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A SCALE OF LINEAR SPACES RELATED TO THE L_p SCALE

BY

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1. Introduction and preliminary results

In [7] the Banach space $L_1(\mathbf{R}) \cap L_2(\mathbf{R})$ was investigated. In the present paper we consider a range of spaces of which $L_1(\mathbf{R}) \cap L_2(\mathbf{R})$ is one member. This scale of Banach spaces is closely related both to the L_p scale and to Hilbert space. Subspace structure and other linear topological properties of the scale are investigated.

Let (Ω, Σ, m) be a measure space and let $L_p(\Omega)$ be the usual Lebesgue space with the norm

$$\|f\|_p = \left(\int_\Omega |f|^p \, dm\right)^{1/p} \quad (0 < p < \infty)$$

and

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(\omega)|: \omega \in \Omega\}.$$

For $0 < p \leq \infty$, let $Y_p(\Omega)$ be the collection of all measurable f such that

$$\|f\|_{Y_p} = \|f^*I(0,1)\|_p + \|f^*I(1,\infty)\|_2 < \infty$$

(here f^* denotes the decreasing rearrangement of $|f|$), and for $0 < n < \infty$ let $M_p(\Omega)$ consist of all f such that

$$\|f\|_{M_p} = \|f^*I(0,1)\|_2 + \|f^*I(1,\infty)\|_p < \infty$$

Finally, let $M_\infty(\Omega)$ be the closure of $L_2(\Omega)$ with respect to the norm

$$\|f\|_{M_\infty} = \|f^*I(0,1)\|_2.$$

Observe that if Ω is a probability space then

$$Y_p(\Omega) = L_p(\Omega) \quad \text{and} \quad M_p(\Omega) = L_2(\Omega),$$

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and if $\Omega = \{1, 2, \dots\}$ with the counting measure then

$$Y_p(\Omega) = l_2 \quad \text{and} \quad M_p(\Omega) = l_p.$$

The main purpose of this paper is to investigate the M_p scale for $\Omega = (0, \infty)$ with Lebesgue measure. In this case

$$M_p(\Omega) = L_p(0, \infty) \cap L_2(0, \infty) \quad \text{for } 0 < p \le 2$$

and

$$M_p(\Omega) = L_p(0, \infty) + L_2(0, \infty) \quad \text{for } 2 \le p < \infty,$$

while $Y_p(\Omega)$ is just the reverse. Henceforth we shall write L_p for $L_p(0, \infty)$ etc., because there will be no ambiguity.

The study of Y_p as a Banach space has been effectively reduced to the study of L_p by the following fundamental theorem.

THEOREM A [8]. *Let $1 < p < \infty$. Then the Banach space L_p has exactly two representations as a rearrangement invariant function space on $(0, \infty)$, namely L_p and Y_p.*

The M_p spaces, however, are isomorphically distinct from the L_p scale, though there are many similarities in their Banach space structure. In particular, many results about L_p have a counterpart for M_q, where p and q are Hölder conjugate indices. It turns out, for instance, that the space M_∞ resembles L_1: for example, in its subspace structure and in the fact that M_∞ does not embed into a separable dual space or a space with unconditional basis. The linear structure of the M_p spaces is examined in Sections 4 to 7 below, with the ranges $0 < p < 1$, $1 \le p < 2$, $2 < p < \infty$, and $p = \infty$ receiving separate treatment.

There is also a close affinity between the M_p spaces and Hilbert space. One aspect of this is brought out in the second section, in which some members of the M_p scale are constructed by complex interpolation between the two representations of L_p as an r.i. space on $(0, \infty)$. To be precise, if the parameter θ of the method is chosen so that $[L_{p_0}, L_{p_1}]_\theta = L_2$, with $p_0 < 2 < p_1 < \infty$, then $[L_{p_0}, Y_{p_1}]_\theta = M_p$ for some p.

Two embedding theorems are proved in the third section. In the main result a generalization of an inequality of Rosenthal for sums of independent random variables is used to construct an embedding of M_p into the Lebesgue-Bochner space $L_2(l_p)$. The impossibility of such an embedding into $l_2(l_p)$ is also demonstrated.

We conclude this section with some easy or standard facts about the M_p spaces which are needed later. The reader is referred to [16] and [17] for the Banach space and Banach lattice terminology used throughout the article.

PROPOSITION 1.1. *Let $1 < p \leq \infty$. Then up to an equivalent norm $(M_p)^* = M_q$, where $1/p + 1/q = 1$, with the usual duality $(f, g) = \int_0^\infty fg\,dt$.*

PROPOSITION 1.2. (a) *Let $0 < p \leq 2$. Then M_p is the $(2/p)$-concavification of $Y_{4/p}$.*
(b) *Let $2 \leq p < \infty$. Then M_p is the $(p/2)$-convexification of $Y_{4/p}$.*

Proof. Plainly, $\|f\|_{M_p} \sim \|\,|f|^{p/2}\|_{Y_{4/p}}^{2/p}$.

PROPOSITION 1.3. *Let $0 < p < \infty$ and let*

$$\phi_p(t) = \begin{cases} t^p & (0 \leq t \leq 1) \\ t^2 & (1 \leq t < \infty) \end{cases}.$$

Then M_p is equal to the Orlicz space $L_{\phi_p}(0, \infty)$.

