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# SUBNORMAL AND ASCENDANT SUBGROUPS WITH RANK RESTRICTIONS

BY

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Let $\mathfrak{F}_r^*$ denote the class of groups with finite abelian section rank (thus $H \in \mathfrak{F}_r^*$ if and only if every elementary abelian $p$-section of $H$ is finite for all primes $p$). If $H$ and $K$ are subnormal $\mathfrak{F}_r^*$-subgroups of the group $G$, then the subgroup $J = \langle H, K \rangle$ is not necessarily subnormal in $G$ (even if $G$ belongs to $\mathfrak{F}_r^*$). This is shown in [4, §5], using a construction due to Zassenhaus and Hall. However, it is proved in the same paper that if $G \in \mathfrak{F}_r^*$ then $J$ is ascendant in $G$. Here we make the following improvement on that result.

THEOREM. *Let $H_1, H_2, \ldots, H_n$ be subnormal $\mathfrak{F}_r^*$-subgroups of the group $G$, and let $J = \langle H_1, H_2, \ldots, H_n \rangle$. Then*
(a) *$J$ belongs to $\mathfrak{F}_r^*$,*
(b) *$J$ is ascendant in $G$.*

The number of subgroups $H$ must be finite for either of these conclusions to hold, as may be seen by considering an infinite elementary abelian $p$-group in the case of (a), and a group of type $G = C_p \operatorname{wr} C_{p\infty}$ in the case of (b). (In the latter, the self-normalizing "top group" is a union of cyclic $p$-groups, each of which is subnormal in $G$.) Further, if we weaken the hypothesis of the theorem by replacing "subnormal" by "ascendant", then the conclusion (a) does not follow.

*Example.* Let $p$ be a fixed prime. For each $i = 1, 2, \ldots,$ let $H_i$ be a group with presentation

$$\langle h_{i,i}, h_{i,2}, \ldots : (h_{i,1})^p = 1, (h_{i,j+1})^p = h_{i,j} \ (j = 1, 2, \ldots) \rangle.$$

Then each $H_i$ is clearly a group of type $C_{p\infty}$ and thus of rank 1. Let $H$ denote the direct product of the $H_i$, $i = 1, 2, \ldots,$ and let $x$ be the automorphism of $H$ defined by

$$x : (h_{i,j}) \to (h_{i,j})(h_{i+1,j-1}); \qquad i, j = 1, 2, \ldots$$

(where, for each $i$, $(h_{i+1,0})$ is interpreted as the identity element). Note that $x$

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is indeed an automorphism, its inverse being determined by the assignment

$$h_{i,j} \to h_{i,j} h_{i+1,j-1}^{-1} h_{i+2,j-2} \cdots h_{i+j-1,1}^{(-1)^{j-1}}.$$

Let $G$ be the natural split extension of $H$ by $\langle x \rangle$. Then it is easy to see that $H_1 \triangleleft^2 G$ and that, for each $i, j = 1, 2, \ldots, h_{i,j}$ belongs to the $j$th term of the upper central series of $G$. So $\langle x \rangle$ is ascendant in $G$, but $G = \langle H_1, x \rangle$ has an abelian subgroup of infinite $p$-rank, namely $H$.

The above example shows that the class of groups with finite rank does not form an ascendant coalition class, thus answering a question left open in [1]. The same applies to the class of solvable groups with finite rank. The question as to whether $(b)$ still holds with the weakened hypothesis referred to above is left unanswered here.

In proving the theorem, we require a residual property of certain groups in the class $\mathfrak{F}_r^*$. This is given in the following lemma, which is little more than an observation based on a well-established fact concerning radical $\mathfrak{F}_r^*$-groups. Note that radical groups in $\mathfrak{F}_r^*$ are hyperabelian (see [2, Vol. 2, p. 128]).

