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G^∞-FIBER HOMOTOPY EQUIVALENCE

BY

S.Y. HUSSEINI[1]

Introduction and preliminaries

Let G be a compact, connected Lie group, and V, W two complex G-modules. Denote the unit spheres by SV, SW. In this article we shall be concerned with maps over BG,

$$\begin{array}{ccc} EG \times_G (SV, SV^G) & \xrightarrow{f} & EG \times_G (SW, SW^G) \\ & \searrow \quad BG \quad \swarrow & \end{array}$$

where $EG \to BG$ is a universal G-bundle. Such maps are studied in [9]. It can be easily seen that they are exactly those induced by equivariant maps $EG \times SV \to SW$, i.e., by the so-called G^∞-maps $SV \to SW$. We shall say that f is a *G^∞-equivalence* if, and only if, f is the degree-one map on the fibers. Note that according to Dold's theorem [8] a G^∞-equivalence is a fiber-homotopy equivalence, and therefore it admits a G^∞-equivalence as an inverse. Also note that, in the equivariant case, the notion of a G^∞-equivalence is just the notion of quasi-equivalence introduced in [13]. We shall say that the G^∞-equivalence $SV \to SW$ is *special* if, and only if, it induces a T^∞-equivalence

$$(SV, SV^T, SV^G) \to (SW, SW^T, SW^G),$$

where $T \subset G$ is a maximal torus. It is easy to see that a degree-one G-map is special [11].

In this article we first study how V and W are related to each other, given that SV and SW are G^∞-equivalent. The answer is formulated in terms of an appropriate K-theoretic degree, with values in the completion $R(G)^\wedge$ of the representation ring, defined in the manner of [12] and [7] and denoted by $\deg_G f$. We shall say that $\deg_G f$ is *rational* if, and only if, it lies in $R(G)$. It will be shown in §2 below that $\deg_G f$ is rational if $V \cong W$ and f is a G^∞-equivalence, or if f is equivariant. However, the inverse of a degree-one

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G-map, which is always a G^∞-equivalence, need not have a rational degree (cf. Example (6.4) of [13] and (9.5.1) of [7] and Proposition (2.2) below).

We first show (Theorem (1.1)) that if there are special G^∞-equivalences $SV \rightleftarrows SW$ with rational degrees, then V and W are equivalent up to conjugancy. Theorem (2.2) of [16] follows as a special case. Next we consider the sphere bundles $SV \to B$, $SV \to B$ of the complex G-vector bundles $V \to B$, $W \to B$ over the trivial G-complex B. Given a G^∞-fiber homotopy equivalence $SV \to SW$ over B, we show (Theorem (2.3)) that the summands of V and W defined naturally by the irreducible G-modules are stably equivalent, again up to conjugamcy. This latter result is useful in the study of the question of the injectivity of the equivariant J-homomorphism and whether the image is a direct summand. As an illustration we state a result on the injectivity which generalizes those of [3], [6], [10] and [11].

1. Statement of results

Let (X, A) be a compact G-pair. Put

$$\mathscr{K}_G^*(X, A) = K^*(EG \times_G (X, A)),$$

where K^* is the K-theory based on the Bott-spectrum [1], [15]. Note that, for nice enough spaces, $\mathscr{K}_G^*(X, A)$ is the completion of the equivariant K-theory of Segal [14] (Theorem (2.1) of [2]). Also note that $\mathscr{K}_G^*$ defines an equivariant K-theory on the category of compact G-spaces and G^∞-maps.

Now let V and W be two complex G-modules, and denote by SV and SW the unit spheres with respect to some invariant Hermitian metrics. The *$\mathscr{K}_G^*$-degree* of a G^∞-map $f\colon SV \to SW$ is, by definition, the quantity $\deg_G f$ in $\mathscr{K}_G^*(\text{Point}) = K^*(BG) \cong R(G)^\wedge$ such that

$$\mathscr{K}_G^*(f)(\mu_W) = \deg_G(f) \cdot \mu_V$$

where μ_W and μ_V are the Thom-classes of

$$EG \times_G W \to BG \quad \text{and} \quad EG \times_G V \to BG$$

respectively. This is of course completely analogous to the notion of an equivariant degree defined in [12] and §9.7 of [7] for equivariant maps, and reduces to it in that case. Thus $\deg_G f$ is rational if f is a G-map.