Let $(h_n)_{n=1}^\infty$ be the Haar system on $[0,1]$ defined by $h_1(t) \equiv 1$ and

$$h_{2^k+l}(t) = \begin{cases} 1 & \text{if } t \in \left[(2l-2)2^{-k-1}, (2l-1)2^{-k-1}\right] \\ -1 & \text{if } t \in \left[(2l-1)2^{-k-1}, 2l.2^{-k-1}\right] \\ 0 & \text{otherwise} \end{cases}$$

for $k \geq 0$ and $1 \leq l \leq 2^k$. For $k \geq 1$, let $(h_n^k)_{n=1}^\infty$ be the Haar system translated to the interval $[k-1, k]$ and let $H = (\tilde{h}_n)_{n=1}^\infty$ be the diagonal ordering of $(h_n^k)_{k=1}^\infty{}_{n=1}^\infty$. Finally, let G be the subsequence obtained from H be deleting the terms $h_1^1, h_1^2, \ldots$.

PROPOSITION 1.4 [7]. (a) *Let $1 \leq p \leq \infty$. Then H is a Schauder basis for M_p.*
(b) *Let $0 < p \leq \infty$ and suppose that M_p is isomorphic to a subspace of a quasi-Banach space X having a Schauder basis $(x_n)_{n=1}^\infty$. Then G is equivalent to a block basis of $(x_n)_{n=1}^\infty$.*

2. The M_p spaces and complex interpolation

In this section we show how the M_p spaces arise as interpolation spaces in the ordinary method of complex interpolation. In particular, if the parameter θ is chosen so that L_2 is the result of interpolating between L_{p_0} and L_{p_1}, then M_p (for some p) is the corresponding interpolation space between L_{p_0} and Y_{p_1}.

The reader is referred to [3] and [23] for the method of complex interpolation and the Riesz convexity theorem.

THEOREM 2.1. *Let* $0 < p_0 < 2 < p_1 < \infty$ *and* $0 < \theta < 1$. *Then*

$$[L_{p_0}, Y_{p_1}]_\theta = M_p,$$

where

$$\frac{1-\theta}{p_0} + \frac{1}{p_1} = \frac{1}{2} \quad \textit{and} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{2}.$$

In particular, M_p *admits such a representation provided* $1 < p < 2$.

Proof. $Y_{p_1} = L_{p_1} \cap L_2$ for $p_1 \geq 2$, and so

$$\begin{aligned}[L_{p_0}, Y_{p_1}]_\theta &= [L_{p_0}, L_{p_1} \cap L_2]_\theta \\ &\subseteq [L_{p_0}, L_{p_1}]_\theta \cap [L_{p_0}, L_2]_\theta \\ &= L_2 \cap L_p \\ &= M_p.\end{aligned}$$

To complete the proof we must show that $M_p \subseteq [L_{p_0}, Y_{p_1}]_\theta$. Let

$$g = \sum_{j=1}^{n} a_j I(E_j)$$

be a simple integrable function (here $|E|$ denotes the Lebesgue measure of a set E) with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $|E_1| \geq 1$: call such a function g "flat". Plainly, $\|g\|_{M_p} \sim \|g\|_p$. Consider the vector-valued analytic function

$$g(z) = \sum_{j=1}^{n} a_j^{\alpha(z)} I(E_j) \quad \text{where } \alpha(z) = p\left(\frac{1-z}{p_0} + \frac{z}{2}\right).$$

Since $g(z)$ is always flat it follows that

$$\|g(1+it)\|_{Y_{p_1}} \sim \|g(1+it)\|_2 \quad \text{for } -\infty < t < \infty.$$

The usual proof of the Riesz convexity theorem now gives

$$\begin{aligned}&\max\left\{\|g(it)\|_{p_0}, \|g(1+it)\|_{Y_{p_1}}: -\infty < t < \infty\right\} \\ &\quad \sim \max\left\{\|g(it)\|_{p_0}, \|g(1+it)\|_2: -\infty < t < \infty\right\} \\ &\quad \leq C\|g\|_p \sim C\|g\|_{M_p}.\end{aligned}$$

Now suppose that

$$h = \sum_{j=1}^{n} b_j I(E_j)$$

is a non-negative simple function with $\sum_{j=1}^{n} |E_j| \le 1$, so that $\|h\|_{M_p} \sim \|h\|_2$. Consider the vector-valued analytic function

$$g(z) = \sum_{j=1}^{n} b_j^{\beta(z)} I(E_j) \quad \text{where } \beta(z) = 2\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right).$$

Using the fact that $\|h(1+it)\|_{Y_{p_1}} \sim \|h(1+it)\|_{p_1}$ we obtain

$$\max\left\{ \|h(it)\|_{p_0}, \|h(1+it)\|_{Y_{p_1}} : -\infty < t < \infty \right\} \le C\|h\|_{M_p}.$$

Finally, since any non-negative integrable simple function f can be expressed in the form $f = g + h$, with g and h as above, the inclusion $M_p \subset [L_{p_0}, Y_{p_1}]_\theta$ follows easily.

The following corollary is an immediate consequence of the duality theorem for complex interpolation [3, p. 98].

COROLLARY 2.2. *Suppose that* $1 < p_0 \le 2 \le p_1 < \infty$ *and that*

$$\frac{1}{2} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad (0 < \theta < 1).$$

Then

$$[Y_{p_0}, L_{p_1}]_\theta = M_p \quad \textit{with } \frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p_1}.$$

In particular, M_p *admits such a representation for* $2 < p < 4$.

The proofs of the following two interpolation theorems are similar to that of Theorem 2.1 and have been omitted. The duality statements have been left to the reader.

