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WEIL'S GROUP CHUNK THEOREM: A TOPOLOGICAL SETTING

BY

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Introduction

A. Weil showed that a birational group law that is only partially defined can be extended to an algebraic group [15], [16]. We show below (§1) that a similar construction can be carried out in a topological setting, where the topology is not necessarily the Zariski topology. In §2 we enrich our topological spaces with sheaves and prove a version of Weil's theorem for this "structured" setting. We then derive Weil's original theorem as well as two variations, covering the cases of "quasi-algebraic" group chunks and "differentially algebraic" group chunks (§3). We note that our theorem, though quite general, does not include the scheme theoretic version of Weil's theorem given in [1].

Our version of Weil's theorem can be applied to resolve a problem arising in model theory. The question whether all groups that are first order definable in algebraically closed fields are isomorphic to algebraic groups is connected with work of Cherlin, Poizat, and Zil'ber (cf. [10, [11]). As we show here (§4) a positive solution follows from the group chunk theorem. In characteristic 0, Weil's theorem suffices, and in characteristic $p > 0$ the quasi-algebraic version is needed, together with a theorem of Serre [13] characterizing quasi-algebraic groups.

Another approach to the latter problem was given by Hrushovski (unpublished); there are expositions of his treatment in [2], [12]. It is similar in spirit to our approach, with two variations: since in this application the abstract group is already given to us as a definable group, he omits the first step in the proof of Weil's theorem, and passes directly to the introduction of a topology and structure sheaf; secondly where we combine our generalized Weil theorem with a result of Serre, Hrushovski uses the idea of Serre's proof to reduce to an algebraic group chunk. Hrushovski also gave a generalization of Weil's group chunk theorem in quite a different direction in his thesis [4].

Our group chunk theorem is inspired by unpublished notes of W. van der Kallen [5], and in particular we follow his approach to the construction of the enveloping group as an abstract group in §1. In topologizing the group and equipping it with a sheaf we adopt a different approach which works in greater generality.

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1. Weil's theorem in a topological setting

(1.1) Homogeneous groups.

A homogeneous group is a group G carrying a topology for which inversion and all left multiplication maps are continuous; then right multiplication maps are also continuous.

This notion covers both topological groups and algebraic groups equipped with the Zariski topology.

(1.2) Group chunks.

Let X be a topological space. By a *dense X-homeomorphism* we mean a homeomorphism between two dense open subsets of X.

A *group chunk* is a nonempty topological space X equipped with multiplication and inversion maps p, i defined on sets $U \subseteq X \times X$, $V \subseteq X$ respectively, satisfying the following conditions, in which we write xy and x^{-1} in place of $p(x, y)$ and $i(x)$:

Ia For $x \in X$ the left multiplication map $\lambda_x = p(x, \cdot)$ is a dense X-homeomorphism; in particular its domain is dense open in X.

Ib For $x \in V$ the right multiplication map $\rho_x = p(\cdot, x)$ is a dense X-homeomorphism; in particular its domain is dense open in X.

II The inversion map i is a dense X-homeomorphism; in particular V is dense open in X.

III For $x \in X$ the set $\{z \in X : (xz)z^{-1} \text{ is defined}\}$ is a nonempty open subset of X.

IV For $x, y, z \in X$ the identities $(xy)z = x(yz)$, $(xz)z^{-1} = x$, $z^{-1}(zx) = x$ hold whenever both sides are defined.

The canonical example is as follows. Let G be a homogeneous group, X a dense open subset of G, $U = \{(x, y) \in X \times X : xy \in X\}$, $V = X \cap X^{-1}$, p and i the restrictions of multiplication and inversion respectively.

(1.3) The main theorem.