Following the notation of p. 192 and p. 195 of [4], let $K \subset (LT)^*$ be a Weyl chamber, $I = \ker\{\exp_T\colon LT \to T\}$ the integral lattice, and $I^* = \{\alpha \in (LT)^* \mid \alpha(I) \subset \mathbf{Z}\}$ the lattice of integral forms. For $\omega \in \bar{K} \cap I^*$, denote by M_ω the irreducible G-module whose highest weight is ω (p. 242 of [4]). Let us note that the evaluation morphism

$$\operatorname{Hom}_G(M_\omega, V) \otimes M_\omega \to V$$

induces naturally an isomorphism

$$V \cong \sum_{\omega} V^{\omega} \otimes M_{\omega}$$

of G-modules, where $V^{\omega} = \mathrm{Hom}_G(M_{\omega}, V)$ and ω ranges over $\bar{K} \cap I^*$. Finally for $\omega \in \bar{K} \cap I^*$, let $\bar{\omega} = \sigma(-\omega)$, where σ is the element of the Weyl group $W_G(T)$ of G relative to T which takes $-K$ to K, (p. 261 of [4]).

THEOREM (1.1). *Suppose that*

$$SV \underset{g}{\overset{f}{\rightleftarrows}} SW$$

are special G^{∞}-equivalences such that $\det_G f$ *and* $\det_G g$ *are rational. Then*

$$\dim_{\mathbf{C}} V^{\omega} + \dim_{\mathbf{C}} V^{\bar{\omega}} = \dim_{\mathbf{C}} W^{\omega} + \dim_{\mathbf{C}} W^{\bar{\omega}}$$

for all $w \in \bar{K} \cap I^*$.

The proof is given in §2 below.

As $\dim_{\mathbf{C}} V^{\infty}$ is the multiplicity of M_{ω} in M, and as M_{ω} and $M_{\bar{\omega}}$ are equivalent as real G-modules, the following is an immediate corollary.

COROLLARY (1.2). *V and W are isomorphic as real G-modules.*

The special case when f and g are G-maps is proved in [11]. Also, the case when f and g are the G-maps and $V^G = \{0\} = W^G$ is proved in [16], Theorem (2.2).

Next let us consider the complex G-vector bundles

$$V \cong \sum_{\omega} V^{\omega} \otimes M_{\omega} \to B, \quad W \cong \sum_{\omega} W^{\omega} \otimes M_{\omega} \to B,$$

where $\omega \in \bar{K} \cap I^*$, M_{ω} is the irreducible G-module whose highest weight is ω and $V^{\omega} = \mathrm{Hom}_G(B \times M_{\omega}, V)$. The base-space B is by assumption a trivial G-space.

THEOREM (1.3). *Suppose that*

$$SV \xrightarrow{f} SW$$

is a special G^{∞}-equivalence over B, and that B is a connected finite cell complex.

Then

$$V^\omega + (V^{\bar\omega})^* \to B, \quad W^\omega + (W^{\bar\omega})^* \to B$$

are stably equivalent as vector bundles, for all of $0 \neq \omega \in \bar{K} \cap I^*$.

The proof is given in §3 below.

The preceding theorem yields the following information on the equivariant J-homomorphism. Consider the complex G-module

$$V = V_0 + \sum_\omega V^\omega \otimes M_\omega,$$

where ω ranges over the set $\Omega = \{\omega_1, \dots, \omega_k\} \subset \bar{K} \cap (LT)^*$ of non-zero maximal weights of V, and $V^\omega = \mathrm{Hom}_G(M_\omega, V)$. Thus $V^G = V_0$. Define $\mathrm{Map}^0_{G^\infty}(SV)$ to be the space, with the compact-open topology, of special G^∞-equivalences. Then the sub-space of linear maps is $U_{m_1} \times \cdots \times U_{m_k}$, where $m_k = \dim_{\mathbf{C}} V^{\omega_k}$. Passing to limits and classifying spaces, we obtain the map

$$J\colon (BU)^{xk} \to B\,\mathrm{Map}^0_{G^\infty}(SV^{\oplus\infty})$$

where $(BU)^{xk}$ is the k-fold product of the classifying space of the infinite unitary group, and $\mathrm{Map}^0_{G^\infty}(SV^{\oplus\infty})$ is the limit of $\mathrm{Map}^0_{G^\infty}(SV)$ as $m_1, \dots, m_k \to \infty$.