PROPOSITION 2.3. *Suppose that* $0 < p_0 \le 2 \le p_1 \le \infty$ *and that*

$$\frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{2} \quad (0 < \theta < 1).$$

Then

$$[M_{p_0}, L_{p_1}]_\theta = Y_p \quad \textit{where } \frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p_1}.$$

PROPOSITION 2.4. *Suppose that* $0 < p_0 \leq p_1 \leq 2$ *and that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad (0 < \theta < 1).$$

Then

$$[M_{p_0}, M_{p_1}]_\theta = M_p.$$

3. A probabilistic interpretation

This section is mainly devoted to a proof of the following theorem. Recall that if X is a Banach space, then $L_p(X)$ is the usual space of X-valued Bochner-measurable functions f on $[0,1]$ with the norm

$$\|f\|_{L_p(X)} = \left(\int_0^1 \|f(t)\|^p \, dt\right)^{1/p}.$$

THEOREM 3.1. *Let* $0 < p < \infty$. *Then* M_p *is isomorphic to a subspace of* $L_2(l_p)$. *If* $1 < p < \infty$ *then this subspace may be taken to be complemented in* $L_2(l_p)$.

The proof is very similar to the proof of the isomorphism between Y_p and L_p given in [5]. In particular, we require two preliminary results (cf. [5, Prop. 3.1]).

PROPOSITION 3.2. *Let* $1 < p < \infty$ *and let* $(\mathscr{F}_k)_{k=1}^n$ *be increasing sub-σ-fields. There exists* C_p *such that*

$$\left\|\left(E(f_k|\mathscr{F}_k)\right)_{k=1}^n\right\|_{L_2(l_p^n)} \leq C_p \left\|(f_k)_{k=1}^n\right\|_{L_2(l_p^n)}$$

for all $n \geq 1$ *and for all* $f_1, f_2, \ldots f_n$ *in* $L_2(0,1)$.

Proof. This is a minor variation of a theorem of Stein ([22] or [8, p. 243]), but for completeness we give the idea of the proof. The operator

$$(f_k)_{k=1}^n \to \left(E(f_k|\mathscr{F}_k)\right)_{k=1}^n$$

is plainly bounded from $L_p(l_p^n)$ to $L_p(l_p^n)$ for $1 \leq p < \infty$. An argument involving Doob's maximal inequality proves that the operator is bounded from $L_p(l_\infty^n)$ to $L_p(l_\infty^n)$ for $1 < p < \infty$. Now a standard interpolation theorem shows that the operator is bounded from $L_2(l_p^n)$ to $L_2(l_p^n)$ for $p > 2$. By duality it is also bounded for $1 < p < 2$.

COROLLARY 3.3. *Let $1 < p < \infty$ and let $(\mathscr{F}_k)_{k=1}^n$ be independent sub-σ-fields. Then there exists C_p such that*

$$\|(E(f_k|\mathscr{F}_k))_{k=1}^n\|_{L_2(l_p^n)} \le C_p\|(f_k)_{k=1}^n\|_{L_2(l_p^n)}$$

for all $n \ge 1$ and for all $f_1, f_2, \dots, f_n$ in $L_2(0,1)$.

Proof. Let G_k and $\tilde{G}_k$ be the sub-σ-fields generated by $\bigcup_{i=1}^k \mathscr{F}_i$ and $\bigcup_{i=n-k}^n \mathscr{F}_i$ respectively. Applying Proposition 3.2 first with respect to $(G_k)_{k=1}^n$ and then with respect to $(\tilde{G}_k)_{k=1}^n$ gives the result.

Let $(f_n)_{n=1}^\infty$ be a sequence of functions on $[0,1]$. In the following proof $\Sigma_{n=1}^\infty \oplus f_n$ denotes the function f on $(0,\infty)$ defined by $f(n-1+t) = f_n(t)$ for $n \ge 1$ and for $0 < t \le 1$.

Proof of Theorem 3.1. It is convenient to replace $[0,1]$ by the measure-equivalent space $\Omega = [0,1]^{\mathbf{N}}$ and to denote a typical element of Ω by $s = (s_1, s_2, \dots)$. Consider the linear mapping T: $M_p \to L_2(l_p)$ defined by

$$T\left(\sum_{n=1}^\infty \oplus f_n\right) = (f_n(s_n))_{n=1}^\infty.$$

We show that T is an isomorphic embedding for $0 < p < 2$. First observe that

$$\left\|\sum_{n=1}^\infty \oplus f_n\right\|_{M_p} \sim \left\|\sum_{n=1}^\infty \oplus\ |f_n|^{p/2}\right\|_{Y_{4/p}}^{2/p} \sim \left\|\left(\sum_{n=1}^\infty |f_n(s_n)|^p\right)^{1/2}\right\|_{L_{4/p}(0,1)}$$

by Rosenthal's moment inequality [19] for sums of independent random variables in $L_{4/p}$ (note that $4/p > 2$). (See [5] for more on this interpretation of Rosenthal's inequality.) Then

$$\left\|\left(\sum_{n=1}^\infty |f_n(s_n)|^p\right)^{1/2}\right\|_{4/p}^{p/2} = \left\|\left(\sum_{n=1}^\infty |f_n(s_n)|^p\right)^{1/p}\right\|_2$$
$$= \|(f_n(s_n))_{n=1}^\infty\|_{L_2(l_p)}.$$

Consider the mapping P: $L_2(l_p) \to L_2(l_p)$ defined by

$$P\left((f_n)_{n=1}^\infty\right) = (E(f_n|s_n))_{n=1}^\infty.$$

Then P is a self-adjoint projection onto the range of T, and by Corollary 3.3,

P is bounded for $1 < p < \infty$. It now follows from the fact that P is self-adjoint that T is an isomorphic embedding for the whole range $0 < p < \infty$ and that P is a bounded projection onto a subspace isomorphic to M_p for $1 < p < \infty$.

