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# HARMONIC REFLECTIONS WITH RESPECT TO SUBMANIFOLDS

BY

S. DONNINI, G. GIGANTE AND L. VANHECKE

## 1. Introduction

Geodesic symmetries on a Riemannian manifold $(M, g)$ are local diffeomorphisms which play an important role in the treatment of the geometry of $(M, g)$. Locally symmetric Riemannian manifolds are manifolds with *isometric* local geodesic symmetries. In [3] it is proved that "isometric" may be replaced by "*harmonic*" to characterize these spaces. Further, (local) reflections with respect to a curve are considered in [11] and the case of *harmonic reflections* has been studied in [1], [10].

All the results in these papers show that there is a strong relation between harmonic and isometric reflections. The main purpose of this paper is to clarify this relation. More precisely, the study of reflections with respect to a submanifold has been started in [2], [9]. In this paper we study harmonic reflections with respect to a submanifold and we will show that in the analytic case *a reflection with respect to a submanifold is harmonic if and only if it is an isometry*. As a corollary we obtain a result for *holomorphic* and *anti-holomorphic reflections* on a quasi-Kähler manifold.

## 2. Preliminaries

In this section we give a short description of the basic material we shall use in the rest of the paper. (See [6], [7] for more details.)

Let $(M, g)$ be a Riemannian manifold of class $C^\infty$ and $B$ a (connected) topologically embedded submanifold which is relatively compact. Let $m \in B$ and let $\{E_1, \ldots, E_n\}$, $n = \dim M$, be a local orthonormal frame field of $(M, g)$ defined along $B$ in a neighborhood of $m$. Let $q = \dim B$ and specialize the moving frame such that $E_1, \ldots, E_q$ are tangent vector fields and $E_{q+1}, \ldots, E_n$ are normal vector fields. Further, let $(y^1, \ldots, y^q)$ be a system of coordinates in a neighborhood of $m$ in $B$ such that

$$\frac{\partial}{\partial y^i}(m) = E_i(m), \quad i = 1, \ldots, q,$$

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and let $(x^1, \ldots, x^n)$ be a system of Fermi coordinates with respect to $m$, $(y^1, \ldots, y^q)$ and $(E_{q+1}, \ldots, E_n)$. These coordinates are defined in an open neighborhood $U_m$ of $m$ in $M$. More precisely we have

$$x^i\left(\exp_\nu\left(\sum_{q+1}^{m} t_\beta E_\beta\right)\right) = y^i, \quad i = 1, \ldots, q,$$

$$x^\alpha\left(\exp_\nu\left(\sum_{q+1}^{m} t_\beta E_\beta\right)\right) = t^\alpha, \quad \alpha = q+1, \ldots, n,$$

where $\nu = T^\perp B$ is the normal bundle of $B$.

Choose a fixed normal unit vector $u$ at $m$, $u \in T_m^\perp B \subset T_m M$, and consider the geodesic $\gamma(t) = \exp_m(tu)$. We have

$$\gamma(0) = m, \quad \gamma'(0) = u.$$

We specialize now the frame field $\{E_1, \ldots, E_n\}$ in such a way that

$$E_n(m) = u = \gamma'(0).$$

Next, consider the frame field $\{e_1(t), \ldots, e_n(t)\}$ along $\gamma(t)$ obtained by parallel transport of $\{E_1(m), \ldots, E_n(m)\}$. Further, let $Y_i, Y_a$, $i = 1, \ldots, q$, $a = q+1, \ldots, n-1$, denote the Jacobi vector fields along $\gamma$ with initial conditions

$$Y_i(0) = E_i(m), \quad Y_i'(0) = \nabla_u \frac{\partial}{\partial x^i},$$

$$Y_a(0) = 0, \quad Y_a'(0) = E_a(m),$$

where $\nabla$ denotes the Riemannian connection of $(M, g)$. Note that

$$(1) \qquad Y_i(t) = \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}, \quad Y_a(t) = t\left.\frac{\partial}{\partial x^a}\right|_{\gamma(t)}.$$

Define the endomorphism-valued function $t \mapsto D_u(t)$ by

$$(2) \qquad Y_\alpha(t) = D_u(t) e_\alpha, \quad \alpha = 1, \ldots, n-1.$$

Then the Jacobi equation yields

$$(3) \qquad D_u'' + R \circ D_u = 0$$

where $t \mapsto R(t)$ is the endomorphism-valued function on $(\gamma'(t))^\perp \subset T_{\gamma(t)}M$ defined by

$$R(t)x = R_{\gamma'(t)x}\gamma'(t), \quad x \in (\gamma'(t))^\perp.$$

$R$ denotes the Riemann curvature tensor on $(M, g)$ defined by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all tangent vectors $X, Y$ of $M$.