Let $f\colon B \to (BU)^{xk}$ be a map, and denote by f_ω the component corresponding to $\omega \in \Omega$. Also denote by $c\colon BU \to BU$ the classifying map of the dual of the universal bundle, and put $f^c_\omega = f_\omega \circ c$.

COROLLARY (1.4). *The composite*

$$G \xrightarrow{f} BU^{xk} \xrightarrow{J} B\,\mathrm{Map}^0_{G^\infty}(SV^{\oplus\infty})$$

is null-homotopic if, and only if, $f_\omega + f^c_{\bar\omega}$ *is null-homotopic for all* $\omega \in \Omega$, *where addition is that induced by the Whitney sum.*

Similar results are proved for the map

$$(BU)^{xk} \xrightarrow{J'} B\,\mathrm{Map}^0_G(SV^{\oplus\infty})$$

in [3], Theorem (11.1), and in [6], [10], and [11], with B a sphere. When B is just a finite complex, the case when $G = S^3$ or S^1 and the action is free is established in [3]. This latter result is used there (in [3]) to prove that the image of J' on the homotopy groups is a direct summand. Corollary (1.4) plays a similar role for the general case.

2. Proof of Theorem (1.1)

Consider V and W as T-modules, and denote by $V(\lambda)$ and $W(\lambda)$ the weight spaces of V and W that correspond to $\lambda \in I^*$. The key step in the proof is the following.

Assertion (2.1): *For all* $\lambda \in I^*$,

$$\dim_{\mathbf{C}} V(\lambda) + \dim_{\mathbf{C}} V(-\lambda) = \dim_{\mathbf{C}} W(\lambda) + \dim_{\mathbf{C}} W(-\lambda).$$

Assuming (2.1) for the time being, let us prove Theorem (1.1). Put

$$P = \left\{\lambda \in \bar{K} \cap I^* \mid \dim_{\mathbf{C}} V(\lambda) + \dim_{\mathbf{C}} V(-\lambda) \neq 0\right\}.$$

As $-\lambda$ and $\bar{\lambda}$ belong to the same orbit of the Weyl group $W_G(T)$, we note that

$$\dim_{\mathbf{C}} V(\bar{\lambda}) = \dim_{\mathbf{C}} V(-\lambda) \quad \text{and} \quad \dim_{\mathbf{C}} W(\bar{\lambda}) = \dim_{\mathbf{C}} W(-\lambda).$$

Hence,

$$\begin{aligned} P &= \left\{\lambda \in \bar{K} \cap I^* \mid \dim_{\mathbf{C}} V(\lambda) + \dim_{\mathbf{C}} V(\bar{\lambda}) \neq 0\right\} \\ &= \left\{\lambda \in \bar{K} \cap I^* \mid \dim_{\mathbf{C}} W(\lambda) + \dim_{\mathbf{C}} W(\bar{\lambda}) \neq 0\right\}. \end{aligned}$$

Now let $\omega \in P$ be a maximal element with respect to the usual order [4, Definition (2.2), p. 250]. Then either M_ω or $M_{\bar{\omega}}$ is a G-summand of V. Similarly, either M_ω or $M_{\bar{\omega}}$ is a $G-$ summand of W. Thus proceeding inductively, we can show that

$$\dim_{\mathbf{C}} V^\omega + \dim_{\mathbf{C}} V^{\bar{\omega}} = \dim_{\mathbf{C}} W^\omega + \dim_{\mathbf{C}} W^{\bar{\omega}} \quad \text{for all } \omega \in \bar{K} \cap I^*,$$

which is what is to be proved.