COROLLARY 3.4. *Let $0 < p < \infty$ and let $(X_n)_{n=1}^{\infty}$ be non-negative independent random variables in $L_2(\Omega)$. Then*

$$\left\| \left(\sum_{n=1}^{\infty} X_n^p \right)^{1/p} \right\|_2 \sim \left\| \sum_{n=1}^{\infty} \oplus X_n \right\|_{M_p}.$$

The latter result is actually equivalent to the following extension of Rosenthal's inequality to the range $0 < p < \infty$ discovered by Johnson and Schechtman.

COROLLARY 3.5 [9]. *Let $0 < p < \infty$ and let $(X_n)_{n=1}^{\infty}$ be independent symmetric random variables in $L_p(\Omega)$. Then*

$$\left\| \sum_{n=1}^{\infty} X_n \right\|_p \sim \left\| \sum_{n=1}^{\infty} \oplus X_n \right\|_{Y_p}.$$

Remark 3.6. In [1] Aldous proved that if $L_p(X)$ embeds into a Banach space with unconditional basis then X is a UMD space, thus disproving the conjecture (based on Paley's theorem on the unconditionality of the Haar system) that for $1 < p < \infty$, $L_p(X)$ has an unconditional basis if X has one. Since M_1 does not embed into a Banach space with unconditional basis [7, Theorem 6], it follows from Theorem 3.1 that $L_2(l_1)$ has the same property, thus verifying Aldous' theorem in the special case $X = l_1$.

We show next that there is no isomorphic embedding from M_p into $l_2(l_p)$.

THEOREM 3.7. *Let $0 < p < \infty$ $(p \neq 2)$. Then M_p is not isomorphic to a subspace of $(\Sigma \oplus l_p)_2$.*

The case $2 < p < \infty$ will follow easily from later results (see Corollary 5.6), and so we shall assume that $0 < p < 2$.

PRELIMINARY LEMMA. *Suppose that $(x_i)_{i=1}^{N}$ is a sequence in $(\Sigma \oplus l_p)_2$ such that*

$$C^{-1}\left(\sum_{i=1}^{N} a_i^2 \right)^{1/2} \le \left\| \sum_{i=1}^{N} a_i x_i \right\| \le C\left(\sum_{i=1}^{N} a_i^2 \right)^{1/2}.$$

Suppose further that $x_i = (x_{i,1}, x_{i,2}, \dots)$ *for* $1 \le i \le N$, *where* $x_{i_1, j}$ *and* $x_{i_2, j}$ *are disjointly supported vectors in* l_p *whenever* $i_1 \ne i_2$ *and* $j \ge 1$. *Then given* $\varepsilon > 0$ *and* $M \ge 1$ *there exists* N_1, *depending only on* ε *and* M, *and a subset I of* $\{1, 2, \dots, N\}$ *of cardinality* $N - N_1$ *such that* $\|x_{i,j}\| \le \varepsilon$ *for all* $i \in I$ *and for all* $i \le j \le M$.

Proof of Lemma. Fix j with $1 \le j \le M$ and suppose that $\|x_{i,j}\| > \varepsilon$ for all $i \in I(j)$. Let card(A) denote the cardinality of a finite set A. Then

$$C(\operatorname{card}(I(j)))^{1/2} \ge \Big\| \sum_{i \in I(j)} x_i \Big\| \ge \Big(\sum_{i \in I(j)} \|x_{i,j}\|^p \Big)^{1/p} \ge (\operatorname{card}(I(j)))^{1/p} \varepsilon.$$

Thus

$$\operatorname{card}(I(j)) \le \left(\frac{C}{\varepsilon} \right)^{2p/(2-p)}.$$

Take I to be $\{1, 2, \dots, N\} \setminus (\bigcup_{j=1}^{M} I(j))$. Thus

$$\operatorname{card}(I) \ge N - N_1 \quad \text{where } N_1 = M \left(\frac{C}{\varepsilon} \right)^{2p/(2-p)}.$$

Proof of Theorem 3.7. Suppose that M_p embeds isomorphically into $(\Sigma \oplus l_p)_2$. By Proposition 1.4, G is equivalent to a block basis with respect to the obvious diagonally ordered basis of $(\Sigma \oplus l_p)_2$; to avoid awkward notation identify G with this block basis. Suppose that $h_i^k = (x_{i,1}^k, x_{i,2}^k, \dots)$ and note that, for each $k \ge 1$, $(h_i^k)_{i=2^{m+1}+1}^{i=2^{m+2}}$ is isometric to the unit vector basis of $l_2^{2^{m+1}}$. Suppose that $k \ge 1$, $M \ge 1$, and $\varepsilon > 0$ are given. Then by the Preliminary Lemma we may choose m_k and

$$I(k) \subseteq \{2^{m_k+1} + 1, \dots, 2^{m_k+2}\}$$

such that card$(I(k)) = 2^{m_k}$ and $\|x_{i,j}^k\| < 2^{-m_k/2}\varepsilon$ for all $i \in I(k)$ and $1 \le j \le M$. A straightforward inductive argument now shows that m_k and $I(k)$ may be chosen so that the closed linear span of $\{h_i^k \colon 1 \le k < \infty,\ i \in I(k)\}$ is isomorphic to $(\Sigma \oplus l_2^{2^{m_k}})_2 = l_2$. But the sequence $(\Sigma_{i \in I(k)} h_i^k)_{k=1}^{\infty}$ is equivalent in M_p to the unit vector basis of l_p, which is a contradiction because l_p does not embed into l_2.