Let $(X; p, i)$ be a group chunk. Then:

(1) *There is a homogeneous group G and a homeomorphism $h : X \to h[X] \subseteq G$ such that $h[X]$ is dense open in G and $h(xy) = h(x)h(y)$ for $(x, y) \in$* dom *p. Call the pair (G, h) a realization of $(X; p, i)$.*

(2) *If (G^*, h^*) is a second realization of (X, p, i) then there is a unique homomorphism $\alpha : G \to G^*$ of abstract groups such that $ah = h^*$, and this map is then a homeomorphism of G with G^*.*

We remark that in this theorem, if $U \subseteq X \times X$ is open with the product topology on $X \times X$, and $p : U \to X$ is continuous then the resulting group G

is a topological group//; this is because any homogeneous group whose multiplication has at least one point of continuity on $G \times G$ is a topological group.

For the remainder of this section a group chunk $(X; p, i)$ is given, and we prove the theorem in several stages. Let $U = \operatorname{dom} p$, $V = \operatorname{dom} i$.

(1.4) The group G.

Call two dense X-homeomorphisms *equivalent* if they agree on an open dense subset of X. Let $\mathcal{G}$ be the set of equivalence classes of dense X-homeomorphisms. Then $\mathcal{G}$ carries a natural group structure induced by composition. For $x \in X$ let $h(x)$ be the equivalence class of the left multiplication λ_x, and let G be the subgroup of $\mathcal{G}$ generated by $h[X]$. G will be equipped with a topology below (§1.7). Clearly, the axioms imply that $h(xy) = h(x)h(y)$ for $(x, y) \in U$, and $h(x^{-1}) = h(x)^{-1}$ for $x \in V$.

We remark that if i' is the restriction of i to a dense open subset of V, then $(X; p, i')$ is a group chunk and our construction of (G, h) yields the same result when applied to this group chunk. Thus whenever it is convenient we may shrink V in this fashion.

(1.5) Generic identities in X.

We will say that a property $P(x)$ holds for generic $x \in X$ if the set of x for which P holds contains an open dense subset of X. Note the following left invariance of genericity: if $P(x)$ holds for generic x, and $y \in X$, then $P(yx)$ also holds for generic x, by Axiom Ia.

We now define the notion "*$\bar{x}$ represents y*" for $\bar{x} = (x_1, \ldots, x_n)$ with $x_1, \ldots, x_n$, $y \in X$. We proceed by induction on n. For $n = 1$ this means $x_1 = y$. For $n > 1$ it means that there is m with $1 \leq m < n$ and there are elements y_1, y_2 represented by $(x_1, \ldots, x_m)$ and $(x_{m+1}, \ldots, x_n)$ respectively, so that $y_1 y_2 = y$.

LEMMA 1. *Let $x_1, \ldots, x_n \in X$.*

(1) *If $\bar{x}$ represents y then for generic z,*

$$yz = x_1(x_2(\ldots(x_n z)\ldots))$$

(*and in particular both sides are defined*).

(2) *$\bar{x}$ represents at most one element of X.*

(3) *If x, x', $y \in X$ with $xy = x'y$ then $x = x'$. In particular the map $h : X \to G$ is injective.*

Proof. (1) As the maps λ_x are dense X-homeomorphisms, for fixed $(x_1, x_2) \in \operatorname{dom} p$ the law $x_1(x_2 z) = (x_1 x_2)z$ holds generically on the basis of Axiom IV and the left invariance of genericity. The general case follows similarly by induction on n.

(2) Suppose $\bar{x}$ represents both y_1 and y_2. By part (1), $y_1 z = y_2 z$ for generic z. Hence by Axiom III there is $z \in X$ for which $(y_1 z)z^{-1}$ is defined and $y_1 z = y_2 z$. By Axiom IV, $y_1 = (y_1 z)z^{-1} = (y_2 z)z^{-1} = y_2$ for such a z.

(3) By part (1), $x(yz) = x'(yz)$ for generic z, so $xz = x'z$ for generic z, and as in the proof of part (2) we obtain $x = x'$. ∎

LEMMA 2. *Let $V' = i^{-1}[V]$, and let $i' = i \upharpoonright V'$. Then V' is dense open in V, $i' : V' \to V'$, and $i'^2 = 1$.*

Proof. If $x \in V'$, then $i(i(x))$ is defined and it suffices to check that $i(i(x)) = x$. For generic v we have $i(i(x))v = i(i(x))(x^{-1}(xv)) = xv$, so by Lemma 1, part 3: $i(i(x)) = x$.