To prove Assertion (2.1), let

$$\Lambda = \left\{\lambda \in I^* \mid V(\lambda) \neq \{0\}\right\} \quad \text{and} \quad \Gamma = \left\{\gamma \in I^* \mid W(\gamma) \neq \{0\}\right\}.$$

Regarding λ as a homomorphism $T \to S^1$, we can identify it with

$$K^0(B\lambda)(t) \in K^0(BT),$$

where $t = \xi^* - 1$ and ξ^* is the dual of the Hopf-bundle over BS^1. By definition, let

$$|\Lambda| = \prod_{\lambda \in \Lambda} (\lambda)^{m_\lambda}, \qquad |\Gamma| = \prod_{\gamma \in \Gamma} (\gamma)^{m_\gamma}$$

where $m_\lambda = \dim_{\mathbf{C}} V(\lambda)$, $m_\gamma = \dim_{\mathbf{C}} W(\gamma)$.

The first step in the proof of Assertion (2.1) is the computation of $\deg_T(f)$.

PROPOSITION (2.2). $|\Gamma| = \deg_T(f) \cdot |\Lambda|$.

Proof. Put $V_0 = V^T$. We shall prove only the case when $V_0 \neq \{0\}$, the other being similar. It is easy to see that

$$\mathscr{K}_T^*(SV_0) \cong K^*(BT) \otimes K^*(SV_0),$$

and that

$$\mathscr{K}_T^*(SV) \cong K^*(BT) \otimes K^*(SV).$$

Let $V' \subset V$ be the T-orthogonal complement of V_0 in V, and denote by $\beta \in \mathscr{K}_T^*(SV, SV_0)$ the Thom-class of the normal bundle of SV' in SV. The Thom Isomorphism Theorem implies that

$$\mathscr{K}_T^*(SV, SV_0) \cong \mathscr{K}_T^*(V')[\beta].$$

Moreover the homomorphism $T \to U_{m'}$, $m' = \dim_{\mathbf{C}} V'$, defined by the T-module V', and the naturality of the Euler class imply that

$$\mathscr{K}_T^*(V') \cong K^*(BT)/(|\Lambda|),$$

where $(|\Lambda|)$ is the ideal generated by $|\Lambda|$. Since $\mathscr{K}_T^*(SV)$ is torsion-free as a $K^*(BT)$-module, we see immediately that the exact sequence of (SV, SV_0) becomes the short exact sequence

$$\text{(2.3)} \qquad \begin{aligned} 0 \to K^*(BT) \otimes K^*(SV) &\xrightarrow{|\Lambda|\cdot} K^*(BT) \otimes K^*(SV_0) \\ &\to \mathscr{K}_T^*(SV, SV_0) \to 0. \end{aligned}$$

Similarly, the sequence of (SW, SW_0) is

$$\text{(2.4)} \qquad \begin{aligned} 0 \to K^*(BT) \otimes K^*(SW) &\xrightarrow{|\Gamma|\cdot} K^*(BT) \otimes K^*(SW_0) \\ &\to \mathscr{K}_T^*(SW, SW_0) \to 0. \end{aligned}$$

Now let

$$\mu_{V_0} \in K_T^*(DV_0, SV_0) \quad \text{and} \quad \mu_{W_0} \in K_T^*(DW_0, SW_0)$$

be the Thom classes of

$$ET \times_T V_0 \to BT \quad \text{and} \quad ET \times_T W_0 \to BT,$$

respectively, and define

$$[SV_0] \in \mathscr{K}_T^*(SV_0), \quad [SV] \in \mathscr{K}_T^*(SV), \quad [SW_0] \in \mathscr{K}_T^*(SW_0)$$

and

$$[SW] \in \mathscr{K}_T^*(SW)$$

to be the elements whose coboundaries are the Thom classes of the corresponding vector bundles. As $|\Lambda|$ and $|\Gamma|$ are the equivalent Euler classes of V_0 in V and W_0 in W, it follows that the first morphisms of (2.3) and (2.4) take $[SV]$ to $|\Lambda| \cdot [SV_0]$ and $[SW]$ to $|\Gamma| \cdot [SW_0]$. Finally, the given map f induces a map of (2.4) to (2.3). By naturality we see that $|\Gamma| = (\deg_G f)|\Lambda|$ as required.

