by

$$\gamma(a, b) = \frac{a(a+1)}{2} + b.$$

Also, write $Y^n = X^{a,q}$. Notice that

$$G^{n+1} = Y^{n+1} \oplus G^n$$

and we have

$$T = \bigoplus_{n \geq 0} Y^n.$$

Finally, we write the complementary filtration as $\tilde{G}^a = \bigoplus_{j>a} Y^j$, so that for each n we have the induced splitting

$$T = G^n \oplus Y^{n+1} \oplus (\tilde{G}^{n+1}).$$

For $n \leq 5$ we have

$$\begin{tikzcd}
R = F^0 \arrow[d, equal] & & F^1 \arrow[d, equal] & & & F^2 \arrow[d, equal] \\
R = F^{0,0} \arrow[r] \arrow[d, equal] & F^{1,0} \arrow[r] \arrow[d, equal] & F^{1,1} \arrow[r] \arrow[d, equal] & F^{2,0} \arrow[r] \arrow[d, equal] & F^{2,1} \arrow[r] \arrow[d, equal] & F^{2,2} \arrow[d, equal] \\
R = G^0 \arrow[r] & G^1 \arrow[r] & G^2 \arrow[r] & G^3 \arrow[r] & G^4 \arrow[r] & G^5
\end{tikzcd}$$

The coderivation ∂ and the twisting cochain τ are given by

$$\partial = \partial_2 + \partial_3 + \dots, \quad \tau = \tau_2 + \tau_3 + \dots$$

where ∂_n and τ_n are given inductively: On G^0 and G^1 they are zero and

$$\begin{aligned}\tau_2 &= \alpha\pi_1,\\ \tau_{n+1} &= \tau_n + x_{n+1} \quad \text{for } n > 1,\\ \partial_2 &= \omega(-d_M\pi_1),\\ \partial_{n+1} &= \partial_n + y_{n+1} \quad \text{for } n > 1.\end{aligned}$$

The "partial twisting cochain" x_{n+1} is given by

$$x_{n+1} = \begin{cases} 0 & \text{on } G^n \\ -\nu\Gamma_n & \text{on } Y^{n+1} \\ 0 & \text{on } \tilde{G}^{n+1} \end{cases}$$

and the coderivation y_{n+1} is given by

$$\pi_1 y_{n+1} = \begin{cases} 0 & \text{on } G^n \\ -\beta\Gamma n & \text{on } Y^{n+1} \\ 0 & \text{on } \tilde{G}^{n+1}. \end{cases}$$

Γ_n is the function given by

$$\Gamma_n = D(\tau_n) - \tau_n \cup \tau_n \quad \text{(twisting cochain obstruction)}.$$

Notice that $\tau_n|_{T^a} = \tau_a$ for $a < n$. We have

$$\begin{aligned}\Gamma_2 &= d\tau_2 + \tau_2\partial_2 - \tau_2 \cup \tau_2\\ &= d\alpha\pi_1 + \alpha\pi_1\partial_2 - \alpha\pi_1 \cup \alpha\pi_1\\ &= d\alpha\pi_1 + \alpha(-d_m\pi_1) - \alpha\pi_1 \cup \alpha\pi_1\\ &= d\alpha\pi_1 - d\alpha\pi_1 - \alpha\pi_1 \cup \alpha\pi_1 \quad \text{(since } \alpha \text{ is a chain map)}\\ &= -\alpha\pi_1 \cup \alpha\pi_1.\end{aligned}$$

It follows that, in general,

$$\begin{aligned}\tau_3 &= \alpha\pi_1 + \nu(\alpha\pi_1 \cup \alpha\pi_1),\\ \partial_3 &= \omega(-d_M\pi_1 + \beta(\alpha\pi_1 \cup \alpha\pi_1)) \quad \text{on } G^3.\end{aligned} \tag{2.3.2}$$

Also note that for $n \geq \gamma(a, a)$,

$$\tau_n[m_1, \ldots, m_a] = \tau_{\gamma(a,a)}[m_1, \ldots, m_a].$$

2.4. Basic perturbation. We say that SDR-data is filtered if there are increasing filtrations on the objects involved which are bounded below and the maps are filtration preserving.