As noted earlier, we may replace V by the dense open subset V' defined in the preceding lemma. In other words after a change of notation we assume V is i-invariant and $i^2 = 1$.

(1.6) Generation.

LEMMA 3. *Let A be a nonempty open subset of V. Then $h[A] \cdot h[V] = G$.*

Proof. It suffices to show that $h[X]h[V]^2 \subseteq h[A]h[V]$. Let $x \in X$, $y, z \in V$. For generic v, xv and $(v^{-1}y)z$ are defined and in V. For v in a nonempty open set, $xv \in A$. Hence for v in some nonempty open set,

$$h(x)h(y)h(z) = h(xv)h\big((v^{-1}y)z\big) \in h[A]h[V].$$ ∎

(1.7) The topology.

For $x \in V$ define $\phi_x : V \to G$ by $\phi_x(y) = h(x)h(y)$. A set $\mathcal{O} \subseteq G$ will be taken to be open if and only if its preimages under all the ϕ_x are.

LEMMA 4. *G is a homogeneous group.*

Proof. We first check the continuity of inversion. Let $\mathcal{O}$ be open in G, $x \in V$, $y \in \phi_x^{-1}[\mathcal{O}^{-1}]$, that is $h(y)^{-1}h(x)^{-1} \in \mathcal{O}$. Take $v \in V$ such that $(v^{-1}y^{-1})x^{-1}$ is defined and in V. Then $h(y^{-1})h(x^{-1}) = \phi_v((v^{-1}y^{-1})x^{-1})$, so on a neighborhood of y a similar formula holds, and as ϕ_v is continuous by definition of the topology, we have $h(y_1^{-1})h(x^{-1}) \in \mathcal{O}$ for y_1 in a neighborhood of y, as required.

Similarly we check that $g^{-1}\mathcal{O}$ is open for $g \in G$, $\mathcal{O}$ open. As $h[X]$ generates G we may suppose $g = h(a)$, $a \in X$. Then for $x \in V$ we have

$$y \in \phi_x^{-1}\big[g^{-1}\mathcal{O}\big] \quad \text{iff} \quad h(a)h(x)h(y) \in \mathcal{O}.$$

If a, x, y are fixed with $y \in \phi_x^{-1}[g^{-1}\mathcal{O}]$ and $v \in V$ is taken so that av^{-1}, $(vx)y$ are defined and in V, then we have

$$h(a)h(x)h(y) = h(av^{-1})h((vx)y) \in \mathcal{O}$$

and the same holds for y' in a neighborhood of y. Also

$$(vx)y' \in \phi_{av^{-1}}[\mathcal{O}]$$

for y' in a neighborhood of y, so we have a neighborhood of y in $\phi_x^{-1}[g^{-1}\mathcal{O}]$. ■

LEMMA 5. *For $x \in V$ the map $\phi_x : V \to \phi_x[V]$ is a homeomorphism onto an open set.*

Proof. As h is injective, ϕ_x is injective. Continuity is immediate. Thus we need only show that for $A \subseteq V$ open, $\phi_x[A]$ is open in G; that is $\phi_y^{-1}[\phi_x[A]]$ is open in V for $y \in V$. Suppose therefore that $z \in \phi_y^{-1}[\phi_x[A]]$, say $\phi_y(z) = \phi_x(a)$, with $a \in A$. With $v \in V$ chosen so that $x^{-1}(y(zv))$ and av are defined, they are then equal. For z' in a neighborhood of z,

$$x^{-1}(y(z'v)) = a'v \quad \text{for some} \quad a' \in A.$$

Then all such z' are in $\phi_y^{-1}[\phi_x[A]]$, as required. ■

LEMMA 6. *$h : X \to G$ is a homeomorphism with dense open image.*

Proof. Let $x \in X$. By Axioms III, IV we have $x = uv$ for some $u, v \in V$. Then uV is a neighborhood of x on which $h = \phi_u \circ \lambda_u^{-1}$ (in the notation of Axiom Ia). Thus h is an injective map which is locally a homeomorphism onto an open set. Thus h is a homeomorphism with open image.