The second step in the proof of assertion (2.1) is the computation of $|\Lambda|$ and $|\Gamma|$. So choose an isomorphism $\tau\colon T \to S^1 \times \cdots \times S^1$ of T with the r-fold product S^1, with $r = \dim T$. Regarding the components $\tau_1, \ldots, \tau_r$ of τ as elements of $K^*(BT)$, we see immediately that $K^*(BT) \cong R[[\tau_1, \ldots, \tau_r]]$, where $R = K^*(\text{Point})$, and the latter is isomorphic to $\mathbf{Z}[u, u^{-1}]$ [1, p. 13]. A homomorphism $\lambda_i\colon T \to S^1$ induces in turn a homomorphism

$$L(\lambda_i)^*\colon L(S^1)^* \to L(T)^*$$

of the duals of the Lie algebras, which can be expressed in the form

$$L(\lambda_i)^*(dt) = \sum_{i=1}^{r} \lambda_{ij}\, d\tau_j, \quad 1 \le i \le k,$$

where $[\lambda_{ij}]$ is an integral matrix. An easy computation shows that

$$\lambda_i = \left(\sum_{j=1}^{r} (\tau_j + 1)^{\lambda_{ij}} \right) - 1. \tag{2.5}$$

Similarly,

$$\gamma_i = \left(\prod_{j=1}^{r} (\tau_j + 1)^{\gamma_{ij}} \right) - 1. \tag{2.6}$$

Now set $x_j = (\tau_j + 1)$ for $j = 1, \ldots, r$, and consider the equation

$$|\Gamma| = (\deg_T f) \cdot |\Lambda|. \tag{2.7}$$

Since $\deg_T(f) \in R(T)$ by assumption, and since $|\Lambda|, |\Gamma| \in R(T)$, then (2.7) is an equation in $R(T)$. The third step in the proof of Assertion (2.1), is to exploit the divisibility of $|\Gamma|$ by $|\Lambda|$. Recall that

$$R(T) = \mathbf{Z}\left[X_1, \ldots, X_r; (X_1 \ldots X_r)^{-1}\right].$$

Assume that the elements of $|\Lambda| \subset (LT)^*$ are ordered so that $|\lambda_i| \geq |\lambda_j|$ for $1 \leq i < j \leq k$, where $|\lambda_i|^2 = \Sigma_{j=1}^r (\lambda_{ij})^2$. For every integer s, define

$$\alpha_s \colon \mathbf{Z}\left[X_1, \ldots, X_r; (X_1 \ldots X_r)^{-1}\right] \to \mathbf{Z}[X; X^{-1}]$$

to be the homomorphism which takes X_j to $X^{(s\lambda_{1j} + a_j)}$, where a_j is the coefficient of τ_j is the sum $\tau_0 = a_1\tau_1 + \cdots + a_r\tau_r$, with the coefficients $a_j \in \mathbf{Z}$ chosen so that $(\tau_0, \gamma) \neq 0$ for all $\gamma \in \Gamma$. Putting $\mu_s = s\lambda_1 + \tau_0$, we see easily that

$$\alpha_s(\lambda_1) = X^{(\lambda_1, \mu_s)} - 1,$$
$$\alpha_s(|\Gamma|) = \prod_\gamma \left(X^{(\gamma, \mu_s)} - 1\right)^{m_\gamma}, \quad \gamma \in \Gamma$$