COROLLARY 3.8. *Let $0 < p < \infty$ ($p \neq 2$). Then $L_2(l_p)$ is not isomorphic to a subspace of $l_2(l_p)$.*

Proof. This is immediate from Theorems 3.1 and 3.7. For $0 < p < 1$ the result is a very special case of a theorem of Kalton [12, Theorem 4.2].

Remark 3.9. The proof of Theorem 3.7 was inspired by the proof of Theorem 6.1 in [15].

In the light of Theorem 3.1 one could be excused for conjecturing that M_p and $L_2(l_p)$ are isomorphic Banach spaces. This is not the case, however, as we shall now show. I am indebted to the referee for this observation.

PROPOSITION 3.10. *Let $0 < p < 2$. Then $(\Sigma \oplus l_p)_2$ is not isomorphic to a subspace of M_p.*

Proof. Suppose on the contrary that $(\Sigma \oplus l_p)_2$ may be identified with a subspace of M_p. Then the $L_2(0, \infty)$ and the M_p topologies do not coincide on $(\Sigma_{k=n}^{\infty} \oplus l_p)_2$ for any $n \geq 1$. Let $(\varepsilon_k)_{k=1}^{\infty}$ be a null sequence of positive numbers. By an obvious inductive argument there exist an increasing sequence of integers $(n_k)_{k=1}^{\infty}$ and vectors

$$b_k \in \left(\sum_{n_k+1}^{n_{k+1}} \oplus l_p \right)_2$$

such that $\|b_k\|_p = 1$ and $\|b_k\|_2 < \varepsilon_k$. Clearly, $(b_k)_{k=1}^{\infty}$ spans a subspace of $(\Sigma \oplus l_p)_2$ isomorphic to l_2. Yet by the argument of Theorem 4.1 below, $(b_k)_{k=1}^{\infty}$ spans a subspace isomorphic to l_p provided $(\varepsilon_k)_{k=1}^{\infty}$ decreases rapidly to zero. This contradiction completes the proof.

COROLLARY 3.11. *Let $0 < p < \infty$ ($p \neq 2$). Then M_p is not isomorphic to $L_2(l_p)$.*

4. The spaces M_p ($1 \leq p < 2$)

Many of the results in this section about the space M_p ($1 < p < 2$) correspond to well-known theorems about the space L_q ($2 < q < \infty$).

THEOREM 4.1. *Let $1 \leq p \leq 2$ and let X be a subspace of M_p. Then X is isomorphic to a Hilbert space and complemented in M_p or X contains a subspace isomorphic to l_p and complemented in M_p.*

Proof. First suppose that the M_p and the $L_2(0,\infty)$ topologies agree on X. Then X is isomorphic to a Hilbert space and the orthogonal projection from $L_2(0,\infty)$ onto X restricts to a bounded projection from M_p onto X. If the topologies do not agree, then given $\varepsilon > 0$ there exists $f \in X$ such that $\|f\|_p = 1$ and $\|f\|_2 < \varepsilon$. By Hölder's inequality

$$\| fI(0,M) \|_p^p \leq M^{(2-p)/2} \|f\|_2^p \leq M^{(2-p)/2} \varepsilon^p.$$

So by an inductive procedure one can construct functions $(f_n)_{n=1}^{\infty}$ in X and disjoint compactly supported functions $(g_n)_{n=1}^{\infty}$ in M_p such that

$$\|g_n\|_p = 1, \quad \|g_n\|_2 \leq \varepsilon_n \quad \text{and} \quad \|f_n - g_n\|_{M_p} \leq \varepsilon_n,$$

where $(\varepsilon_n)_{n=1}^{\infty}$ is any decreasing sequence of positive numbers. Then $(g_n)_{n=1}^{\infty}$ is equivalent in M_p to the unit vector basis of l_p, and its closed linear span is the range of a contractive projection on $L_p(0,\infty)$ whose restriction to M_p is a bounded projection. A standard perturbation argument now shows that $(f_n)_{n=1}^{\infty}$ spans a complemented subspace of M_p isomorphic to l_p provided ε_n decreases rapidly to zero.

THEOREM 4.2. *Let $1 \leq p \leq 2$. Then the only rearrangement invariant (r.i.) function space on $[0,1]$ to embed isomorphically into M_p is $L_2(0,1)$.*

Proof. Suppose that X is an r.i. space on $[0,1]$ that is isomorphic to a subspace of $M_p (1 \leq p \leq 2)$. Since M_p is a 2-concave Banach lattice it follows that X is also 2-concave, and so $\|f\|_X \leq C\|f\|_2$ for all $f \in L_2(0,1)$ and some constant C (e.g., [17, p. 133]). Now recall that $M_p = L_{\phi_p}(0,\infty)$ (Proposition 1.3), and so by [8, p. 169 and p. 198] either $X = L_2(0,1)$ or

$$\|f\|_X \geq \frac{1}{K}\|f\|_{M_p} = \frac{1}{K}\|f\|_2$$

for all $f \in L_2(0,1)$ and some constant K. In the latter case, we have

$$\frac{1}{K}\|f\|_2 \leq \|f\|_X \leq C\|f\|_2.$$

Thus $X = L_2(0,1)$ in any case.