To obtain the initial conditions for $D_u(t)$, where $u$ is fixed, we use some facts about submanifolds. Denote by $\tilde{\nabla}$ the Riemannian connection of $B$. Further, let $X, Y$ be tangent vector fields and $N$ a unit normal vector field along an open domain in $B$. Then we have the orthogonal decompositions

$$\nabla_X Y = \tilde{\nabla}_X Y + T_X Y, \quad \nabla_X N = T(N)X + \nabla_X^{\perp} N,$$

where $T_X Y = T(X, Y)$ is the second fundamental form operator of $B$, $T(N)$ the shape operator of $B$ corresponding to the normal vector $N$, and $\nabla^{\perp}$ is the normal connection along $B$. Note that

$$g(T(N)X, Y) = -g(T(X,Y), N).$$

Also, we use the operator $\perp$ defined in [6], [7] by

$$\perp_X N = \nabla_X^{\perp} N.$$

Now using the initial conditions for $Y_\alpha$, we obtain the following initial conditions (in the matrix form with respect to the basis $\{E_1, \dots, E_{n-1}\}_m$ of $(u)^{\perp} \subset T_m M$):

$$D_u(0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad D_u'(0) = \begin{pmatrix} T(u) & 0 \\ -{}^t\!\perp(u) & I \end{pmatrix} \tag{4}$$

where

$$T(u)_{ij} = g(T(u)E_i, E_j)(m),$$
$$\perp(u)_{ia} = g(\perp_{E_i} E_a, E_n)(m).$$

In the rest of the paper we will consider the local diffeomorphism

$$\varphi_B \colon p \mapsto \varphi_B(p), \quad \exp_m(tu) \mapsto \exp_m(-tu)$$

for $u \in T_m B$, $\|u\| = 1$. $\varphi_B$ is called *the (local) reflection with respect to the submanifold B*. Using the Fermi coordinates, $\varphi_B$ is locally given by

$$\varphi_B \colon (x^1, \dots, x^q, x^{q+1}, \dots, x^n) \mapsto (x^1, \dots, x^q, -x^{q+1}, \dots, -x^n).$$

## 3. Harmonic reflections

Let $(M, g)$ and $(N, h)$ be two Riemannian manifolds with metrics $g$ and $h$ and let $f: (M, g) \to (N, h)$ be a smooth map. Then the covariant differential $\nabla(df)$ is called the *second fundamental form* and the *tension field* of $f$, denoted by $\tau(f)$, is the trace of $\nabla(df)$. We say $f$ is *harmonic* if $\tau(f) = 0$.

To express this condition analytically, let $U \subset M$ be a domain with coordinates $(x^1, \dots, x^m)$ and $V \subset M$ a domain with coordinates $(y^1, \dots, y^n)$. Then $f$ can be locally represented by $y^\alpha = f^\alpha(x^1, \dots, x^m)$, $\alpha = 1, \dots, n$. Further we have

$$(5) \qquad \nabla(df)^\gamma_{ij} = \frac{\partial^2 f^\gamma}{\partial x^i \, \partial x^j} - {}^M\Gamma^k_{ij} \frac{\partial f^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta}(f) \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j},$$

$i, j = 1, \dots, m$ and $j = 1, \dots, n$. Here ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbols for $(M, g)$ and $(N, h)$ respectively. Hence, $f$ is harmonic if and only if

$$(6) \qquad \tau(f)^\gamma = g^{ij}(\nabla(df))^\gamma_{ij} = 0.$$

For more details about harmonic maps we refer to [4], [5].

From these remarks we now get easily, using Fermi coordinates,

THEOREM 1. *The local reflection $\varphi_B$ with respect to the submanifold $\varphi_B$ is harmonic if and only if*

$$(7) \qquad \tau(\varphi_B)^k(p) = \left\{ g^{ij}\nabla(d\varphi_B)^k_{ij} + 2g^{ia}\nabla(d\varphi_B)^k_{ia} + g^{ab}\nabla(d\varphi_B)^k_{ab} \right\}(p) = 0,$$

$$(8) \qquad \tau(\varphi_B)^c(p) = \left\{ g^{ij}\nabla(d\varphi_B)^c_{ij} + 2g^{ia}\nabla(d\varphi_B)^c_{ia} + g^{ab}\nabla(d\varphi_B)^c_{ab} \right\}(p) = 0,$$