Now we check that $h[X]$ is dense in G. Let $\mathcal{O}$ be nonempty and open in G, and by Lemma 3 take $u, v \in V$ with $h(u)h(v) \in \mathcal{O}$. Thus $\phi_u^{-1}[\mathcal{O}]$ is nonempty. For generic z, uz exists, hence also for some $z \in \phi_u^{-1}[\mathcal{O}]$, uz exists, and $h(uz) = h(u)h(z) \in \mathcal{O}$. ■

(1.8) Uniqueness.

LEMMA 7. *Let $h_1 : X \to G_1$ where G_1 is a group and $h_1(xy) = h_1(x)h_1(y)$ for $(x, y) \in$ dom p. Then there is a unique group homomorphism $a : G \to G_1$ so that $ah = h_1$.*

Proof. It suffices to show that if

$$h(x_1)\dots h(x_n) = h(y_1)\dots h(y_m)$$

with the x_i, $y_j \in X$, then h_1 satisfies the corresponding relation. For generic $z \in X$, $x_1(\ldots(x_n z)\ldots)$ and $y_1(\ldots(y_m z)\ldots)$ are defined, and in view of the injectivity of h they are equal. Hence by applying the homomorphism law for h_1 and canceling, our claim follows. ■

Now to complete the proof of part 2 of the theorem, it is clear that if (G, h) and (G^*, h^*) are two realizations of $(X; p, i)$ then they are isomorphic as abstract groups by a unique isomorphism

$$a : G \xrightarrow{\cong} G^*$$

compatible with h, h^*. It will suffice now to check the continuity of a. Let $\mathcal{O} \subseteq G^*$ be open. For $x \in V$ we must check that $\phi_x^{-1}a^{-1}[\mathcal{O}]$ is open in V. We have

$$\phi_x^{-1}a^{-1}[\mathcal{O}] = \{v \in V : h_1(x)h_1(v) \in \mathcal{O}\} = V \cap h_1^{-1}[h_1(x^{-1})[\mathcal{O}]],$$

which is open. ■

2. Weil's theorem for locally affine group chunks

We now develop a form of Weil's theorem for group chunks with a structure sheaf. To include cases like differential algebraic groups it is useful to formulate this quite generally.

(2.1) k-spaces.
Let k be an abelian group. A *k-space* will be a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a sheaf of abelian groups on X [3], where for open $U \supseteq V$ in X we require that $\mathcal{F}(U)$ be a subgroup of k^U, and that the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ be induced by the restriction from k^U to k^V.

A *morphism* of k-spaces $(X, \mathcal{F})$, $(Y, \mathcal{G})$ will be a continuous map $a : X \to Y$ such that for V open in Y, $\mathcal{G}(V) \circ a \subseteq \mathcal{F}(a^{-1}V)$. Typically k will be a field and each group $\mathcal{F}(U)$ will be a subalgebra of the k-algebra k^U.

(2.2) k-groups and k-group chunks.
A *k-group* is a group G equipped with a topology and a sheaf $\mathcal{F}$ such that $(G, \mathcal{F})$ is a k-space and the inverse and left multiplication maps are morphisms from $(\mathcal{G}, \mathcal{F})$ to $(\mathcal{G}, \mathcal{F})$. In particular G is a homogeneous group.

We call $(X, \mathcal{F}; p, i)$ a k-group chunk if $(X; p, i)$ is a group chunk and the inversion, left multiplication, and right multiplication maps are all morphisms, where their domains are equipped with the restriction of $\mathcal{F}$ as structure sheaf.

LEMMA 8. *Let $(X, \mathcal{F}; p, i)$ be a k-group chunk. Let G be the homogeneous group associated with the group chunk according to Theorem* 1, *and identify X*

with $h[X]$. Let $g \in G$, $W \subseteq X$ open with $gW \subseteq X$. Then the map $x \to gx$ induces a morphism from $(W, \mathscr{F} \restriction W)$ to $(X, \mathscr{F})$.