COROLLARY 4.3. *Let $1 \leq p \leq \infty$. Then M_p is not isomorphic to an r.i. space on $[0,1]$.*

PROPOSITION 4.4. *Let $1 \leq p < \infty$. Then the only complemented subspaces of M_p with a symmetric Schauder basis are (up to isomorphism) l_2 and l_p.*

Proof. First suppose that $1 \le p < 2$. Let $(f_n)_{n=1}^\infty$ be disjoint functions in M_p and suppose that $\|f_n\|_p = 1$ for all n. If the sequence $(f_n)_{n=1}^\infty$ is bounded in $L_2(0, \infty)$, then $(f_n)_{n=1}^\infty$ is equivalent in M_p to the unit vector basis of l_p. If $(f_n)_{n=1}^\infty$ is unbounded then we may pass to a subsequence $(f_{n_k})_{k=1}^\infty$ such that $\|f_{n_k}\|_2 \ge k^{1/p}$. Then by Hölder's inequality,

$$\begin{aligned}\left\| \sum_{k=1}^{\infty} a_k f_{n_k} \right\|_p &= \left(\sum_{k=1}^{\infty} |a_k|^p \right)^{1/p} \\ &\le \left(\sum_{k=1}^{\infty} \left(\frac{1}{k} \right)^{2/(2-p)} \right)^{(2-p)/2p} \left(\sum_{k=1}^{\infty} k^{2/p} |a_k|^2 \right)^{1/2} \\ &\le C_p \left\| \sum_{k=1}^{\infty} a_k f_{n_k} \right\|_2 ,\end{aligned}$$

and so $(f_{n_k})_{k=1}^\infty$ is equivalent in M_p to the unit vector basis of l_2. It follows that the only disjoint symmetric sequences in M_p are l_p and l_2. The desired conclusion now follows in the case $1 \le p < 2$ from [8, Lemma 8.10]. The case $2 < p < \infty$ follows by duality.

COROLLARY 4.5. *Let $1 \le p \le \infty$. Then M_p and $L_2(0, \infty)$ are the only r.i. spaces on $(0, \infty)$ which are isomorphic to complemented subspaces of M_p.*

Proof. First suppose that $1 \le p < 2$. Let X be an r.i. space on $(0, \infty)$ which is isomorphic to a complemented subspace of M_p. By Theorem 4.2 the restriction of X to $[0, 1]$ is $L_2(0, 1)$. Thus

$$\|f\|_X \sim \| f^* I(0,1) \|_2 + \psi\left((f^*(n))_{n=1}^\infty \right),$$

where ψ is the symmetric sequence norm associated with the sequence $(I(n-1, n))_{n=1}^\infty$ in X. This sequence corresponds to a complemented subspace of M_p with a symmetric basis, and so by Proposition 4.4, ψ is equivalent to the l_p or l_2 norm. The former easily implies that $X = M_p$ and the latter that $X = L_2(0, \infty)$.

COROLLARY 4.6. *Let $1 \le p \le \infty$. Then the Banach space M_p has a unique representation as an r.i. function space on $(0, \infty)$.*

PROPOSITION 4.7. *Let X be an r.i. space on $(0, \infty)$, and let $1 \le p < 2$. If X is order isomorphic to a sublattice of M_p, then $X = M_p$ or $X = L_2(0, \infty)$.*

Proof. It was shown in the proof of Proposition 4.4 that l_2 and l_p are the only disjoint symmetric sequences in M_p for $p < 2$. If X is order isomorphic to a sublattice of M_p, then the sequence $(I(n-1, n))_{n=1}^{\infty}$ in X corresponds to a disjoint symmetric sequence in M_p. The proof is concluded as in Corollary 4.5.

5. The spaces M_p $(2 < p < \infty)$

As is the case for L_q with $0 < q < 2$ the subspace structure of M_p is quite varied in the range $2 < p < \infty$. We prove below, for example, a counterpart of the well-known embedding of L_r into L_s for $0 < s \le r \le 2$. But first we deduce the appropriate version of Theorem 4.1 for this range. The reader is referred to [14] for the theory of "stable" Banach spaces.

PROPOSITION 5.1. *Let $1 \le p < \infty$. Then M_p is isomorphic to a stable Banach space.*

Proof. By the results of [14] $L_2(l_p)$ is stable for $1 \le p < \infty$. The result now follows from Theorem 3.1 since stability is inherited by subspaces.

COROLLARY 5.2. *Let $2 < p < \infty$ and let X be an infinite-dimensional subspace of M_p. Then X contains a subspace isomorphic to l_r for some r in $[2, p]$.*

Proof. It is easy to verify that M_p has "type 2" and "cotype p" in this range (see [17] for the basic facts about type and cotype), whence it follows that if l_r embeds in M_p then $2 \le r \le p$. Finally, a stable Banach space necessarily contains a copy of l_r for some r.

PROPOSITION 5.3. *Let $2 < p < \infty$ and let X be a complemented subspace of M_p. Then X is isomorphic to l_2 or X contains a subspace isomorphic to l_p which is complemented in M_p.*

Proof. This follows by duality from Theorem 4.1.

PROPOSITION 5.4. *Let $2 < p < q \le \infty$. Let*

$$f_1(t) = \begin{cases} t^{-1/p} & (0 < t \le 1) \\ 0 & (t > 1) \end{cases}$$

and let $f_n(t)$ be the translation of f_1 to the interval $(n-1, n)$. Then $(f_n)_{n=1}^{\infty}$ is equivalent in M_q to the unit vector basis of l_p.