*for $i, j, k = 1, \dots, q$ and $a, b, c = q+1, \dots, n$, where*

$$(9_1) \qquad \nabla(d\varphi_B)^k_{ij}(p) = -\Gamma^k_{ij}(p) + \Gamma^k_{ij}(\varphi_B(p)),$$

$$(9_2) \qquad \nabla(d\varphi_B)^k_{ia}(p) = -\Gamma^k_{ia}(p) - \Gamma^k_{ia}(\varphi_B(p)),$$

$$(9_3) \qquad \nabla(d\varphi_B)^k_{ab}(p) = -\Gamma^k_{ab}(p) + \Gamma^k_{ab}(\varphi_B(p)),$$

$$(9_4) \qquad \nabla(d\varphi_B)^c_{ij}(p) = \Gamma^c_{ij}(p) + \Gamma^c_{ij}(\varphi_B(p)),$$

$$(9_5) \qquad \nabla(d\varphi_B)^c_{ia}(p) = \Gamma^c_{ia}(p) - \Gamma^c_{ia}(\varphi_B(p)),$$

$$(9_6) \qquad \nabla(d\varphi_B)^c_{ab}(p) = \Gamma^c_{ab}(p) + \Gamma^c_{ab}(\varphi_B(p)).$$

It is worthwhile to note that $(9_1)$, $(9_3)$, $(9_5)$ are odd functions and $(9_2)$, $(9_4)$, $(9_6)$ are even functions.

In what follows we will need to compute the Christoffel symbols with respect to a system of Fermi coordinates. Therefore we shall use the well-known expression

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{m} g^{kl}\left\{\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right\}. \tag{10}$$

## 4. First result

To compute (10) we use power series expansions for the components of $g$ with respect to a system of Fermi coordinates. We have

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad g_{ia} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^a}\right), \quad g_{ab} = g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right),$$

$i, j = 1, \dots, q$; $a, b = q+1, \dots, n-1$, and

$$g_{in} = g_{an} = 0, \quad g_{nn} = 1.$$

Using (1), we get for $p = \exp_m(tu)$, $u \in T_m^{\perp}B$,

$$\begin{aligned} g_{ij}(p) &= g(D_u(t)e_i, D_u(t)e_j), \\ g_{ia}(p) &= \frac{1}{t}g(D_u(t)e_i, D_u(t)e_a), \\ g_{ab}(p) &= \frac{1}{t^2}g(D_u(t)e_a, D_u(t)e_b). \end{aligned} \tag{11}$$

Finally, using (3) and the initial values (4), together with a rotation of $\{E_{q+1}, \dots, E_n\}$ at $m$, we obtain with $T = T(u)$, $\perp(u) = \perp$ and $R = R_u.u$:

$$\begin{aligned} g_{ij}(p) &= g(E_i, E_j)(m) + 2tg(TE_i, E_j)(m) + O(t^2), \\ g_{ia}(p) &= -tg({}^t\!\perp E_i, E_a)(m) - \frac{2}{3}t^2 g(RE_i, E_a)(m) + O(t^3), \\ g_{ab}(p) &= g(E_a, E_b)(m) - \frac{t^2}{3}g(RE_a, E_b)(m) + O(t^3), \end{aligned} \tag{12}$$

where $i, j = 1, \dots, q$ and $a, b = q+1, \dots, n$. (Note that $u \in T_m^{\perp}B$, $\|u\| = 1$ is now arbitrary with respect to this basis.)

From this we easily get

$$
\begin{aligned}
& g^{ij}(p) = g(E_i, E_j)(m) - 2tg(TE_i, E_j)(m) + O(t^2), \\
(13) \qquad & g^{ia}(p) = tg({}^t\!\perp E_i, E_a)(m) + \frac{2}{3}t^2 g(RE_i, E_a)(m) + O(t^3), \\
& g^{ab}(p) = g(E_a, E_b)(m) + O(t^2).
\end{aligned}
$$

Finally, using (10), (12) and (13) we get for $u = E_n(m)$:

$$
\begin{aligned}
& \Gamma^n_{ij}(p) = -g(TE_i, E_j)(m) + O(t), \\
(14) \qquad & \Gamma^n_{ia}(p) = g({}^t\!\perp E_i, E_a)(m) + tg(RE_i, E_a)(m) + O(t^2), \\
& \Gamma^n_{ab}(p) = O(t).
\end{aligned}
$$

From this we obtain our first result:

THEOREM 2. *If the reflection $\varphi_B$ with respect to the submanifold $B$ is harmonic, then $B$ is totally geodesic.*

*Proof.* Using (8) and the formulas above we compute the first order term in $\tau(\varphi_B)^n$ for $u = E_n(m)$. After an easy calculation one gets

$$\sum_{i,j=1}^{q} g(TE_i, E_j)^2(m) = 0$$

for all $m \in B$ and all $u \in T_m^\perp B$. Hence $T = 0$ which means that $B$ is totally geodesic.