Proof. Let $g = ab$ with $a, b \in X$. If $gx = y$ with $x, y \in X$ then for generic v we have $a(b(xv)) = yv$. There is such a v so that, in addition, $(yv)v^{-1} = y$; hence

$$[a(b(xv))]v^{-1} = y.$$

For x' in a neighborhood of x we then have

$$[a(b(x'v))]v^{-1} = gx'$$

defined and (by hypothesis) a morphism as a function of x', near x. ■

THEOREM 2. *Let $(X, \mathscr{F}; p, i)$ be a k-group chunk. Let G be the homogeneous group associated with the group chunk according to Theorem 1. Then there is a unique sheaf $\mathscr{G}$ on G making G a k-group and h an isomorphism onto $(h[X], \mathscr{G} \restriction h[X])$.*

Proof. For $g \in G$ let $g\mathscr{F}$ be the direct image of $\mathscr{F}$ under left multiplication by g, a sheaf on gX. We claim there is a unique sheaf on G which restricts to $g\mathscr{F}$ on gX for each g. It suffices to check that for $g, h \in G$, $g\mathscr{F}$ and $h\mathscr{F}$ restrict to the same sheaf on $gX \cap hX$.

Let $W = gX \cap hX$. Multiplication by $(g^{-1}h)$ gives a morphism from

$$\left(h^{-1}W, \mathscr{F} \restriction h^{-1}W\right) \quad \text{to} \quad \left(g^{-1}W, \mathscr{F} \restriction g^{-1}W\right),$$

by the previous lemma. Hence the identity map, factored as $x \to g(g^{-1}h)h^{-1}x$, gives a morphism from $(W, h\mathscr{F} \restriction W)$ to $(W, g\mathscr{F} \restriction W)$, and conversely. ■

(2.3) Affine models and products.

It will be useful to mimic the algebraic case more closely. We assume that certain subsets of k^n have been singled out, for all $n \geq 0$, which will be called *affine sets*, and that to each affine set V a topology t^V and a sheaf $\mathscr{F}^V$ are associated making V (that is, $(V, t^V, \mathscr{F}^V)$) a k-space. These distinguished k-spaces will be referred to as *affine models*. A k-space will be called *affine* if it is isomorphic to an affine model. A k-space is called *locally affine* if each point has an affine open neighborhood.

Our assumptions on the class of affine models are as follows.

- Aff I For every affine model A, all constant functions $A \to k$ lie in $\mathscr{F}^A(A)$.
- Aff II The restriction of an affine model to an open subset is locally affine.
- Aff III If $A \subseteq k^m$, $B \subseteq k^n$ are affine sets then $A \times B \subseteq k^{m+n}$ is an affine set, and using the same notation for the corresponding affine

models, we require that $A \times B$ (with natural projections to A, B) be the product in the category of affine models and morphisms of k-spaces.

In connection with the third property, we will say that a pair of locally affine spaces X, Y have a *set-like product* if the product $X \times Y$ of the underlying sets can be given a (necessarily unique) locally affine structure which makes it their product (with respect to the natural projections) in the category of locally affine spaces with k-morphisms. This forces the topology on $X \times Y$ to contain as open sets the products $U \times V$ for U, V open in X, Y respectively (since $U \times V = \pi_1^{-1}U \cap \pi_2^{-1}V$). It follows from the axioms Aff II, III that any two locally affine spaces have a set-like product, by equipping each subset $U \times V$, where U, V are open affine in X, Y, with the k-structure making it the product of U and V in the category of affine k-spaces, and checking coherence on intersections.

(2.4) Locally affine groups and group chunks.
We define a *locally affine group* to be a group G which is also a locally affine space such that the multiplication and inversion maps are morphisms $G \times G \to G$, $G \to G$ respectively. By Aff I, constant maps $G \to G$ are morphisms and hence G is a k-group.