LEMMA. *Let $G$ be a hyperabelian group in the class $\mathfrak{F}_r^*$, let $p$ be a prime and let $D$ denote the maximal normal (periodic) $p'$-subgroup of $G$. Then $G/D$ is of finite rank and solvable.*

Since every element $x$ of $G$ is either of infinite order or of order divisible by some prime $p$, there is always such a subgroup $D = D(x)$ which does not contain $x$. So we have the following immediate consequence.

COROLLARY. *A hyperabelian group in the class $\mathfrak{F}_r^*$ is residually solvable-of-finite-rank.*

*Proof of the lemma.* We assume $D = 1$. Let $T$ be the maximal normal periodic subgroup of $G$. Then $G/T$ is solvable of finite rank [2, Lemma 9.34] so we may assume $G = T$ (which does not have any non-trivial normal $p'$-subgroups, since the maximal such would be characteristic in $T$ and thus normal in $G$). Let $H$ be the Hirsch-Plotkin radical of $G$. Then $H$ is a Černikov $p$-group (by Corollary 1 to Theorem 6.36 of [2] and the fact that $H$ has no normal $p'$-subgroups). Let $C = C_G(H)$. Then $C \leq H$ [2, Lemma 2.32] and $G/C$ embeds in Aut $H$ and is periodic, and so is itself a (solvable) Černikov group [2, Theorem 3.29]. The result follows.

*Proof of the theorem.* First we observe that the class $\mathfrak{F}_r^*$ is extension-closed (this is easily proved). In particular, the product of a normal $\mathfrak{F}_r^*$-subgroup and a subnormal one is again in $\mathfrak{F}_r^*$, and an easy induction shows that the product of two permuting subnormal $\mathfrak{F}_r^*$-subgroups is in $\mathfrak{F}_r^*$. Then, with the hypotheses of the theorem satisfied, results from [3] (Theorem A and Corollary C1) establish that, for some integer $\lambda$, $J^{(\lambda)}$ is subnormal in $G$ and belongs to $\mathfrak{F}_r^*$. To complete the proof of (a), suppose $J$ is solvable and let $D$ be the maximal normal $p'$-subgroup of $J$, where $p$ is an arbitrary prime. It suffices to prove that $J/D$ has finite rank, and we may suppose $D = 1$. Since $J$ then has no subnormal $p'$-subgroups, neither has any of the subgroups $H_i$. By the lemma, each $H_i$ has finite rank. By [1], $J$ also has finite rank and (a) is proved.

Now let $G = K_0, K_1, \ldots, K_m = J^{(\lambda)}$ denote the successive terms of the normal closure series of $J^{(\lambda)}$ in $G$. Then each $K_i$ is normalized by $J$ and, to establish (b), it suffices to prove that, for arbitrary $i$, $JK_i$ is ascendant in $JK_{i-1} = L$, say. To do this, we first factor by $K_i$ and thus reduce to the case where $J$ is solvable. Now write $N_0 = 1$ and $N_i = H_1^L \ldots H_i^L$, $i = 1, \ldots, n$. We need only show that $JN_{i-1}$ is ascendant in $JN_i$ ($i = 1, \ldots, n$). (Note that $JN_n = J^L$ is normal in $L$.) Again factoring, it suffices to prove that $J$ is ascendant in $JH^L$, where $H$ is some $H_i$. Our next reduction is to the case where $H$ is abelian: this may be carried out by first introducing subgroups $JA_j^L$ where $A_j$ runs through the terms of the derived series of $H$ and then arguing by induction. Then $N = H^L$ is locally nilpotent, and clearly we can make the further assumption that $N$ is either torsion-free or a $p$-group for some prime $p$. In either case, let $D$ denote the maximal normal $p'$-subgroup of $JN$. Then $[D, N] \subseteq D \cap N = 1$ and so $J \cap D \triangleleft JN$. Set $J \cap D = 1$. Then, for each $i$, $H_i \cap D = 1$, so $H_i$ has no subnormal $p'$-subgroups and is therefore of finite rank, by the lemma. By [1], $J$ is therefore subnormal in $JN$, and the proof is complete.

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