2.4.1 BASIC PERTURBATION LEMMA [2], [8], [17]. *Given SDR-data*

$$\left(X \underset{f}{\overset{\nabla}{\rightleftarrows}} Y, \phi\right)$$

and a new differential D on Y, let $t = D - d$ and more generally, $t_n = (t\phi)^{n-1}t$, $n \geq 1$. For each n define new maps:
On X,

(i) $$\partial_n = d + f(t_1 + t_2 + \cdots + t_{n-1})\nabla,$$

(ii) $$\nabla_n = \nabla + \phi(t_1 + t_2 + \cdots + t_{n-1})\nabla.$$

On Y,

(iii) $$f_n = f + f(t_1 + t_2 + \cdots + t_{n-1})\phi,$$

(iv) $$\phi_n = \phi + \phi(t_1 + t_2 + \cdots + t_{n-1})\phi.$$

If the SDR-data is filtered and t lowers filtration, then these maps converge to new SDR-data which we denote by $\partial_\infty, \nabla_\infty, f_\infty, \phi_\infty$:

$$\left((X, \partial_\infty) \underset{f_\infty}{\overset{\nabla_\infty}{\rightleftarrows}} (Y, D), \phi_\infty\right).$$

Furthermore, if the original data satisfies 1.2(i)–(v) *then so does the new data.*

2.5. Eilenberg-Zilber data. The dual of the following construction was given in [7] for Eilenberg-Zilber data. Assume that the SDR-data

$$\left(A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} M, \nu\right)$$

is such that M is an algebra and β is an algebra map. Define a twisting cochain from $\overline{B}M$ to A by

$$\mu = \mu_1 + \mu_2 + \ldots$$

where

$$\mu_0 = 0, \quad \mu_1 = \alpha\pi_1, \quad \mu_n = \sum_{i+j=n} \nu(\mu_i \cup \mu_j).$$

Under the given conditions, we have [7]:

2.5.1 THEOREM. *The coalgebra map induced by μ is a chain homotopy equivalence.*

3. A special case of the construction

3.1. Multiplicative projection. In this section we show that under the hypotheses of (2.5), both the construction of Gugenheim and Stasheff in (2.3.1) and the basic perturbation lemma in (2.4.1) yield the ordinary bar construction and the twisting cochain of (2.5.1).

3.1.1 THEOREM. *Under the hypotheses of* (2.5) *the formula for the twisting cochain in Theorem* (2.3.1) *reduces to the formula for the twisting cochain in* (2.5.1).

Proof. We have $\beta\nu = 0$, $\nu\alpha = 0$ and $\nu\nu = 0$ so that for $n \geq 3$,

$$\beta x_n = 0 \quad \text{and} \quad \nu x_n = 0.$$

This is because the non-trivial part of x_n looks like $x_n = -\nu\Gamma_{n-1}$. It follows that

$$\beta\tau_n = \beta(\tau_2 + x_3 + \cdots + x_n) = \beta\tau_2 = \beta\alpha\pi_1 = \pi_1$$

and

$$\nu\tau_n = \nu(\tau_2 + x_3 + \cdots + x_n) = \nu\tau_2 = \nu\alpha\pi_1 = 0.$$

We will see that ∂_3 is the ordinary (differential Tor) bar construction differential and, by induction, that $\partial_n = \partial_3$, for all $n \geq 3$. For this, consider y_{n+1} which is defined in terms of Γ_n. We have

$$\begin{aligned}
\beta\Gamma_n &= \beta(d\tau_n + \tau_n\partial_n - \tau_n \cup \tau_n) \\
&= d\beta\tau_n + \beta\tau_n\partial_n - \beta(\tau_n \cup \tau_n) \\
&= d\pi_1 + \pi_1\partial_n - \beta(\tau_n \cup \tau_n) \\
&= d\pi_1 + \pi_1\partial_n + (\beta\tau_n) \cup (\beta\tau_n) \quad \text{(since } \beta \text{ is multiplicative)} \\
&= d\pi_1 + \pi_1\partial_n - \pi_1 \cup \pi_1.
\end{aligned}$$