A *locally affine group chunk* is a locally affine space X together with morphisms p, i from open subspaces U, V of $X \times X$, X (respectively) to X, such that $(X; p, i)$ is a group chunk.

Then the corresponding k-group G will be a locally affine group, because U is open in $G \times G$ and the multiplication map is a morphism when restricted to U. (This is analogous to the remark concerning topological groups following Theorem 1.)

3. Examples

(3.1) The algebraic case.
Let k be an algebraically closed field. The affine models are taken to be the Zariski closed sets $A \subseteq k^n$ with the Zariski topology on A and the structure sheaf $\mathcal{O}_A$ of *regular* functions (locally, given by rational functions with nonvanishing denominator). A regular function on A itself is just the restriction to A of a polynomial function (Hilbert's Nullstellensatz; see [9, p. 6] for example). Thus the morphisms between affine models are also given by polynomial maps. Our axioms Aff I–III hold in this case.

A *prevariety* is a locally affine space with a *finite* covering by affine open pieces. An algebraic group over k is a locally affine k-group G which is also a prevariety. (It is then a variety, that is, the diagonal of $G \times G$ is closed, as it is the inverse image of $\{1\}$ under $(x, y) \to x^{-1}y$.) We define an algebraic group chunk as a prevariety X with morphisms p, i from open subspaces U, V of $X \times X$, X (respectively) to X such that $(X; p, i)$ is a group chunk. Weil's theorem is then the existence and uniqueness of the corresponding algebraic group G. In (2.4) we constructed G as a locally affine k-group. It is also the image of $X \times X$ under a continuous map, and $X \times X$ is noetherian (satisfies the d.c.c. on closed subspaces), so G is noetherian and hence a prevariety.

(3.2) The quasi-algebraic case.

Here k is algebraically closed of positive characteristic p. The affine models in the quasi-algebraic context are the Zariski closed affine sets A with their Zariski topology and the structure sheaf $\mathcal{O}_A^{p^{-\infty}}$, the closure of the usual structure sheaf under p-th roots. Hence morphisms are given by p^n-th roots of polynomial maps. The axioms Aff I–III hold, the affine models are still noetherian, and the analog of a prevariety or an algebraic group in this context is called a *perfect prevariety* and a *perfect group* [13], which is then a perfect variety. From any prevariety one obtains a perfect prevariety by closing the structure sheaf under p-th roots. Conversely Serre shows in [5] that any perfect group is obtained in this fashion from an algebraic group. Weil's theorem holds in this setting, and shows that a perfect group chunk (defined in the usual fashion) gives rise to a perfect group, which by Serre's theorem comes from an algebraic group by enlarging the structure sheaf. Because of Serre's theorem perfect groups are also called quasi-algebraic groups.

(3.3) The differentially algebraic case.

Let k be a differentially closed field of characteristic 0, possibly with several commuting derivations. (This is called a constrainedly closed differential field by Kolchin [7].) A closed set for the differential Zariski topology on k^m is the set of common zeros of a collection of m-variable differential polynomials over k. The affine models are the closed sets $A \subseteq k^m$ with the induced differential Zariski topology, and the structure sheaf $\mathcal{O}_A$ now consists of differential rational functions (locally quotients of differential polynomials with nonvanishing denominator). We do not have as simple a description of the global functions in $\mathcal{O}_A(A)$ as in the preceding cases, but it remains true that the morphisms between affine models A and B are given coordinate wise by global functions in $\mathcal{O}_A(A)$.

The conditions Aff I–III may be verified as before. By the Ritt-Raudenbush basis theorem [6] the affine models are again noetherian. Thus we have differential prevarieties and varieties, differential algebraic groups, the differential algebraic group chunks, and Weil's theorem applies, with the same argument as in our first example.

We remark that our notion of a differential algebraic group appears rather different from Kolchin's [7].