Proof. M_q contains the function $g(t) = t^{-1/p}$ $(0 < t < \infty)$, and any sequence $(g_n)_{n=1}^{\infty}$ of disjoint functions having the same distribution as g is obviously equivalent to the l_p basis. Now

$$\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{1/p} \sim \left\|\sum_{n=1}^{\infty}a_n g_n\right\|_{M_p} \geq \left\|\sum_{n=1}^{\infty}a_n f_n\right\|_{M_p},$$

and so to conclude the proof it is enough to show that

$$\left\|\sum_{n=1}^{\infty}a_n f_n\right\|_{M_p} \geq \frac{1}{C}\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{1/p}.$$

Select $a_1, a_2, \ldots, a_n$ such that $\sum_{k=1}^{n}|a_k|^p = 1$ and let $f = \sum_{k=1}^{n}a_k f_k$. Then

$$\begin{aligned}\|f^*I(0,1)\|_2^2 &\geq \sum_{k=1}^{n}|a_k|^2\int_0^{|a_k|^p}t^{-2/p}\,dt\\ &= \frac{p}{p-2}\sum_{k=1}^{n}|a_k|^p\\ &= \frac{p}{p-2},\end{aligned}$$

and so

$$\|f\|_{M_q} \geq \sqrt{\frac{p}{p-2}}\,.$$

COROLLARY 5.5. (a) *Let $2 < p < \infty$. Then M_p is not isomorphic to a subspace of L_p.*

(b) *Let $1 \leq p < 2$ or $p = \infty$. Then M_p is not isomorphic to a complemented subspace of L_p.*

Proof. (a) By [10, p. 169], l_2 and l_p are the only symmetric basic sequences in L_p for $p > 2$. (Also by [8] the only r.i. spaces on $(0, \infty)$ to embed into L_p $(p > 2)$ are L_p and L_2.)

(b) The case $1 < p < 2$ follows by duality from (a), and the case $p = 1$ or $p = \infty$ simply form the fact that l_2 is not isomorphic to a complemented subspace of L_1 or L_∞.

We are now able to conclude the proof of Theorem 3.7.

COROLLARY 5.6. *Let $2 < p < \infty$. Then M_p is not isomorphic to a subspace of $(\Sigma \oplus l_p)_2$.*

Proof. By Proposition 5.4 it is enough to show that l_r is not isomorphic to a subspace of $(\Sigma \oplus l_p)_2$ for $2 < r < p$. This is a fairly routine "gliding hump" argument and will therefore be omitted.

We now use Proposition 5.4 to prove an analogue of the "stable embedding" of L_r into L_p for $1 \le p \le r \le 2$.

THEOREM 5.7. *Let $2 < p < q \le \infty$. Then M_p is order isomorphic to a sublattice of M_q.*

Proof. Let $f_0^{(1)}$ be a countably valued function on $[0,1]$ such that

$$t^{-1/p} \le f_0^{(1)} \le 2t^{-1/p}.$$

We define a mapping T from the simple dyadic functions on $[0,\infty]$ into M_q. Put $T(I(0,1)) = f_0^{(1)}$. Now choose disjoint identically distributed functions $f_{1,1}^{(1)}$ and $f_{1,2}^{(1)}$ such that $f_0^{(1)} = f_{1,1}^{(1)} + f_{1,2}^{(1)}$. Let

$$T\left(I\left(0, \frac{1}{2}\right)\right) = f_{1,1}^{(1)} \quad \text{and} \quad T\left(I\left(\frac{1}{2}, 1\right)\right) = f_{1,2}^{(1)}.$$

Now select identically distributed functions $f_{2,k}^{(1)}$ $(1 \le k \le 4)$ such that

$$f_{1,1}^{(1)} = f_{2,1}^{(1)} + f_{2,2}^{(1)} \quad \text{and} \quad f_{1,2}^{(1)} = f_{2,3}^{(1)} + f_{2,4}^{(1)}$$

and define

$$T\left(I\left(\frac{k-1}{4}, \frac{k}{4}\right)\right) = f_{2,k}^{(1)} \quad (1 \le k \le 4).$$

Continue in this manner to define T on all simple dyadic functions on $[0,1]$. Finally, extend T by translation to all simple dyadic functions on $[0,\infty)$. Observe that T has the following properties: (i) if f and g have the same distribution, then Tf and Tg have the same distribution; (ii) if $|f| \le |g|$ then $|Tf| \le |Tg|$. Now let f be any simple dyadic function on $(0,\infty)$. Write $f = g + h$, where $g^*(t) = f^*(1)$ for $0 \le t \le 1$, $g^*(t) = f^*(t)$ for $t \ge 1$, and $h = f - g$. Then

$$\|Tf\|_{M_q} \sim \max\left(\|Tg\|_{M_q}, \|Th\|_{M_q}\right).$$

Plainly, $\|Th\|_{M_q} = \|Th\|_2 = C\|h\|_2$, where C is an absolute constant. To compute $\|Tg\|$, observe that

$$\sum_{n=1}^{\infty} f^*(n) I(n-1, n) \le g^* \le f^*(1) I(0,1) + \sum_{n=1}^{\infty} f^*(n) I(n, n+1).$$

So

$$\sum_{n=1}^{\infty} f^*(n)T(I(n-1,n)) \le T(g^*)$$

$$\le f^*(1)T(I(0,1)) + \sum_{n=1}^{\infty} f^*(n)T(I(n-1,n)),$$

whence by Proposition 5.4,

$$\|T(g)\|_{M_q} = \|T(g^*)\|_{M_q} \sim \left(\sum_{n=1}^{\infty} |f^*(n)|^p\right)^{1/p}.$$

It follows that

$$\|Tf\|_{M_q} \sim \max\left(\|f^*I(0,1)\|_2, \|f^*I(1,\infty)\|_p\right),$$

and so T extends to a lattice isomorphism from M_p onto a sublattice of M_q.

PROPOSITION 5.8. *Let $2 < r < p \le \infty$. Then $L_r(0,1)$ is isometric to a sublattice of M_p.*

Proof. The function $t^{-1/r}$ belongs to M_p, and so the result follows at once from [8, p. 222].