Also note that

$$\begin{aligned}
\pi_1\partial_3 &= \pi_1(\partial_2 + y_3) \\
&= -d_M\pi_1 + \pi_1 y_3 \\
&= -d_M\pi_1 - \beta\Gamma_2 \\
&= -d_M\pi_1 + \pi_1 \cup \pi_1;
\end{aligned}$$

by the remark at the end of (2.2.1), this gives the bar construction differential. Now assume that $\partial_n = \partial_3$. Then

$$\begin{aligned}\beta\Gamma_n &= d\pi_1 + \pi_1\partial_n - \pi_1 \cup \pi_1 \\ &= d\pi_1 + \pi_1\partial_3 - \pi_1 \cup \pi_1 \\ &= 0 \quad \text{(because } \pi_1 \text{ is a twisting cochain)}.\end{aligned}$$

Thus

$$\partial_{n+1} = \partial_n + 0 = \partial_n = \partial_3.$$

We now show that for each $n \geq 3$,

$$\tau_n = \tau_2 + \nu(\tau_{n-1} \cup \tau_{n-1}) \quad \text{on } G^n.$$

Indeed notice that

$$\tau_3 = \tau_2 + \nu(\tau_2 \cup \tau_2) \quad \text{from (2.3.2) above.}$$

More generally (on G^n),

$$\begin{aligned}\tau_n &= \tau_{n-1} - \nu\Gamma_{n-1} \\ &= \tau_{n-1} - \nu(d\tau_{n-1} + \tau_{n-1}\partial - \tau_{n-1} \cup \tau_{n-1}) \\ &= \tau_{n-1} - \nu\, d\tau_{n-1} + \nu(\tau_{n-1} \cup \nu_{n-1}) \quad (\text{since } \nu\tau_{n-1} = 0)\end{aligned}$$

but $\nu d = 1 - \alpha\beta - d\nu$ so

$$\begin{aligned}\tau_n &= \tau_{n-1} - (\tau_{n-1} - \alpha\beta\tau_{n-1} - d\nu\tau_{n-1}) + \nu(\tau_{n-1} \cup \tau_{n-1}) \\ &= \tau_{n-1} - \tau_{n-1} + \alpha\pi_1 + 0 + \nu(\tau_{n-1} \cup \tau_{n-1}) \quad (\text{since } \beta\tau_{n-1} = \pi_1) \\ &= \alpha\pi_1 + \nu(\tau_{n-1} \cup \tau_{n-1}).\end{aligned}$$

Now for each $a \geq 2$, consider τ_n for $n = \gamma(a, a)$. On T^2 we have

$$\begin{aligned}\tau_5[m_1, m_2] &= \nu(\tau_4 \cup \tau_4)[m_1, m_2] \\ &= \nu m_A(\tau_4 \otimes \tau_4)\Delta([m_1, m_2]) \\ &= \nu(\tau_4(m_1)\tau_4(m_2)) \\ &= \nu(\tau_2(m_1)\tau_2(m_2)) \\ &= \nu(\tau_2 \cup \tau_2)[m_1, m_2] \quad \text{for } m_i \in s\overline{M}\end{aligned}$$

where m_A is the multiplication on A. For $n = \gamma(a, a)$ and $a \geq 3$,

$$\begin{aligned}\tau_n[m_1, \ldots, m_a] &= \nu(\tau_{n-1} \cup \tau_{n-1})[m_1, \ldots, m_a] \\ &= \sum \nu m_A(\tau_{n-1} \otimes \tau_{n-1})([m_1, \ldots, m_i] \otimes [m_{i+1}, \ldots, m_a]) \\ &= \sum \nu m_A(\tau_{n-1}[m_1, \ldots, m_i] \otimes \tau_{n-1}[m_{i+1}, \ldots, m_a]) \\ &= \sum \nu m_A\big(\tau_{\gamma(i,i)}[m_1, \ldots, m_i] \otimes \tau_{\gamma(a-i, a-i)}[m_{i+1}, \ldots, m_a]\big)\end{aligned}$$

so, in the limit, we just get the twisting cochain of (2.5.1).