4. Constructible groups

(4.1) Constructible versus algebraic groups.
A *constructible* subset of a prevariety is a finite union of locally closed sets in the Zariski topology. A map from a constructible subset of one prevariety to a constructible subset of a second prevariety is called constructible if its graph is a constructible subset of the product of the ambient prevarieties. In affine spaces over an algebraically closed field the constructible sets and maps are just the definable ones (Tarski's theorem). The constructible sets and maps form a category. A constructible group is a constructible set G together with a constructible map $p\colon\ G \times G \to G$ making (G, p) a group. Constructible groups are just the groups interpretable in algebraically closed fields by [11], and in this guise they are potentially of considerable importance in pure model theory [10]. We prove here that such groups are in a certain sense disguised versions of algebraic groups, namely:

THEOREM 3. *For each constructible group G there is an algebraic group G_{alg} and a constructible group isomorphism*

$$\iota\colon G_{\text{alg}} \xrightarrow{\cong} G$$

such that for any constructible group homomorphism $\phi\colon G \to H$ with H an algebraic group, $\phi\iota$ is a morphism of perfect groups (in characteristic zero, a morphism of algebraic groups).

The universal property is a rather straightforward matter involving the structure of constructible morphisms between two algebraic groups; cf. Lemma 9 below. We will confine our discussion to the construction of G_{alg}. Our approach is somewhat indirect: from the constructible group extract a quasialgebraic group chunk (in characteristic zero, this just means an algebraic group chunk), thereby throwing away some of the original information, and then by the quasialgebraic version of Weil's theorem (Example 3.2 above) construct a quasialgebraic group G_{alg} agreeing with G on the chunk. The rest then follows from Lemma 7 and Theorem 1, part 2. So it will suffice to find a quasialgebraic group chunk in G.

(4.2) Constructible maps.
We quote a basic result on constructible maps from [8]. Our base field k is algebraically closed throughout.

LEMMA 9. *Let V be an irreducible affine variety and $f\colon\ V \to k^m$ a constructible map. Then there is a nonempty open subset U of V on which f is a morphism of varieties, if* characteristic$(k) = 0$, *and a morphism of perfect varieties if* characteristic$(k) = p > 0$.

DEFINITION. If A is a constructible set then the irreducible components of maximal dimension of A's Zariski closure in its ambient prevariety will be called the *blocks* of A.

LEMMA 10. *Let f: $A \to B$ be a constructible bijection between nonempty constructible sets, and let $A_1, \dots, A_k$ be the distinct blocks of A. Then B also has k blocks $B_1, \dots, B_k$ which can be arranged so that for each i there is nonempty open U_i in A_i with $U_i \subseteq A$ and $f[U_i]$ open in B_i and $f \restriction U_i$ an isomorphism of perfect prevarieties (where in characteristic 0, "perfect prevariety" simply means "prevariety").*

Proof. Let W be the Zariski closure of B. Using Lemma 9 we can find nonempty open subsets U_i of the blocks A_i of A such that $U_i \subseteq A$, $U_i \cap U_j = \varnothing$ for $i \neq j$, and $f \restriction U_i$: $U_i \to W$ is a morphism of perfect prevarieties. Let B_i be the Zariski closure of $f[U_i]$. Then $\dim(B_i) = \dim(A_i)$ and in particular $\dim(B) \geq \dim(A)$. By symmetry $\dim(B) = \dim(A)$, and the B_i are distinct blocks of B. Again by symmetry they are all the blocks of B. ∎

(4.3) The group chunk.
We now fix a constructible group G with multiplication p and inversion i. Let $B_1, \dots, B_k$ be the blocks of G, B their union.

LEMMA 11. *There are sets $V \subseteq G$, $U \subseteq V \times V$ such that the following hold.*
(1) *V is dense open in B, and the sets $V_i = V \cap B_i$ are disjoint and open in V.*
(2) *$i[V] = V$ and $i \restriction V$ is a perfect prevariety isomorphism.*
(3) *U is dense open in $V \times V$, $p[U] \subseteq V$ and $p \restriction U$ is a perfect prevariety morphism.*

Proof. By Lemma 10 there are dense open S, S' in B, both contained in G, so that i induces a perfect prevariety isomorphism between them. Let (initially) $V = S \cap S'$. This gives (2) and if S is small enough, also (1), which implies that the V_i are the irreducible components of V. It follows that $V \times V$ has components of constant dimension, so there is a dense open subset U_1 of $V \times V$ such that $p \restriction U_1$ is a perfect prevariety morphism. We note that for $a \in G$, λ_a can be restricted to a perfect prevariety isomorphism between dense open subsets of V.