6. The space M_∞

In this short section we initiate an examination of the Banach space M_∞. These results are analogous to some well-known facts about L_1 (cf. [10, 18]).

THEOREM 6.1. *M_∞ does not embed into any Banach space with an unconditional basis.*

Proof. By Proposition 5.8, $L_r(0,1)$ is isometric to a subspace of M_∞ for all $2 < r < \infty$. The result now follows from the "reproducibility" of the Haar system and from the fact that the constant of unconditionality of the Haar basis in $L_r(0,1)$ increases without limit as r increases [15].

THEOREM 6.2. *Let X be a subspace of M_∞. Then either X is reflexive or X contains a subspace isomorphic to c_0 and complemented in M_∞.*

Proof. Since M_∞ is an order continuous Banach lattice it follows from [16, p. 35] that every non-reflexive subspace contains c_0 or l_1. But l_1 is not contained in M_∞ since $M_\infty^* = M_1$ is separable. Thus every non-reflexive subspace contains a subspace isomorphic to c_0, and by Sobczyk's theorem [21] such a subspace must be complemented in M_∞.

COROLLARY 6.3. *Let X be a complemented subspace of M_∞. Then either X is isomorphic to l_2 or X contains a subspace isomorphic to c_0 and complemented in M_∞.*

Proof. Since X^* is isomorphic to a subspace of M_1, it follows from Theorem 4.1 that every reflexive complemented subspace of M_∞ is isomorphic to l_2. Theorem 6.2 now concludes the proof.

COROLLARY 6.4. *Let X be an infinite-dimensional complemented subspace ofM_∞ with the Radon-Nikodym property. Then X is isomorphic to l_2.*

7. The spaces M_p $(0 < p < 1)$

The proof of the following result is standard and has been omitted. (See [13] for the definition of Banach envelope.)

PROPOSITION 7.1. *Let $0 < p < 1$. Then the Banach envelopes of M_p is isomorphic to M_1.*

COROLLARY 7.2. *Let $0 < p < 1$ and let T: $M_p \to X$ be a bounded linear operator from M_p into a quasi-Banach space X. Then either T factorizes through M_1 or T fixes a copy of l_2.*

Proof. We use the fact that $M_p = L_{\Phi_p}(0, \infty)$. The result is now an immediate consequence of a theorem of Kalton [11, Theorem 3.4]. (In fact, Kalton's theorem is stated for operators on the Orlicz space $L_\Phi(0,1)$, but the proof readily extends to $L_\Phi(0,\infty)$.)

The next result is proved by checking the above dichotomy for an operator from M_p into l_p and for a projection on M_p.

COROLLARY 7.3. *Let $0 < p < 1$.*
(a) *Every operator from M_p into l_p is compact.*
(b) *Every complemented subspace of M_p contains l_2.*

PROPOSITION 7.4. *Let $0 < p < 1$ and let X be a separable Banach space. Then X is isomorphic to a quotient space of M_p.*

Proof. It is easily verified that the operator $f \to (\int_{n-1}^n f(t)\,dt)_{n=1}^\infty$ defines a quotient mapping from M_p onto l_1. To conclude, recall that every separable Banach space is a quotient of l_1.

A weaker version of Theorem 4.1 is also valid.

PROPOSITION 7.5. *Let $0 < p < 1$ and let X be a subspace of M_p. Then either X is isomorphic to a Hilbert space and complemented in M_p or X contains a subspace isomorphic to l_p.*

Finally, we state a consequence of a further theorem of Kalton [11, Corollary 2.4].

PROPOSITION 7.6. *Let $0 < p < 1$. Then M_p is not isomorphic to a subspace of a quasi-Banach space with a Schauder basis.*

Appendix

Recall that a Banach space X is said to be primary if whenever X is isomorphic to $Y \oplus Z$, then either Y or Z is isomorphic to X. The fact that the M_p spaces are primary for $1 < p < \infty$ is just a special case of the following theorem. The same result for r.i. spaces on $[0, 1]$ was proved in [2].

THEOREM A.1. *Let X be a separable r.i. function space on $(0, \infty)$ whose Boyd indices p_X and q_X satisfy $1 < p_X, q_X < \infty$. Then X is a primary Banach space.*

The proof of Theorem A.1 closely follows the argument of [2] as it is expounded in [17]. Because the necessary changes are fairly routine it will suffice to state without proof some of the preliminary results that are needed, and to leave the verification of Theorem A.1 to the interested reader. The main idea is to use the system H (see Proposition 1.4 above) in place of the Haar system. Throughout X denotes a separable r.i. space on $(0, \infty)$ whose Boyd indices satisfy $1 < p_X, q_X < \infty$.

PROPOSITION A.2 (cf. [17, p. 172]). *X is isomorphic to the Banach space $X(l_2)$.*

PROPOSITION A.3 (cf. [17, p. 172]). *Let Y be a complemented subspace of X which itself contains a complemented subspace isomorphic to X. Then Y is isomorphic to X.*

Let $(\phi_n)_{n=1}^{\infty}$ be a subsequence of the system H and let

$$\sigma_n = \{t \in [n-1, n]: \phi_k(t) \neq 0 \text{ for infinitely many } k\}.$$

The following result is the key ingredient in the proof of Theorem A.1.

PROPOSITION A.4 (cf. [17], p. 178]). *Suppose that there exists $\delta > 0$ such that $|\sigma_n| > \delta$ for infinitely many n. Then $[\phi_n]_{n=1}^{\infty}$ is isomorphic to X.*

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