where $(\cdot, \cdot)$ is the usual inner-product, and $m_\gamma = \dim_{\mathbf{C}} W(\gamma)$, the multiplicity of γ. Let us observe now that (2.7) implies that λ_1 divides $|\Gamma|$ in $R(T)$. Hence, for sufficiently large s, $X^{(\lambda_1, \mu_s)} - 1$ divides $\alpha_s(|\Gamma|)$. Since the prime factors of the polynomials that appear in $\alpha_s(\lambda_1)$ and $\alpha_s(|\Gamma|)$ are the cyclotomic polynomials that correspond to the factors of (λ_1, μ_s) and (γ, μ_s), we see immediately that (λ_1, μ_s) divides (γ_1, μ_s) for some γ_1 in Γ and infinitely many integers $s \geq 0$. Therefore, either $|\gamma_1| > |\lambda_1|$ or $|\gamma_1| = |\lambda_1|$. If $|\gamma_1| > |\lambda_1|$, then arguing as above by using the T^∞-map g, whose $\mathscr{K}_T^*$-degree is in $R(T)$, we would obtain an element $\lambda' \in \Lambda$ such that $|\lambda'| \geq |\gamma_1| > |\lambda_1|$. But this would contradict the maximality of $|\lambda_1|$. Hence $|\gamma_1| = |\lambda_1|$, which implies that $\lambda_1 = \pm\gamma_1$, since $|\lambda_1| = |\gamma_1|$ and (γ_1, μ_s) is a multiple of (λ_1, μ_s) for infinitely many $s \in \mathbf{Z}$.

Finally, repeating the argument for $\Lambda \setminus \{\lambda_1\}$ and $\Gamma \setminus \{\gamma_1\}$, one sees that after a finite number of steps, given $\lambda \in \Lambda$, we can find $\gamma \in \Gamma$ such that $\lambda = \pm\gamma$, and conversely. This proves assertion (2.1) and hence Theorem (1.1).

3. Proof of Theorem (1.3)

The proof proceeds in stages. Let

$$V = \sum_\omega V^\omega \otimes M_\omega \to B \quad \text{and} \quad W = \sum_\omega W^\omega \otimes M_\omega \to B$$

be two complex G-vector bundles over B as in §1, with $\omega \in \overline{K} \cap I^*$. Observe that on adding appropriate G-vector bundles to V and W we can reduce the theorem to the special case where $W^{\omega} \to B$ is the trivial bundle for all $0 \neq \omega \in \bar{K} \cap I^*$. Thus the theorem is equivalent to the following statement.

(3.1) *For all* $0 \neq \omega \in \bar{K} \cap I^*$, *the complex vector bundle* $V^{\omega} + (V^{\bar{\omega}})^* \to B$ *is stably trivial.*

For each $\lambda \in I^*$, put $V(\lambda) = \Sigma_{\omega} m(\lambda, \omega) V^{\omega}$, for $0 \neq \omega \in \bar{K} \cap I^*$ where $m(\lambda, \omega)$ is the multiplicity of λ in M_{ω}. The first step in the proof of (3.1) is to show that it is implied by the following assertion.

(3.2) *For all* $0 \neq \lambda \in I^*$, *the complex vector bundle* $V(\lambda) + V(-\lambda)^* \to B$ *is stably trivial.*

Put

$$P = \left\{ \omega \in \bar{K} \cap I^* \mid \dim_{\mathbf{C}} V^{\omega} + \dim_{\mathbf{C}} (V^{\bar{\omega}})^* \neq 0 \right\},$$

and choose an element $\omega \in P$ such that, for all $\gamma \in P$ with $\gamma > \omega$, the bundle $V^{\gamma} + (V^{\bar{\gamma}})^* \to B$ is stably trivial. Then

$$V(\omega) = V^{\omega} + \sum_{\gamma > \omega} m(\omega, \gamma) V^{\gamma}, \quad V(\bar{\omega}) = V^{\bar{\omega}} + \sum_{\bar{\gamma} > \bar{\omega}} m(\bar{\omega}, \bar{\gamma}) V^{\bar{\gamma}},$$