3.2. Tensor coalgebra of SDR-data. The tensor coalgebra of the SDR-data

$$\left(M \underset{(\beta)}{\overset{(\alpha)}{\rightleftarrows}} A, \nu\right) \text{ is the SDR-data}$$

$$\left(TM \underset{T(\beta)}{\overset{T(\alpha)}{\rightleftarrows}} TA, T(\nu)\right) \quad \text{(see (2.1))}$$

where $T(\alpha)$ and $T(\beta)$ are the obvious maps and

$$T(\nu) = R \oplus A \oplus T_2(\nu) \oplus \cdots \oplus T_n(\nu) \oplus \ldots$$

where

$$\begin{aligned}T_n(\nu) &= \nu \otimes \alpha\beta \otimes \cdots \otimes \alpha\beta \\ &\quad + \cdots + A \otimes \cdots \otimes A \otimes \nu \otimes \alpha\beta \otimes \cdots \otimes \alpha\beta \\ &\quad + \cdots + A \otimes \cdots \otimes A \otimes \nu.\end{aligned}$$

This construction appears in [3]. Note that the differentials involved are just the tensor product differentials denoted by $T(d_M)$ and $T(d_A)$. Sine A is an algebra, we also have the bar-construction differential ∂ on $T(A)$ and as noted in (2.2.1),

$$\partial = T(d_A) + \omega(\pi_1 \cup \pi_1).$$

We now apply the basic perturbation lemma (2.4.1) to the tensor coalgebra data above using $t = \omega(\pi_1 \cup \pi_1)$ to obtain limit SDR-data

$$\left((TM, \partial_\infty) \underset{T(\beta)_\infty}{\overset{T(\alpha)_\infty}{\rightleftarrows}} (TA, \partial), T(\nu)_\infty\right)$$

provided that a suitable filtration on the tensor coalgebra data is given. For now, we will only consider this method in the case that M is an algebra and the projection β is multiplicative. Then we have:

3.2.1 THEOREM. *Under the hypotheses of* (2.5), *the limit tensor coalgebra data satisfies*

(1) $$\partial_\infty = \textit{bar-construction differential on } M,$$

(2) $$T(\beta)_\infty = T(\beta),$$

(3) $$\pi_1 T(\alpha)_\infty = \textit{the twisting cochain of } (2.5).$$

Proof. By definition,

$$\partial_\infty = T(d) + T(\beta)(t_1 + t_2 + \dots)T(\alpha)$$

where

$$t = \omega(\pi_1 \cup \pi_1) \quad \text{and} \quad t_n = \left(tT(\nu)\right)^{n-1} t.$$

Thus t is the bar-construction differential for the ordinary algebra obtained by forgetting the differential of A. Since β is an algebra map, we have

$$T(\beta)\omega(\pi_1 \cup \pi_1) = \omega(\pi_1 \cup \pi_1)T(\beta).$$

Since $T(\beta)T(\nu) = 0$, we obtain

$$\begin{aligned}\partial_\infty &= T(d) + T(\beta)tT(\alpha)\\ &= T(d_M) + \omega(\pi_1 \cup \pi_1)T(\beta)T(\alpha)\\ &= T(d_M) + \omega(\pi_1 \cup \pi_1).\end{aligned}$$

A similar computation shows that $T(\beta)_\infty = T(\beta)$. Now consider the limit inclusion:

$$\begin{aligned}T(\alpha)_\infty &= T(\alpha) + T(\nu)\left(t + (tT(\nu))t + (tT(\nu))^2 t + \dots\right)T(\alpha)\\ &= T(\alpha) + \left((T(\nu)t) + (T(\nu)t)^2 + \dots\right)T(\alpha)\\ &= \left(1 + (T(\nu)t) + (T(\nu)t)^2 + \dots\right)T(\alpha).\end{aligned}$$

A straightforward accounting of the tensor product filtration shows that

$$\pi_1 T(\alpha)_\infty[a_1|\dots|a_n] = \pi_1(T(\nu)t)^{n-1}[a_1|\dots|a_n].$$

Expanding the last term gives precisely $\mu_n[a_1|\dots|a_n]$ where μ_n is the n^{th} term of the twisting cochain in (2.5).