## 5. The main theorem

In the rest of this paper we suppose that $(M, g)$, $B$ and the embedding are analytic. In this case we prove:

THEOREM 3. *The reflection $\varphi_B$ with respect to the submanifold $B$ is harmonic if and only if it is isometric.*

*Proof.* Since an isometry is always harmonic, we have only to prove the converse. To do this we have to prove $\varphi_B^* g = g$, i.e.

$$
(15) \qquad \begin{gathered}
g_{ij}(\varphi_B(p)) = g_{ij}(p), \quad g_{ab}(\varphi_B(p)) = g_{ab}(p), \\
g_{ia}(\varphi_B(p)) = -g_{ia}(p).
\end{gathered}
$$

As in Section 4 we put $p = \exp_m(tu)$, $u \in T_m^\perp B$, $\|u\| = 1$. Then (15) means that we have to prove that the $g_{ij}$, $g_{ab}$ are even functions and the $g_{ia}$ are odd functions of $t$.

To do this we use the following result.

LEMMA 4. *With respect to Fermi coordinates, with $u = E_n(m)$, if the $g_{ij}$ and $g_{ab}$ are even functions up to order $k+1$ and the $g_{ia}$ are odd functions up to order $k+1$, we have*

$$(16)\qquad 2(\Gamma^n_{ab})_{(k)} = (k+3)(g^{ab})_{(k+1)},\quad 2(\Gamma^n_{ij})_{(k)} = (k+1)(g^{ij})_{(k+1)}$$

*if $k$ is even and*

$$(17)\qquad 2(\Gamma^n_{ia})_{(k)} = (k+2)(g^{ia})_{(k+1)}$$

*if $k$ is odd. Here the index indicates the order of the coefficient in the Taylor expansion.*

*Proof.* First, by our hypothesis and the rule for obtaining the elements of the inverse of a matrix, the $g^{ij}$, $g^{ab}$ are even functions of $t$ up to order $k+1$, and the $g^{ia}$ are odd functions of $t$ up to order $k+1$.

Further, using $g^{\alpha\gamma}g_{\gamma\beta} = \delta^\alpha_\beta$, we get

$$(g^{ab})_{(k+1)} = -(g_{ab})_{(k+1)},\quad (g^{ij})_{(k+1)} = -(g_{ij})_{(k+1)}$$

if $k$ is even and

$$(g^{ia})_{(k+1)} = -(g_{ia})_{(k+1)}$$

if $k$ is odd.

Finally, we use (10) and a method similar to that used in Section 4, combined with the symmetry properties of the curvature tensor $R$ and its covariant derivatives, to obtain the formulas (16) and (17). (We omit the straightforward computations.)

Now we return to the proof of the main theorem. Suppose we have the hypotheses of Lemma 4 and let $\varphi_B$ be harmonic. Then it follows from (8) and Lemma 4 that the vanishing of $\tau(\varphi_B)^n_{(2k+1)}$ yields

$$(k+1)\sum_{i,j}(g_{ij})^2_{(k+1)} + (k+3)\sum_{a,b}(g_{ab})^2_{(k+1)} = 0$$

if $k$ is even and

$$\sum_{i,a}(g_{ia})^2_{(k+1)} = 0$$

if $k$ is odd. Using now induction on $k$ ($k = 0$ has been considered in Section 4) we obtain the required result.

## 6. Holomorphic and anti-holomorphic reflections

Let $(M, g, J)$ be an almost Hermitian manifold. Then $(M, g, J)$ is said to be a quasi-Kähler manifold if

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$$

for all tangent vector fields $X$, $Y$. It is proved in [8] that any holomorphic and anti-holomorphic map from a quasi-Kähler manifold to a quasi-Kähler manifold is harmonic. From this and the main theorem we have:

THEOREM 5. *Let $\varphi_B$ be a holomorphic or anti-holomorphic reflection with respect to a submanifold $B$ of a quasi-Kähler manifold $(M, g, J)$. Then $\varphi_B$ is an isometry.*

This extends a result of [2]. Note that it is proved in [2] that when $\varphi_B$ is holomorphic, then $B$ is a holomorphic submanifold. Using a similar procedure as in [2], it is easy to show that when $\varphi_B$ is anti-holomorphic, then $B$ is a totally real submanifold and $\dim B = \frac{1}{2} \dim M$.

## 7. Remark

The proof of our main theorem is much easier when the manifold $(M, g)$ is locally symmetric. In this case one may use the following result.

LEMMA 6 [2]. *Let $(M, g)$ be a locally symmetric Riemannian manifold and $B$ a submanifold. Then the reflection $\varphi_B$ is an isometry if and only if*
(i) *$B$ is totally geodesic and*
(ii) *$R_{uv}u$ is normal to $B$ for all $u, v \in T^{\perp}B$.*

The proof of our result follows then easily from Theorem 2 and only the first step in the induction procedure.

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UNIVERSITÀ DI CAGLIARI
CAGLIARI, ITALY

UNIVERSITÀ DI PARMA
PARMA, ITALY

KATHOLIEKE UNIVERSITEIT LEUVEN
LEUVEN, BELGIUM