since $m(\gamma, \gamma) = 1 = m(\bar{\gamma}, \bar{\gamma})$. But $m(\omega, \gamma) = m(\bar{\omega}, \bar{\gamma})$, for all γ (cf. proof of Proposition (4.1), p. 261 of [4]), since $M_{\bar{\gamma}} = M_{\gamma}^*$, and as $-\omega$ and ω belong to the same $W_G(T)$-orbit, it also follows that $m(-\omega, \bar{\gamma}) = m(\bar{\omega}, \bar{\gamma})$. Hence $V(\omega) + V(-\omega)^* \to B$ is stably equivalent to $V^{\omega} + (V^{\bar{\omega}})^* \to B$, since $V^{\gamma} + (V^{\bar{\gamma}})^* \to B$ is stably trivial for all $\gamma > \omega$. Now (3.2) implies that $V^{\omega} + (V^{\bar{\omega}})^* \to B$ is stably trivial. Arguing by induction, we can deduce (3.1) assuming (3.2).

To prove (3.2), note first of all that $V(\lambda) \cong \mathrm{Hom}_T(B \times C_{\lambda}, V)$, where $\mathbf{C}_{\lambda}$ is the T-irreducible module defined by $\lambda \in I^*$. Now we proceed as in [11], adapting the proof to K-theory. The isomorphism

$$\tau: T \to S^1 \times \cdots \times S^1,$$

defined in §2, induces naturally a splitting $\xi \cong \xi_1 + \cdots + \xi_r$ of ξ as a sum of line bundles, where ξ is the principal T-bundle

$$ET \times SV \to (ET \times SV)/T = ET \times_T SV.$$

Define $t_i \in \mathscr{K}_T^*(SV) = K^*(ET \times_T SV)$ to be $[\xi_i^*] - 1$, where ξ_i^* is the dual of ξ_i, and put

$$P(V) = \prod_{\lambda \neq 0} P(V(\lambda)), \tag{3.3}$$

where $P(V(\lambda)) = \lambda^{m_\lambda} + c_1(V(\lambda))\lambda^{m_\lambda - 1} + \cdots + c_{m_\lambda}(V(\lambda))$ is the K-theoretic Grothendieck defining relation of $V(\lambda) \to B$ evaluated at λ (Theorem (7.1) of [5]). The following result is the K-theoretic analogue of [11]. The proof will be omitted, it being similar.

Put $V_0 = V^T$, $V = V_0 + V^{\perp}$, and denote by β the Thom-class of $SV^{\perp}$ in SV. Regard $P(V)$ as an element in $K^*(B)[[t_1, \dots, t_r]]$, where $t_1, \dots, t_r$ are regarded as indeterminates.

THEOREM (3.4). *The map $ET \times_T SV \to B$ induces an isomorphism*

$$K^*(B)[[t_1, \dots, t_r]]/(P(V))[\beta] \to \mathscr{K}_T^*(SV, SV_0)$$

of $K^*(B)$-modules, where $(P(V))$ is the ideal generated by $P(V)$.

Denote by Λ and Γ the non-zero weights of the representation of T, defined by the fibers of $V \to B$ and $W \to B$, respectively. Then the existence of a T^{∞}-fiber homotopy equivalence over B, f: $SV \to SW$, implies

$$\text{(3.5)} \qquad P(V) = (\deg_T f)^{-1} \cdot |\Gamma|$$

where $\deg_T(f)$ is the $\mathscr{K}_T$-degree of $f|S(V_b)$, with $b \in B$, and V_p is the fiber at b. But, according to Proposition (2.2), $\deg_T(f) = |\Gamma|/|\Lambda|$. Hence equation (3.5) can be written in the form

$$\text{(3.6)} \qquad P(V) = |\Lambda|.$$

Consider first the case when $\dim T = 1$. Since for every $\lambda \in \Lambda$, there is a T-equivalence

$$\begin{array}{ccc} S(V(\lambda) \otimes \mathbf{C}_\lambda) & \longrightarrow & S(V(\lambda)^* \otimes \mathbf{C}_{-\lambda}) \\ & \searrow \quad B \quad \swarrow & \end{array}$$

where $V(\lambda)^* \to B$ is the dual of the $V(\lambda) \to B$, we can adjust the components of $V \to B$ so that the given bundle $V \to B$ becomes

$$V_0 + \sum_{\lambda \neq 0} (V(\lambda) + V(-\lambda)^*) \otimes \mathbf{C}_\lambda \to B$$