4. Resolutions

In this section, we examine Theorem (2.3.1) from the viewpoint of resolutions.

4.1. We need a simple lemma that gives a special condition for two twisted tensor products to be homology equivalent.

4.1.1 LEMMA. *Suppose that C and C′ are connected coalgebras and A and A′ are connected algebras. If the diagram*

$$\begin{array}{ccc} C & \xrightarrow{\tau} & A \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ C' & \xrightarrow{\tau'} & A' \end{array}$$

commutes, where τ and τ' are twisting cochains and F and G are coalgebra and algebra maps that induce isomorphisms in homology, respectively, then the tensor product map $\Psi = F \otimes G$ induces an isomorphism of twisted tensor products

$$H(C \otimes_\tau A) \cong H(C' \otimes_{\tau'} A').$$

Proof. Since Ψ is a chain map of the underlying tensor product complexes, we only need to show that $\Psi(\tau \cap -) = (\tau' \cap -)\Psi$. This is a straightforward computation using the fact that G and F are algebra and coalgebra maps respectively. Now consider the Serre spectral sequence of each twisted tensor product and the map of spectral sequences that Ψ induces. On E^2 terms, this map is just the tensor product and so by the spectral sequence comparison theorem Ψ induces an isomorphism in homology.

4.1.2 REMARK. One application of this lemma is to the data produced by theorem (2.3.1). Let $\tilde{B}M \to A$ be the twisting cochain produced there. By the lemma above, the twisted tensor product $\tilde{B}M \otimes_\tau A$ is homology equivalent to the twisted tensor product $\bar{B}A \otimes_{\pi_1} A$; but this latter complex is just the bar-construction of A and hence acyclic on R, that is, it has zero homology in positive dimensions and homology R in dimension 0. Thus the twisted tensor product

$$\tilde{B}(M) \otimes_\tau A$$

is a differential A-module which is acyclic on R and so it provides a resolution of R over A.

4.2. Distinguished resolutions in a special case. In [6, (2.1)], it is shown that under rather general conditions, a resolution over $H(A)$ can be "perturbed" into a resolution over A. Thus, for example, a resolution of R of the form $X \otimes H(A)$ gives rise to a resolution of the form $X \otimes A$. By explicitly giving a contracting homotopy s for $X \otimes H(A)$, i.e., by specifying SDR-data

$$\left(X \otimes H(A) \underset{\sigma}{\overset{\varepsilon}{\rightleftarrows}} R, s\right)$$

and by specifying SDR-data

$$\left(H(A) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} A, \nu\right)$$

all of the choices in [6, (2.1)] can be made systematically and an analogous remark holds in the more general context given there. For now, we just want to note that in some instances, small complexes for differential Tor can be obtained using the simple twisting chain of (2.5). Resolutions of the ground ring over group rings of two-stage nilpotent groups provide a class of examples. For simplicity, consider the case of two-step torsion free nilpotent groups which are finitely generated. The small complexes given by [6] were worked out explicitly in [12]. On the other hand, for a direct sum of infinite cyclic groups π, consider the multiplicative map

$$\beta\colon \bar{C}(\bar{B}\pi) \to H^*(K(\pi,2); R)$$

which is the dual of the differential Hopf algebra map [6, A.25]. ($\bar{B}\pi$ is the bar construction of the group ring of π over R.) One can complete this to SDR-data

$$\left(R[a_1,\ldots,a_n] \underset{\beta}{\overset{\alpha}{\rightleftarrows}} A, \nu\right) \tag{4.2.1}$$

where $A = \bar{C}(\bar{B}\pi)$ and $H^*(K(\pi,2); R) = R[a_1,\ldots,a_n]$. By composing the twisting chain obtained in (2.5) with the well known coalgebra homology equivalence