Let $U = U_1 \cap p^{-1}[V]$, so U is open in $V \times V$. To see that U is dense in $V \times V$, we check that it meets each $V_i \times V_j$. Fix i, j and

$$(a, b) \in U_1 \cap (V_i \times V_j).$$

Then $a^{-1}V$ contains a nonempty open subset of V_j, hence meets

$$\{x \in V_j: (a, x) \in U_1\}.$$

If x is in this intersection then $(a, x) \in U \cap (V_i \times V_j)$, as desired. ∎

Now we will construct a perfect group chunk, thereby completing the proof of Theorem 3. The key step is the following.

LEMMA 12. *The sets U, V constructed above can be chosen to satisfy the following additional properties for all $a \in V$:*

(4) *The set $\{x \in V: (a, x) \in U\}$ is dense open in V.*

(5) *The set $\{x \in V: (x, a) \in U\}$ is dense open in V.*

(6) *The set $\{x \in V: (a, x) \in U, (ax, x^{-1}) \in U\}$ is dense open in V.*

Proof. Let U, V have the properties (1)–(3) of Lemma 11. If V' is dense open in V with $i[V'] = V'$ and $U' = U \cap (V' \times V') \cap p^{-1}[V']$ then the pair (U', V') inherits properties (1)–(3), as well as (4) (or (5)) if it held originally. Indeed properties (1)–(3) hold as in the proof of Lemma 11, while if U, V satisfy (4) and $a \in V'$, then $\{x \in V': (a, x) \in U\}$ is dense open in V', and $a^{-1}V' \cap V'$ contains a dense open subset of V', so their intersection

$$\{x \in V': (a, x) \in U'\}$$

is dense, and is certainly open.

Now begin with sets U, V satisfying (1)–(3). With the notation of Lemma 11, let V_i' be $\{a \in V: (\{a\} \times V_i) \cap U \neq \varnothing\}$. Then V_i' is dense open in V: it is easily seen to be open, and it meets each V_j. Let $V_0 = \cap V_i'$,

$$V' = i[V_0] \cap V_0, \quad U' = U \cap (V' \times V') \cap p^{-1}[V'].$$

Then U', V' still has properties (1)–(3), and we check (4): if $a \in V'$ then $(\{a\} \times V_i)$ meets U for each i, so $(a, x) \in U$ for a dense open set of $x \in V$, and (4) follows.

Similarly we may achieve (1)–(5) by a second shrinking operation. Finally we achieve (6) as follows. Let the injective morphism $\phi: U \to V \times V$ be defined by $\phi(a, x) = (ax, x^{-1})$. Since U has the $B_i \times B_j$ as its blocks, it follows from Lemma 10 that $\phi[U]$ has the same blocks, which are the blocks of $U \cap \phi[U]$ as well. As ϕ maps $\phi^{-1}[U]$ bijectively onto $U \cap \phi[U]$, the blocks of $\phi^{-1}[U]$ are also the same. So $\phi^{-1}[U] \cap (V_i \times V_j) \neq \varnothing$ for all i, j. Let π_1: $V \times V \to V$ be the first projection, $V_0 = \pi_1[\phi^{-1}[U]]$, $V' = V_0 \cap i[V_0]$, U' defined correspondingly. For $a \in V'$ the set

$$\{x \in V: (a, x) \in U, \phi(a, x) \in U\}$$

meets each V_i, hence is open dense in V. Intersecting with $\{x \in V' \colon ax \in V'\}$ we see that $\{x \in V' \colon (a, x) \in U', (ax, x^{-1}) \in U'\}$ is dense (and clearly open) in V'. This is (6).

It follows from (1)–(6) that $(V; p \restriction U, i \restriction V)$ is a perfect group chunk, as required. ■

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