where $\lambda \in \Lambda$ ranges over the positive elements. (Recall that when $\dim T = 1$, $I^* \cong \mathbf{Z}$.) Denote the positive elements of Λ by $\{\lambda_1, \dots, \lambda_k\}$, and assume that λ_1 is the smallest element. Now consider the equivariant Grothendieck polynomial

$$P(V) = \prod_{i=1}^{k} \left\{ (X^{\lambda_i} - 1)^{m_i} + c_1(V_1')(X^{\lambda_i} - 1)^{m_i - 1} + \cdots \right\}$$

where $V_i' = V(\lambda_i) + V(-\lambda_i)^*$, $m_i = m_{\lambda_i} = \dim_{\mathbf{C}} V_i'$, and $X = t + 1$. Collecting the terms that involve the first Chern classes of the components V_i', we obtain the expression

$$\sum_i c_1(V_i')(X^{\lambda_1} - 1)^{m_1} \dots (X^{\lambda_i} - 1)^{m_i - 1} \dots (X^{\lambda_k} - 1)^{m_k}.$$

The leading coefficient of $c_1(V_1')$ is the monomial

$$X^{\lambda_1(m_1 - 1)} X^{\lambda_2 m_2} \dots X^{\lambda_k m_k}$$

and, since λ_1 is the smallest element of $\{\lambda_1, \dots, \lambda_k\}$, it follows easily that this monomial does not occur anywhere else in $P(V) - |\Lambda|$. Thus the equation $P(V) = |\Lambda|$ implies that $c_1(V_1') = 0$. This means that V_1' is stably trivial and, hence, $c_j(V_1') = 0$ for $j + 1, \dots, m_1$. Therefore $P(V_1') = (X^{\lambda_1} - 1)^{m_1}$ and, after dividing the equation $P(V) = |\Lambda|$ by $(X^{\lambda_1} - 1)^{m_1}$, which is the same as $(\lambda_1)^{m_1}$, we obtain a similar equation involving one less character. Proceeding inductively, we prove the theorem in the special case when $\dim T = 1$.

Now let us turn to the general case when $\dim T \neq 1$. Choose an element $\lambda_1 \in \Lambda$ of maximal length as in §1 and a character

$$\alpha = \sum_{i=1}^{r} a_i \tau_i \quad \text{in } I^* \subset (LT)^*$$

such that

(i) $(\alpha, \lambda_1) > (\alpha, \lambda')$ for all $\lambda' \in \Lambda$, and
(ii) $(\alpha, \mu) \neq 0$ for all $\mu \in \Lambda$.

The element $\alpha = \sum_{i=1}^{r} a_i \tau_i$ defines a homomorphism $\varphi_\alpha \colon S^1 \to T$ which takes $e^{2\pi i \tau}$ to the tuple $(e^{2\pi i a_1 \tau}, \dots, e^{2\pi i a_r \tau_r})$. Considering the bundle

$$V_0 + \sum_{i=1}^{k} V(\lambda_i) \otimes \mathbf{C}_{\lambda_i} \to B$$

as an S^1-bundle by means of the homomorphism φ_α, we can conclude, because of condition (ii) above, that $V^{S^1} = V_0$. Put

$$\Lambda' = \{(\lambda_i, \alpha) | i = 1, \dots, k\},$$

and by definition let $\lambda_1', \dots, \lambda_p'$ be its distinct elements. Write V in the form

$$V_0 + \sum_{i=1}^{p} V(\lambda_i') \otimes \mathbf{C}_{\lambda_i'} \to B.$$

(This is just the decomposition of V as an S^1-bundle.) It is easy to see that condition (i) above implies that $V(\lambda_1') = V(\lambda_1)$ and $V(-\lambda_1') = V(-\lambda_1)$. Proceeding as in the special case when $\dim T = 1$, we prove that $V(\lambda_1) + V(-\lambda_1)^* \to B$ is stably trivial. Now, continuing inductively, we finish the proof of the theorem for T with $\dim T > 1$.

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University of Wisconsin
Madison, Wisconsin