$$E[u_1,\ldots,u_n] \to \bar{B}R[u_1,\ldots,u_n]$$

given by "shuffle product", we obtain a small complex

$$E[u_1,\ldots,u_n] \otimes_\eta A$$

equivalent to the bar construction $\bar{B}A \otimes A$ for differential Tor. In our experience, any homotopy ν arising in this context is of the same order of complexity as the cup-1 products used in [6]. Similar remarks apply to the case of more general two-stage nilpotent groups. See [14] for generalizations which apply to n-stage systems. See [12] and [13] for explicit calculations of the cup-1 products for π.

4.3. Cohomology exterior algebra. We conclude with a brief example using the Eilenberg-Maclane contraction [3] for the integers, **Z**, over **Z**:

$$\left(E[x] \underset{f}{\overset{\nabla}{\rightleftarrows}} \bar{B}Z, \phi\right)$$

where

$$\nabla(x) = [1],$$

$$f[n_1|\dots|n_k] = \begin{cases} 0 & \text{if } k > 1 \\ n_1 & \text{if } k = 1, \end{cases}$$

$$\phi[n_1|\dots|n_k] = \begin{cases} \sum_{i=1}^{n_1-1} [1|i|n_2|\dots|n_k] & \text{if } n_1 > 1 \\ 0 & \text{if } n_1 = 1 \\ \sum_{i=1}^{|n_1|} -[1|-i|n_2|\dots|n_k] & \text{if } n_1 \le 1. \end{cases}$$

Note that we are using the normalized bar construction for the additive group Z which is generated by the expressions $[n_1|\dots|n_k]$ for non-zero $n_i \in Z$. We will consider the dual of this data denoted by

$$\left(E[z] \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bar{C}Z, \nu\right)$$

where $\bar{C}Z$ is the complex of "normalized cochains" consisting of certain functions $Z^k \to Z$ [15, IV.5.7] and

$$\alpha(z) = Z \quad \text{identity function},$$

$$\beta(g) = \begin{cases} 0 & \text{if } g\colon Z^i \to Z, i > 1, \\ g(1)z & \text{if } g \in C^1. \end{cases}$$

We will work out the twisting cochain (2.5) in this case. First, note that

$$\tau_n[z|\dots|z] \in C^1.$$

This follows by counting degrees. Write this function as f_n. Now note that

$$\begin{aligned}\nu(\tau_n \cup \tau_k)[z|\dots|z] &= \nu\mu(\tau_n \otimes \tau_k)\psi[z|\dots|z]\\ &= \nu\mu(\tau_n[z|\dots|z] \otimes \tau_k[z|\dots|z])\\ &= \nu(f_n \cup f_k)\end{aligned}$$

where μ denotes cup-product of the functional cochains f_n and f_k, i.e.,

$$\mu(f_n \otimes f_k) = f_n \cup f_k \quad \text{where } (f_n \cup f_k)(i, j) = f_n(i) f_k(j).$$

By dualizing the formula for the homotopy ϕ above, and using the usual convention for signs, we have

$$f_2(i) = -\binom{i}{2}.$$

Furthermore, by induction,

$$\nu(f_n \cup f_k) = 0 \quad \text{if } n > 1$$

so that we have the recursive equations

$$\begin{aligned}f_1(i) &= i,\\ f_n(i) &= -\sum_{j=1}^{i-1} f_{n-1}(j) \quad \text{for } j > 0\end{aligned}$$

and these are the binomial coefficient functions

$$f_n(i) = (-1)^{n+1}\binom{i}{n}.$$

Consider the well-known coalgebra homology isomorphism

$$Z[e] \to \bar{B}E[z].$$

By composing the binomial twisting cochain with this map, we obtain a "small model" for $\bar{B}(\bar{C}Z) \otimes \bar{C}Z$. By tensoring data we can construct a model for $\bar{B}A \otimes A$ of the form $Z[e_1, \dots, e_n[\otimes A$ where $A = \bar{C}Z^n$. Analogous results for other arithmetic rings can also be obtained.

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